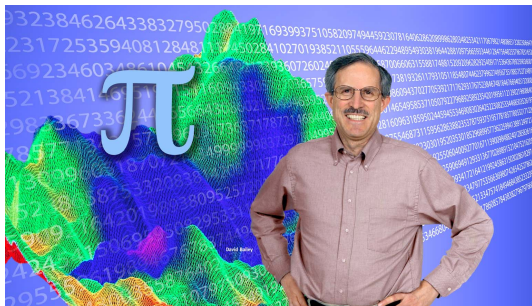


Pi and normality

David H. Bailey

Lawrence Berkeley National Laboratory (retired) and
Computer Science Department, University of California, Davis



New York Times PiDay 2007 (March 14, 2007) crossword puzzle

The New York Times Crossword

Edited by Will Shortz

No. 0314

Across

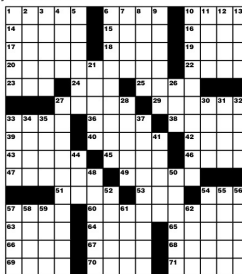
- 1 Enlighten
6 A couple CBS spinoffs
10 1972 Broadway musical
14 Metal giant
15 Evict
16 Area
17 Surface again, as a road
18 Pirate or Padre, briefly
19 Camera feature
20 Barracks artwork, perhaps
22 River to the Ligurian Sea
23 Keg necessity
24 "... he drove out of sight"
25 ___ St. Louis, Ill.
27 Preen
29 Greek peak
33 Vice president after Hubert
36 Patient wife of Sir Geraint
38 Action to an ante
39 Gain ___
40 French artist Odilon ___
42 Grape for winemaking
43 Single-dish meal
45 Broad valley
46 See 21-Down
47 Artery inserts
49 Offspring
51 Mexican mouse catcher
53 Medical procedure, in brief
54 "Wheel of Fortune" option
57 Animal with striped legs
60 Editorial

Down

- 63 It gets bigger at night
64 "Hold your horses!"
65 Idiots
66 Europe/Asia border river
67 Suffix with launder
68 Leaning
69 Brownback and Obama, e.g.: Abbr.
70 Rick with the 1976 #1 hit "Disco Duck"
71 Yegg's targets

Down

- 1 Mastodon trap
2 "Mefistofele" soprano
3 Misbehave
4 Pen
5 More pleased
6 Treated with disdain
7 Enterprise crewman
8 Rhone feeder
9 Many a webcast
10 Mushroom, for one
11 Unfortunate
12 Nevada's state tree
13 Disney fish
21 Colonial figure with 46-Across
26 Poker champion Ungar
27 Self-medicating excessively
28 March 14, to mathematicians



Puzzle by Peter A. Collins

- 30 Book part
31 Powder, e.g.
32 007 and others: Abbr.
33 Drains
34 Stove feature
35 Feet per second, e.g.
37 Italian range
41 Prefix with surgery
44 Captain's announcement, for short
48 Tucked away
50 Stealthy fighters
52 Sedative
54 Letter feature
55 Jam
56 Settles in
57 Symphony or sonata
58 Japanese city bombed in W.W. II
59 Beelike
61 Evening, in ads
62 Religious artwork

ANSWER TO PREVIOUS PUZZLE

ARFS ACHE ORGAN
CORK TREX KERRY
ODAY LAIT STATS
LEN RANDOM IPSE
DOCTOR KILDARE
RAG SIESTA
TYRONE POWER OHM
RUED ALI IDES
IMP HOLD THE MAYO
BABA ORE
IRISH COUNTRIES
PERT TIRADE DXC
ARMED TATI YIPE
CLARE ITEN DOWN

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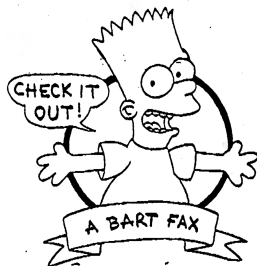
Answers to crossword

