The Greatest Mathematical Discovery?

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1 Introduction

Question: What mathematical discovery more than 1500 years ago:

- Is one of the greatest, if not the greatest, single discovery in the field of mathematics?
- Involved three subtle ideas that eluded the greatest minds of antiquity, even geniuses such as Archimedes?
- Was fiercely resisted in Europe for hundreds of years after its discovery?
- Even today, in historical treatments of mathematics, is often dismissed with scant mention, or else is ascribed to the wrong source?

Answer: Our modern system of positional decimal notation with zero, together with the basic arithmetic computational schemes, which were discovered in India prior to 500 CE.

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2 Why?

As the 19th century mathematician Pierre-Simon Laplace explained:

> *It is India that gave us the ingenious method of expressing all numbers by means of ten symbols, each symbol receiving a value of position as well as an absolute value; a profound and important idea which appears so simple to us now that we ignore its true merit. But its very simplicity and the great ease which it has lent to all computations put our arithmetic in the first rank of useful inventions; and we shall appreciate the grandeur of this achievement the more when we remember that it escaped the genius of Archimedes and Apollonius, two of the greatest men produced by antiquity.* [7, pg. 527]

French Historian Georges Ifrah describes the significance in these terms:

> *Now that we can stand back from the story, the birth of our modern number-system seems a colossal event in the history of humanity, as momentous as the mastery of fire, the development of agriculture, or the invention of writing, of the wheel, or of the steam engine.* [11, pg. 346–347]

As Laplace noted, the scheme is anything but "trivial," since it eluded the best minds of the ancient world, even superhuman geniuses such as Archimedes. Archimedes saw far beyond the mathematics of his time, even anticipating numerous key ideas of modern calculus and numerical analysis. He was also very skilled in applying mathematical principles to engineering and astronomy. Nonetheless he used a cumbersome Greek numeral system for calculations. Archimedes' computation of π, a *tour de force* of numerical interval analysis, was performed without either positional notation or trigonometry [2, 13].

Perhaps one reason this discovery gets so little attention today is that it is very hard for us to appreciate the enormous difficulty of using Greco-Roman numerals, counting tables and abacuses. As Tobias Dantzig (father of George Dantzig, the inventor of linear programming) wrote,

> *Computations which a child can now perform required then the services of a specialist, and what is now only a matter of a few minutes [by hand] meant in the twelfth century days of elaborate work.* [6, pg. 27]

Michel de Montaigne, Mayor of Bordeaux and one of the most learned men of his day, confessed in 1588 (prior to the widespread adoption of decimal arithmetic in Europe) that in spite of his great education and erudition, "*I cannot yet cast account either with penne or Counters.*" That is, he could not do basic arithmetic [11, pg. 577]. In a similar vein, at about the same time a wealthy German merchant, consulting a scholar regarding which European university offered the best education for his son, was told the following:

> *If you only want him to be able to cope with addition and subtraction, then any French or German university will do. But if you are intent on your son going on to multiplication and division—assuming that he has sufficient gifts—then you will have to send him to Italy.* [11, pg. 577]

We observe in passing that Claude Shannon (1916–2001) constructed a mechanical calculator wryly called *Throback 1* at Bell Labs in 1953, which computed in Roman, so as to demonstrate that it was possible, if very difficult, to compute this way.

Indeed, the development of modern decimal arithmetic hinged on three key abstract (and certainly non-intuitive) principles [11, pg. 346]:

(a) Graphical signs largely removed from intuitive associations;

(b) Positional notation; and

(c) A fully operational zero—filling the empty spaces of missing units and at the same time representing a null value.

Some civilizations succeeded in discovering one or two of these principles, but none of them, prior to early-first-millennium India, found all three and then combined them with effective algorithms for practical computing. The Mayans came close—before 36 BCE they had devised a place-value system that included a zero. However, in their system successive positions represented the mixed sequence $(1, 20, 360, 7200, 144000, \cdots)$ rather than the purely base-20 sequence $(1, 20, 400, 8000, 160000, \cdots)$, which precluded the possibility that their numerals could be used as part of an efficient system for computation.

3 History

So who exactly discovered the Indian system? Sadly, there is no record of the individual who first discovered the scheme, who, if known, would surely rank among the greatest mathematicians of all time. As Dantzig notes, "the achievement of the unknown Hindu who some time in the first centuries of our era discovered [positional decimal arithmetic] assumes the proportions of a world-event. Not only did this principle constitute a radical departure in method, but we know now that without it no progress in arithmetic was possible" [6, pg. 29–30].

