Solution to Monthly Problem 11650

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In the June-July 2012 issue of the *American Mathematical Monthly*, Michael Becker asks to evaluate [1]

$$\int_{x=0}^{\infty}\int_{y=x}^{\infty} e^{-(x-y)^2}\sin^2(x^2+y^2)\frac{x^2-y^2}{(x^2+y^2)^2}\,\mathrm{d}y\,\mathrm{d}x. \tag{1}$$

In fact, this problem can be solved by either *Mathematica* or *Maple* nearly "out of the box," after making a change of variables to polar coordinates, and then simplifying the resulting expression. In *Mathematica*, the entire problem can be solved by typing

```
Integrate[r * Simplify[Exp[-(x - y)^2] * Sin[x^2 + y^2]^2 *
  (x^2 - y^2)/(x^2 + y^2)^2 /. x -> r*Cos[t] /. y -> r*Sin[t]],
  {t, Pi/4, Pi/2}, {r, 0, Infinity}]
```

which produces the output

```
1/16 (-4 ArcTan[1/2] - Log[5])
```

or, in other words,

$$\frac{1}{16}\left(-4\arctan(1/2)-\log(5)\right) = -0.21650177177733280247\ldots \tag{2}$$

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For us, the most significant challenge was to find a way to numerically evaluate the integral, as a check to *Mathematica*'s analytic evaluation. We were able to do this, to a few decimal digits accuracy, by using the QD software, available from the first author's website [2].

References

[1] Michael Becker, "Problem 11650," *American Mathematical Monthly*, vol. 119 (Jun–Jul 2012), pg. 522.

[2] Yozo Hida, Xiaoye S. Li and David H. Bailey, "Algorithms for quad-double precision floating point arithmetic," *15th IEEE Symposium on Computer Arithmetic*, IEEE Computer Society, 2001, pg. 155-162. This software is available at `http://crd-legacy.lbl.gov/~dhbailey/mpdist`.