ANSWER TO PREVIOUS PUZZLE

T	E	A	C	H		C	S	I	S		π	P	π	N
A	L	C	O	A		O	U	S	T		Z	O	N	E
R	E	T	O	P		N	L	E	R		Z	O	O	M
π	N	U	P	π	C	T	U	R	E		A	R	N	O
T	A	P		E	R	E		E	A	S	T			
			P	R	I	M	P		M	T	O	S	S	A
S	π	R	O		E	N	I	D		U	P	π	N	G
A	L	A	P		R	E	D	O	N		π	N	O	T
P	O	T	π	E		D	A	L	E		N	E	W	S
S	T	E	N	T	S		Y	O	U	N	G			
			G	A	T	O		M	R	I		S	π	N
O	K	A	π		O	π	N	I	O	N	π	E	C	E
P	U	π	L		W	A	I	T		J	E	R	K	S
U	R	A	L		E	T	T	E		A	T	I	L	T
S	E	N	S		D	E	E	S		S	A	F	E	S

I once received a strange fax

- ▶ In October 1992, I received this fax from the Simpsons TV show.
- ▶ They wanted the 40,000th digit of π .
- ▶ I faxed back the result: it is a "1."
- ▶ This was used in the Simpsons show, dated 6 May 1993, "Marge in Chains."



TO: DAVID BAILEY
FROM: JACQUELINE ATKINS
DATE: 10/9/92
NUMBER OF PAGES: 1

FAX (310) 203-3852

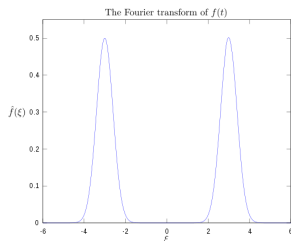
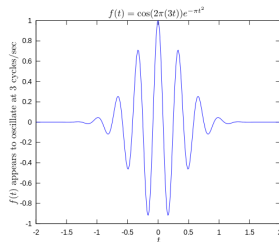
PHONE (310) 203-3959

A Professor at UCLA told me that you might be able to give me the answer to: What is the 40,000th digit of π ?

We would like to use the answer in our show. Can you help?

Smartphones and π

- ▶ Every smartphone or mobile phone crucially relies on computations (e.g., the fast Fourier transform) that involve π to resolve microwave signals.



- ▶ π appears in the fundamental equations of quantum mechanics, which are used to design smartphone electronics. For example, Heisenberg's uncertainty principle:

$$\left(\int_{-\infty}^{\infty} s^2 |f(s)|^2 ds \right) \left(\int_{-\infty}^{\infty} t^2 |f(t)|^2 dt \right) \geq \frac{\|f\|_2^4}{16\pi^2}$$

- ▶ π appears in the equations of general relativity, used in GPS:

$$R_{\mu\nu} - \frac{Rg_{\mu\nu}}{2} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

“Exact” value of $\pi = 17 - 8\sqrt{3} = 3.1435935394 \dots$

In November 2016, the *IOSR Journal of Mathematics* published the following paper:

Abstract: *It is believed that π (π) is a transcendental number. In author's opinion, it is not the fact. The paper aims at showing that π (π) is an algebraic number with exact value $17 - 8\sqrt{3}$. The derivation of this value is supported by several geometrical constructions, arithmetic calculations and use of some simple algebraic formulae.*

Another author has published 8 papers insisting that $\pi = (14 - \sqrt{2})/4 = 3.1464466094 \dots$, in supposedly “peer-reviewed” journals.

- ▶ “Exact value of π ($17 - 8\sqrt{3}$),” *IOSR J. of Mathematics*, vol. 12 (Nov.-Dec. 2016), <http://www.iosrjournals.org/iosr-jm/papers/Vol12-issue6/Version-1/B1206010408.pdf>.

IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X, Volume 12, Issue 6 Ver. I (Nov. - Dec.2016), PP 04-08
www.iosrjournals.org

Exact Value of π ($17 - 8\sqrt{3}$)

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Abstract: *It is believed that π (π) is a transcendental number. In author's opinion, it is not the fact. The paper aims at showing that π (π) is an algebraic number with exact value $17 - 8\sqrt{3}$. The derivation of this value is supported by several geometrical constructions, arithmetic calculations and use of some simple algebraic formulae.*