The earliest document that exhibits familiarity with decimal arithmetic, and which at the same time can be accurately dated, is the Indian astronomical work *Lokavibhaga* ("Parts of the Universe") [14]. Here, for example, we find numerous large numbers, such as $14236713, 13107200000$ and 70500000000000000, as well as detailed calculations such as $(14230249 - 355684)/212 = 65446\frac{13}{212}$ [14, pg. 70, 79, 131, 69]. Methods for computation were not presented in this work — the author evidently presumed that the reader understood decimal arithmetic. Near the end of the *Lokavibhaga*, the author provides detailed astronomical data that enable modern scholars to confirm, in two independent ways, that this text was written on 25 August 458 CE (Julian calendar). The text also mentions that it was written in the 22nd year of the reign of Simhavarman, which corresponds to 458 CE. As Ifrah points out, this information not only allows us to date the document with precision, but also proves its authenticity [11, pg. 417].

An even more ancient source employing positional decimal arithmetic is the Bakhshali manuscript, a copy of a ancient mathematical treatise that gives rules for computing with fractions. British scholar Rudolf Hoernle, for instance, has noted that the document was written in the "Shloka" style, which was replaced by the "Arya" style prior to 500 CE, and furthermore that it was written in the "Gatha" dialect, which was largely replaced by Sanskrit, at least in secular writings, prior to 300 CE. Also, unlike later documents, it used a plus sign for negative numbers and did not use a dot for zero (although it used a dot for empty position). For these and other reasons scholars have concluded that the original document most likely was written in the third or fourth century [10]. One intriguing item in the Bakhshali manuscript is the following approximation for the square root [5]:

$$\sqrt{a^2 + x} \quad \approx \quad a + \frac{x}{2a} - \frac{\left(\frac{x}{2a}\right)^2}{2\left(a + \frac{x}{2a}\right)}. \tag{1}$$

In 510 CE, the Indian mathematician Aryabhata presented schemes not only for various arithmetic operations, but also for square roots and cube

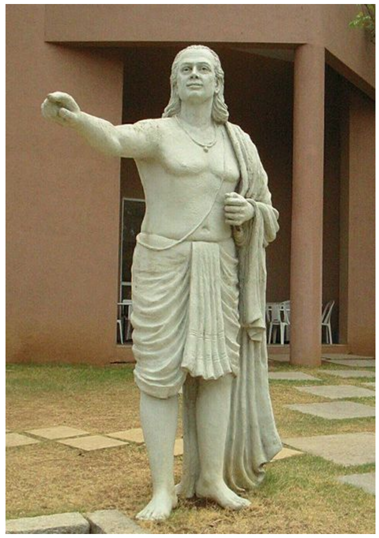

Figure 1: Statue of Aryabhata on the grounds of IUCAA, Pune, India (courtesy Wikimedia).

roots. Additionally, Aryabhata gave a decimal value of $\pi = 3.1416$. Aryabhata's "digital" algorithms for computing square roots and cube roots are illustrated in Figures 2 and 3 (based on [1, pg. 24–26]). A statue of Aryabhata, on display at the Inter-University Centre for Astronomy and Astrophysics (IUCAA) in Pune, India, is shown in Figure 1.

In the centuries that followed, the Indian system was slowly disseminated to other countries. In China, there are records as early as the Sui Dynasty (581–618 CE) of Chinese translations of the *Brahman Arithmetical Classic*, although sadly none of these copies have survived [9].

The Indian system was introduced in Europe by Gerbert of Aurillac in the tenth century. He traveled to Spain to learn about the system first-hand from Arab scholars, then was the first Christian to teach mathematics using decimal arithmetic, all prior to his brief reign as Pope Sylvester II (999–1002 CE) [3, pg. 5]. Little progress was made at the time, though, in part because of clerics who, in the wake of the crusades, rumored that Sylvester II had been a sorcerer, and that he had sold his soul to Lucifer during his travels to Islamic Spain. These accusations persisted until 1648, when papal authorities who reopened his tomb reported that Sylvester's body had not, as suggested in historical accounts, been dismembered in penance for Satanic practices [3, pg. 236]. Sylvester's reign was a turbulent time, and he died after a short reign. It is worth speculating how history would have been different had this remarkable scientist-Pope lived longer.

In 1202 CE, Leonardo of Pisa, also known as Fibonacci, reintroduced the Indian system into Europe with his book *Liber Abaci*. However, usage of the system remained limited for many years, in part because the scheme was considered "diabolical," due in part to the mistaken impression that it originated in the Arab world (in spite of Fibonacci's clear descriptions of the "nine Indian figures" plus zero). Indeed, our modern English word "cipher" or "cypher," which is derived from the Arabic *zephirum* for zero, and which alternately means "zero" or "secret code" in modern usage, is likely a linguistic memory of the time when using decimal arithmetic was deemed evidence of involvement in the occult [11, pg. 588-589].

Decimal arithmetic began to be widely used by scientists beginning in the 1400s, and was employed, for instance, by Copernicus, Galileo, Kepler and Newton, but it was not universally used in European commerce until 1800, at least 1300 years after its discovery. In limited defense of the Greco-Roman system, it is harder to alter Roman entries in an account book or the sum payable in a cheque, but this does not excuse the continuing practice of performing arithmetic using Roman numerals and counting tables.

