Solution to Monthly Problem 11650

David H. Bailey*    Jonathan M. Borwein†

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In the June-July 2012 issue of the American Mathematical Monthly, Michael Becker asks to evaluate [1]

\[ \int_0^\infty \int_0^\infty e^{-(x-y)^2} \sin^2(x^2 + y^2) \frac{x^2 - y^2}{(x^2 + y^2)^2} \, dy \, dx. \]  

(1)

In fact, this problem can be solved by either Mathematica or Maple nearly “out of the box,” after making a change of variables to polar coordinates, and then simplifying the resulting expression. In Mathematica, the entire problem can be solved by typing

\begin{verbatim}
Integrate[r * Simplify[Exp[-(x - y)^2] * Sin[x^2 + y^2]^2 * 
    (x^2 - y^2)/(x^2 + y^2)^2 /. x -> r*Cos[t] /. y -> r*Sin[t]],
    {t, Pi/4, Pi/2}, {r, 0, Infinity}]
\end{verbatim}

which produces the output

\[ \frac{1}{16} (\log(5) - 4 \arctan(1/2)) \]

or, in other words,

\[ \frac{1}{16} (-4 \arctan(1/2) - \log(5)) = -0.21650177177733280247 \ldots \]  

(2)

*Lawrence Berkeley National Laboratory, Berkeley, CA 94720, DHBailey@lbl.gov. Supported in part by the Director, Office of Computational and Technology Research, Division of Mathematical, Information, and Computational Sciences of the U.S. Department of Energy, under contract number DE-AC02-05CH11231.

†Laureate Professor and Director Centre for Computer Assisted Research Mathematics and its Applications (CARMA), University of Newcastle, Callaghan, NSW 2308, Australia. Distinguished Professor, King Abdulaziz University, Jeddah 80200, Saudi Arabia. Email: jonathan.borwein@newcastle.edu.au.
For us, the most significant challenge was to find a way to numerically evaluate the integral, as a check to Mathematica’s analytic evaluation. We were able to do this, to a few decimal digits accuracy, by using the QD software, available from the first author’s website [2].

References