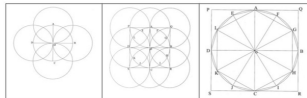
I. Introduction

Determination of the ratio of circumference of circle with its diameter has been a problem of much more interest in the field of mathematics since thousands of years. The ratio of circumference of circle/diameter is denoted by a Greek letter π . It is believed that the constant π (π) is a transcendental number. Its value has been computed correct up to several trillions of digits. For practical purpose the value of π is taken approximately equal to 22/7, 355/113 etc. However, the eminent mathematicians of the world suffer from a serious drawback while using the methods such as use of infinite series, trigonometry, division of area of circle into infinite symmetric parts in solid polygon and many others, as we cannot measure the end point quantities exactly. This causes an obvious limitation on obtaining exact value of π .

The sections which follow, describe the derivation for “Exact” value of π which is a creation of the author. In author's opinion the exact value of π is $17 - 8\sqrt{3}$.

I have made many proofs & here I am giving one of them.

Basic figures



Why compute π ?

- ▶ Question: Do we need to know π to thousands or millions of digits in everyday science and engineering?

Answer: No. 10–15 digits suffice for most scientific calculations.

- ▶ However, some research problems in mathematics and physics require hundreds or thousands of digits.
- ▶ I have personally done computations that required π to 64,000-digit precision.
- ▶ Billions and even trillions of digits have been computed by mathematicians, in part to explore the unanswered question “Are the digits of π ‘random’?”

The first 1000 decimal digits of π

3.14159265358979323846264338327950288419716939937510582097494459230781
6406286208998628034825342117067982148086513282306647093844609550582231
7253594081284811174502841027019385211055596446229489549303819644288109
7566593344612847564823378678316527120190914564856692346034861045432664
8213393607260249141273724587006606315588174881520920962829254091715364
3678925903600113305305488204665213841469519415116094330572703657595919
5309218611738193261179310511854807446237996274956735188575272489122793
8183011949129833673362440656643086021394946395224737190702179860943702
7705392171762931767523846748184676694051320005681271452635608277857713
4275778960917363717872146844090122495343014654958537105079227968925892
3542019956112129021960864034418159813629774771309960518707211349999998
3729780499510597317328160963185950244594553469083026425223082533446850
3526193118817101000313783875288658753320838142061717766914730359825349
0428755468731159562863882353787593751957781857780532171226806613001927
876611195909216420198...

Pre-computer history of π calculations

Name	Year	Digits
Archimedes	-250?	3
Ptolemy	150?	3
Liu Hui	265?	5
Aryabhata	480?	5
Tsu Ch'ung Chi	480?	7
Madhava	1400?	13
Al-Kashi	1429	14
Romanus	1593	15
Van Ceulen	1615	35
Sharp and Halley	1699	71
Machin	1706	100
Strassnitzky and Dase	1844	200
Rutherford	1853	440
W. Shanks	1874	*707
Ferguson (mechanical calculator)	1947	808

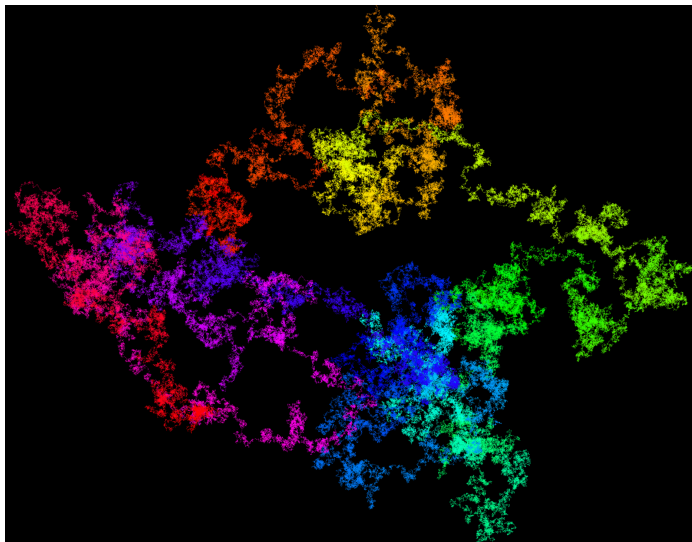
*Only the first 527 were correct.