The Arabic world, by comparison, was much more accepting of the Indian system—in fact, as mentioned briefly above, the West owes its knowledge of the scheme to Arab scholars. One of the first to popularize the method was al-Khowarizmi, who in the ninth century wrote at length about the Indian system and also described algebraic methods for the solution of quadratic equations. In 1424, Al-Kashi of Samarkand, "*who could calculate as eagles can fly*" computed 2π in *sexagecimal* (good to an equivalent of 16 decimal digits) using $3{\cdot}2^{28}$-gons and a base-60 variation of Indian positional arithmetic [2, Appendix on Arab Mathematics]:

$$2\pi \quad\approx\quad 6 + \frac{16}{60^1} + \frac{59}{60^2} + \frac{28}{60^3} + \frac{01}{60^4} + \frac{34}{60^5} + \frac{51}{60^6} + \frac{46}{60^7} + \frac{14}{60^8} + \frac{50}{60^9}.$$

This is a personal favorite of ours: re-entering it on a computer centuries later and getting the predicted answer still produces goose-bumps.

4 Modern History

It is disappointing that this seminal development in the history of mathematics is given such little attention in modern published histories. For example, in one popular work on the history of mathematics, although the author describes Arab and Chinese mathematics in significant detail, he mentions the discovery of positional decimal arithmetic in India only in one two-sentence passage [4, pg. 253]. Another popular history of mathematics mentions the discovery of the "Hindu-Arabic Numeral System," but says only that

> *Positional value and a zero must have been introduced in India sometime before A.D. 800, because the Persian mathematician al-Khowarizmi describes such a completed Hindu system in a book of A.D. 825.* [8, pg. 23]

A third historical work briefly mentions this discovery, but cites a 662 CE Indian manuscript as the earliest known source [12, pg. 221]. A fourth reference states that the combination of decimal and positional arithmetic "*appears in China and then in India*" [16, pg. 67]. None of these authors devotes more than a few sentences to the subject, and, more importantly, none suggests that this discovery is regarded as particularly significant.

In partial defense of these histories, though, it must be acknowledged that all historians work from other sources, and only within the past few years, with the advent of the Internet, has it been possible to readily access original and translated original documents with the click of a mouse.

In any event, we entirely agree with Dantzig, Ifrah and others that the discovery of positional decimal arithmetic, by an unknown scholar in early first millennium India, is a mathematical development of the first magnitude. The fact that the system is now taught and mastered in grade schools worldwide, and is implemented (in binary) in every computer chip ever manufactured, should not detract from its historical significance. To the contrary, these same facts emphasize the enormous advance that this system represents, both in simplicity and efficiency, as well as the huge importance of this discovery in modern civilization.

Perhaps some day we will finally learn the identity of this mysterious Indian mathematician. If we do, we surely must accord him or her the same accolades that we have granted to Archimedes, Newton, Gauss and Ramanujan.

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4	5	4	6	8	0	4	9	6	$\lfloor\sqrt{45}\rfloor = 6$
3	6								$6^2 = 36$
	9	4						6 7	$\lfloor 94/(2\cdot 6)\rfloor = 7$
	8	4							$7\cdot(2\cdot 6) = 84$
	1	0	6						
		4	9						$7^2 = 49$
		5	7	8				6 7 4	$\lfloor 578/(2\cdot 67)\rfloor = 4$
		5	3	6					$4\cdot(2\cdot 67) = 536$
			4	2	0				
				1	6				$4^2 = 16$
			4	0	4	4		6 7 4 3	$\lfloor 4044/(2\cdot 674)\rfloor = 3$
			4	0	4	4			$3\cdot(2\cdot 674) = 4044$
						0	9		
							9		$3^2 = 9$
							0		Finished; result = 6743

Figure 2: The Aryabhata algorithm for computing square roots.

4	5	4	9	9	2	9	3	3	$\lfloor\sqrt[3]{45}\rfloor = 3$
2	7								$3^3 = 27$
1	8	4						3 5	$\lfloor 184/(3\cdot 3^2)\rfloor = 6$ (too high, so take 5)
1	3	5							$5\cdot(3\cdot 3^2) = 135$
	4	9	9						
	2	2	5						$3\cdot 5^2\cdot 3 = 225$
	2	7	4	9					
		1	2	5					$5^3 = 125$
	2	6	2	4	2			3 5 7	$\lfloor 26242/(3\cdot 35^2)\rfloor = 7$
	2	5	7	2	5				$7\cdot(3\cdot 35^2) = 25725$
			5	1	7	9			
			5	1	4	5			$3\cdot 7^2\cdot 35 = 5145$
					3	4	3		
					3	4	3		$7^3 = 343$
							0		Finished; result = 357

Figure 3: The Aryabhata algorithm for computing cube roots.