Computer-era π calculations

Name	Year	Digits
Miyoshi and Kanada	1981	2,000,036
Kanada-Yoshino-Tamura	1982	16,777,206
Gosper	1985	17,526,200
Bailey	1986	29,360,111
Kanada et. al	1987	134,217,700
Kanada and Tamura	1989	1,073,741,799
Chudnovskys	1994	4,044,000,000
Kanada and Takahashi	1997	51,539,600,000
Kanada and Takahashi	1999	206,158,430,000
Kanada-Ushiro-Kuroda	2002	1,241,100,000,000
Takahashi	2009	2,576,980,377,524
Bellard	2009	2,699,999,990,000
Kondo and Yee	2010	5,000,000,000,000
Trueb	2016	22,459,157,718,361

If 22 trillion digits were printed in 12-point type, they would stretch nearly to Mars.

A random walk on the first 100 billion base-4 digits of π



This dataset can be explored online: <http://gigapan.com/gigapans/106803>

Some formulas for computing π

$$\pi = \frac{3\sqrt{3}}{4} - 24 \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{(2n+3)(2n-1)4^{2n+1}} \quad (\text{Newton, 1660})$$

$$\frac{\pi}{4} = 4 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)5^{2n+1}} - \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)239^{2n+1}} \quad (\text{Machin, 1730})$$

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{n=0}^{\infty} \frac{(4n)!(1103 + 26390n)}{(n!)^4 396^{4n}} \quad (\text{Ramanujan, 1930})$$

Set $a_0 = 6 - 4\sqrt{2}$ and $y_0 = \sqrt{2} - 1$. Iterate

$$y_{k+1} = \frac{1 - (1 - y_k^4)^{1/4}}{1 + (1 - y_k^4)^{1/4}}$$

$$a_{k+1} = a_k(1 + y_{k+1})^4 - 2^{2k+3}(1 + y_{k+1} + y_{k+1}^2).$$

Then a_k converge quartically to $1/\pi$: **each iteration quadruples the number of correct digits.**

20 iterations are sufficient to compute π to 2.9 trillion digits, provided all iterations are done to this precision. (Jonathan and Peter Borwein, 1984)

How does one do arithmetic to extremely high precision?

Computing π or anything else to extremely high precision requires special software:

- ▶ High-precision numbers are stored as a sequence of computer words.
- ▶ Addition and subtraction are performed using relatively simple methods.
- ▶ Multiplication is performed using a **fast Fourier transform**, which is thousands or even millions of times faster than conventional methods.
- ▶ Division and square roots are performed using **Newton iterations**, based on multiplication and addition.
- ▶ Exponential and trigonometric functions are evaluated using special algorithms.

Software packages to perform these operations are readily available on the Internet, or by using systems such as *Mathematica*, *Maple* or *Sage*.

The PSLQ integer relation algorithm

Given a vector (x_n) of real numbers, an integer relation algorithm finds integers (a_n) such that

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$$

(to within the precision of the arithmetic being used), or else finds bounds within which no relation can exist.

Helaman Ferguson's **PSLQ algorithm** is the most widely used integer relation algorithm.

Integer relation detection (using PSLQ or any other algorithm) requires very high numeric precision, both in the input data and in the operation of the algorithm.

1. H. R. P. Ferguson, D. H. Bailey and S. Arno, "Analysis of PSLQ, an integer relation finding algorithm," *Mathematics of Computation*, vol. 68, no. 225 (Jan 1999), 351–369.
2. D. H. Bailey and D. J. Broadhurst, "Parallel integer relation detection: Techniques and applications," *Mathematics of Computation*, vol. 70, no. 236 (Oct 2000), 1719–1736.

Helaman Ferguson's "Umbilic Torus SC" sculpture at Stony Brook Univ.



Computing binary digits of $\log(2)$ beginning at an arbitrary position

1996 result: Consider this well-known formula for $\log(2)$:

$$\log(2) = \sum_{n=1}^{\infty} \frac{1}{n2^n} = 0.10110001011100100001011111101111101000111001111011 \dots_2$$

Note that the binary digits of $\log 2$ beginning after position d can be written as $\{2^d \log 2\}$, where $\{\cdot\}$ denotes fractional part. Thus we can write:

$$\begin{aligned} \{2^d \log(2)\} &= \left\{ \sum_{n=1}^d \frac{2^{d-n}}{n} \right\} + \left\{ \sum_{n=d+1}^{\infty} \frac{2^{d-n}}{n} \right\} \\ &= \left\{ \sum_{n=1}^d \frac{2^{d-n} \bmod n}{n} \right\} + \left\{ \sum_{n=d+1}^{\infty} \frac{2^{d-n}}{n} \right\} \end{aligned}$$

We have inserted “ $\bmod n$ ” since we are only interested in the fractional part when divided by n . Now note that the numerator $2^{d-n} \bmod n$ can be calculated very rapidly using the [binary algorithm for exponentiation](#).

The binary algorithm for exponentiation

Problem: What is $3^{17} \bmod 10$? (i.e., what is the last decimal digit of 3^{17} ?)

Algorithm A:

$3^{17} = 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 = 129140163$,
so answer = 3.

Algorithm B (faster): $3^{17} = (((((3^2)^2)^2)^2) \times 3 = 129140163$, so answer = 3.

Algorithm C (fastest):

$3^{17} = (((((3^2 \bmod 10)^2 \bmod 10)^2 \bmod 10)^2 \bmod 10) \times 3 \bmod 10 = 3$.

Note that in Algorithm C, we never have to deal with integers larger than $9 \times 9 = 81$,
so the entire operation can be performed very rapidly on a computer.

General BBP-type formulas

The same “trick” that was used for $\log(2)$ can be applied for any real constant α that can be written in the form

$$\alpha = \sum_{n=0}^{\infty} \frac{p(n)}{b^n q(n)}$$

where p and q are integer polynomials, $\deg p < \deg q$, and q has no zeroes for $n \geq 0$, or as a linear sum of such formulas.

What other well-known mathematical constants can be written by such a formula?

Can π be written in this form? None was known at the time (1996).

The BBP formula for π

In 1996, a PSLQ program discovered this new formula for π :

$$\pi = \sum_{k=0}^{\infty} \frac{1}{16^k} \left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6} \right)$$

Indeed, this formula permits one to compute base-16 (or binary) digits of π beginning at an arbitrary starting position. The proof is simple.

This is the first known instance of a computer program discovering a fundamentally new formula for π .

BBP-type formulas (also discovered using PSLQ) are now known for numerous other mathematical constants.

Sadly, there is no such similar formula for base-10 digits of π .

- D. H. Bailey, P. B. Borwein and S. Plouffe, "On the rapid computation of various polylogarithmic constants," *Mathematics of Computation*, vol. 66 (Apr 1997), 903–913.

Some other BBP-type formulas found using PSLQ

$$\pi^2 = \frac{1}{8} \sum_{k=0}^{\infty} \frac{1}{64^k} \left(\frac{144}{(6k+1)^2} - \frac{216}{(6k+2)^2} - \frac{72}{(6k+3)^2} - \frac{54}{(6k+4)^2} + \frac{9}{(6k+5)^2} \right)$$

$$\pi^2 = \frac{2}{27} \sum_{k=0}^{\infty} \frac{1}{729^k} \left(\frac{243}{(12k+1)^2} - \frac{405}{(12k+2)^2} - \frac{81}{(12k+4)^2} - \frac{27}{(12k+5)^2} \right. \\ \left. - \frac{72}{(12k+6)^2} - \frac{9}{(12k+7)^2} - \frac{9}{(12k+8)^2} - \frac{5}{(12k+10)^2} + \frac{1}{(12k+11)^2} \right)$$

$$\pi^2 \log(2) = \frac{1}{32} \sum_{k=0}^{\infty} \frac{1}{4096^k} \left(\frac{18432}{(24k+2)^3} - \frac{69120}{(24k+3)^3} + \frac{18432}{(24k+4)^3} + \frac{25344}{(24k+6)^3} + \frac{27648}{(24k+8)^3} \right. \\ + \frac{8640}{(24k+9)^3} + \frac{1152}{(24k+10)^3} + \frac{2880}{(24k+12)^3} + \frac{288}{(24k+14)^3} + \frac{1080}{(24k+15)^3} + \frac{1728}{(24k+16)^3} \\ \left. + \frac{396}{(24k+18)^3} + \frac{72}{(24k+20)^3} - \frac{135}{(24k+21)^3} + \frac{18}{(24k+22)^3} \right)$$

- David H. Bailey, "A compendium of BBP-type formulas for mathematical constants," updated 15 Aug 2017, <http://www.davidhbailey.com/dhbpapers/bbp-formulas.pdf>.

The BBP formula for π in action

In July 2010, Tsz-Wo Sze used a variant of the BBP formula to compute the base-16 digits of π starting at position **500 trillion** (corresponding to binary position **2 quadrillion**). The run required 16 billion CPU-seconds of computing.

The result was checked by repeating the calculation to find digits starting at position **500 trillion + 1**. The two results were (in base-16 digits):

```
0 E6C1294A ED40403F 56D2D764 026265BC A98511D0 FCFFAA10 F4D28B1B
  E6C1294A ED40403F 56D2D764 026265BC A98511D0 FCFFAA10 F4D28B1B
```

Note that the results precisely overlap. The probability that two randomly generated 56-long strings of base-16 digits perfectly agree is approximately 3.7×10^{-68} .

Philosophical question: What is more securely established?:

- ▶ The computational assertion that the 500 trillionth hex digit of π is “0.”
- ▶ A theorem whose proof is hundreds of pages long, relies on dozens of earlier results (any one of which, if later found to be in error, would render the main theorem invalid), and which has been read in detail by only a handful of mathematicians worldwide?

In some cases, at least, computation is at least as compelling as formal proof.

BBP-type formulas for π^2

Whereas only base-2 (binary) BBP-type formulas exist for π , there are both binary (base-2) and ternary (base-3) formulas for π^2 , both discovered by PSLQ:

$$\pi^2 = \frac{9}{8} \sum_{k=0}^{\infty} \frac{1}{64^k} \left(\frac{16}{(6k+1)^2} - \frac{24}{(6k+2)^2} - \frac{8}{(6k+3)^2} - \frac{6}{(6k+4)^2} + \frac{1}{(6k+5)^2} \right)$$
$$\pi^2 = \frac{2}{27} \sum_{k=0}^{\infty} \frac{1}{729^k} \left(\frac{243}{(12k+1)^2} - \frac{405}{(12k+2)^2} - \frac{81}{(12k+4)^2} - \frac{27}{(12k+5)^2} \right. \\ \left. - \frac{72}{(12k+6)^2} - \frac{9}{(12k+7)^2} - \frac{9}{(12k+8)^2} - \frac{5}{(12k+10)^2} + \frac{1}{(12k+11)^2} \right)$$

We decided to use these formulas to compute base-64 and base-729 digits of π^2 , starting at position **ten trillion**.

BBP-type formula for Catalan's constant

We also decided to calculate digits of Catalan's constant:

$$G = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} = 0.91596559417722 \dots$$

which is closely related to π^2 :

$$\frac{\pi^2}{8} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = 1.2337005501362 \dots$$

We employed this formula, which we discovered using the PSLQ algorithm:

$$\begin{aligned} G = \frac{1}{4096} \sum_{k=0}^{\infty} \frac{1}{4096^k} & \left(\frac{36864}{(24k+2)^2} - \frac{30720}{(24k+3)^2} - \frac{30720}{(24k+4)^2} - \frac{6144}{(24k+6)^2} - \frac{1536}{(24k+7)^2} \right. \\ & + \frac{2304}{(24k+9)^2} + \frac{2304}{(24k+10)^2} + \frac{768}{(24k+14)^2} + \frac{480}{(24k+15)^2} + \frac{384}{(24k+11)^2} + \frac{1536}{(24k+12)^2} \\ & \left. + \frac{24}{(24k+19)^2} - \frac{120}{(24k+20)^2} - \frac{36}{(24k+21)^2} + \frac{48}{(24k+22)^2} - \frac{6}{(24k+23)^2} \right). \end{aligned}$$

Andrew Mattingly, Glenn Wightwick, and the IBM BlueGene

For the actual computations, Jonathan Borwein and I turned to our colleagues Andrew Mattingly and Glenn Wightwick at IBM Australia, who were willing to help with programming and tuning. They received permission from IBM to use an IBM BlueGene supercomputer for this purpose.



Our results — two double-checking runs each

1. *Base-64 digits of π^2 beginning at position 10 trillion (a base-64 digit is a pair of base-8 digits):*

```
75|60114505303236475724500005743262754530363052416350634  
  |60114505303236475724500005743262754530363052416350634
```

2. *Base-729 digits of π^2 beginning at position 10 trillion (a base-729 digit is a triplet of base-9 digits):*

```
001|12264485064548583177111135210162856048323453468  
   |12264485064548583177111135210162856048323453468
```

3. *Base-4096 digits of Catalan's constant beginning at position 10 trillion (a base-4096 digit is a quadruplet of base-8 digits):*

```
0176|34705053774777051122613371620125257327217324522  
     |34705053774777051122613371620125257327217324522
```

These runs required 22.1 billion CPU-seconds.

New calculation of base-16 digits of π

In December 2016, Daisuke Takahashi finished the computation of hexadecimal (base-16) digits of π beginning at position **100 quadrillion**, or 10^{17} .

The run used Bellard's formula (a variation of the BBP formula for π). Both the main run and the verification run each required 320 hours on 512 nodes of a Fujitsu cluster at the Joint Center for Advanced High Performance Computing (JCAHPC) in Japan.

The hexadecimal digits of π from position 10^{17} to $10^{17} + 15$ are: **A937EB59439E485E**

Are the digits of π random?

Given a positive integer b , a real number α is **normal base b** if every m -long string of digits appears in the base- b expansion of α with limiting frequency $1/b^m$. It can be shown that almost all real numbers are normal base b , for all bases b .

These constants are widely believed to be normal base b , for all bases b :

- ▶ $\pi = 3.14159265358979323846 \dots$
- ▶ $e = 2.7182818284590452354 \dots$
- ▶ $\sqrt{2} = 1.4142135623730950488 \dots$
- ▶ $\log(2) = 0.69314718055994530942 \dots$
- ▶ Every irrational algebraic number (this conjecture is due to Borel).

But there are no proofs of normality for any of the above — not even for $b = 2$ and $m = 1$ (i.e., equal numbers of zeroes and ones in the binary expansion).

Until recently, normality proofs were known only for a few constants, such as Champernowne's constant $= 0.12345678910111213141516 \dots$ (normal base 10).

One (very weak) result for algebraic numbers

If x is algebraic of degree $d > 1$, then its binary expansion through position n must have at least $Cn^{1/d}$ 1-bits, for all sufficiently large n and for some C that depends on x .

Simple case: The first n binary digits of $\sqrt{2}$ must have at least \sqrt{n} one bits.

In this case, the result follows by noting that the one-bit count of the product of two integers is less than or equal to the product of the one-bit counts of the two integers. The more general result above requires a more sophisticated approach.

$$\begin{array}{r} \\ \times \\ \hline 1001101101 \end{array}$$

However, note that these results are still a far cry from even single-digit normality.

- D. H. Bailey, J. M. Borwein, R. E. Crandall and C. Pomerance, "On the Binary Expansions of Algebraic Numbers," *Journal of Number Theory Bordeaux*, vol. 16 (2004), pg. 487–518.

BBP formulas and normality

Consider a general BBP-type constant (i.e., a formula that permits the BBP “trick”):

$$\alpha = \sum_{n=0}^{\infty} \frac{p(n)}{b^n q(n)},$$

where p and q are integer polynomials, $\deg p < \deg q$, and q has no zeroes for $n \geq 0$.

Richard Crandall (deceased 2012) and I proved that α is normal base b if and only if the sequence $x_0 = 0$, and

$$x_n = \left\{ bx_{n-1} + \frac{p(n)}{q(n)} \right\},$$

is equidistributed in the unit interval. Brackets $\{\cdot\}$ denote fractional part, as before.

Here **equidistributed** means that the sequence visits each subinterval (c, d) with limiting frequency $d - c$.

- D. H. Bailey and R. E. Crandall, “On the random character of fundamental constant expansions,” *Experimental Mathematics*, vol. 10 (Jun 2001), 175–190.

Two specific examples: $\log(2)$ and π

Consider the sequence $x_0 = 0$ and

$$x_n = \left\{ 2x_{n-1} + \frac{1}{n} \right\}$$

Then $\log(2)$ is normal base 2 if and only if (x_n) is equidistributed in the unit interval.

Similarly, consider the sequence $y_0 = 0$ and

$$y_n = \left\{ 16y_{n-1} + \frac{120n^2 - 89n + 16}{512n^4 - 1024n^3 + 712n^2 - 206n + 21} \right\}$$

Then π is normal base 16 (and hence normal base 2) if and only if (y_n) is equidistributed in the unit interval.

Sadly, we have not yet been able to prove equidistribution for either sequence.

Curiously, the sequence (y_n) , when mapped to the 16 divisions of the unit interval, appears to generate, digit by digit, the entire base-16 expansion of π , error-free.

A class of provably normal constants

Crandall and I also proved that the following constant is normal base 2:

$$\begin{aligned}\alpha_{2,3} &= \sum_{n=1}^{\infty} \frac{1}{3^n 2^{3^n}} \\ &= 0.041883680831502985071252898624571682426096 \dots_{10} \\ &= 0.000010101011100011100011100011110110100001 \dots_2\end{aligned}$$

This constant was proven normal by Stoneham in 1971, but we have extended this to the case where $(2, 3)$ are any pair (p, q) of relatively prime integers, and also to a larger, uncountably infinite class.

The original proof is difficult, but a subsequent proof using a “hot spot lemma” (via ergodic theory) is quite simple.

1. D. H. Bailey and R. E. Crandall, “Random generators and normal numbers,” *Experimental Mathematics*, vol. 11 (2002), 527–546.
2. D. H. Bailey and M. Misiurewicz, “A strong hot spot theorem,” *Proceedings of the American Mathematical Society*, vol. 134 (2006), 2495–2501.

A (weak) hot spot theorem

The previous result on the normality of Stoneham numbers can now be proven much more easily using this result from ergodic theory:

The (weak) hot spot theorem: Given the real constant α , if there exists some B such that for every subinterval $[c, d)$ of $[0, 1)$,

$$\limsup_{m \geq 1} \frac{\#\{0 \leq j \leq m \mid \{b^j \alpha\} \in [c, d)\}}{m(d - c)} \leq B$$

then α is b -normal.

In other words, if α is not b -normal, then:

- ▶ There is some interval $[c_1, d_1)$ that is visited 10 times too often by shifts of the base- b expansion of α ;
- ▶ There is some other interval $[c_2, d_2)$ that is visited 100 times too often;
- ▶ There is some other interval $[c_3, d_3)$ that is visited 1000 times too often; etc.

However, one cannot conclude that these intervals are necessarily nested.

- ▶ L. Kuipers and H. Niederreiter, *Uniform Distribution of Sequences*, Dover, 1974, 77.

A strong hot spot theorem

In 2006 Michal Misiurewicz and I proved a stronger version of this result, using methods of ergodic theory:

The strong hot spot theorem: Let $0.x_1x_2\dots x_n$ be the base- b expansion of x out to position n , so that $[c_n(x), d_n(x)) = [0.x_1x_2\dots x_n, 0.x_1x_2\dots(x_n+1))$ is the n -long digit interval containing x . If for every x in $[0, 1)$,

$$\liminf_{n \geq 1} \limsup_{m \geq 1} \frac{\#\{0 \leq j < m \mid \{b^j \alpha\} \in [c_n(x), d_n(x))\}}{mb^{-n}} < \infty$$

then α is b -normal.

In other words: If α is not b -normal, then there is at least one “hot spot,” namely some $x \in [0, 1)$ such that shifts of the base- b expansion of α visit all sufficiently small digit neighborhoods of x too often, by an arbitrarily large factor.

Conversely, if one can establish that there is no such “hot spot,” then α is b -normal.

- D. H. Bailey and M. Misiurewicz, “A strong hot spot theorem,” *Proceedings of the American Mathematical Society*, vol. 134 (2006), 2495-2501.

Binary digits of $\alpha_{2,3}$ versus base-6 digits of $\alpha_{2,3}$

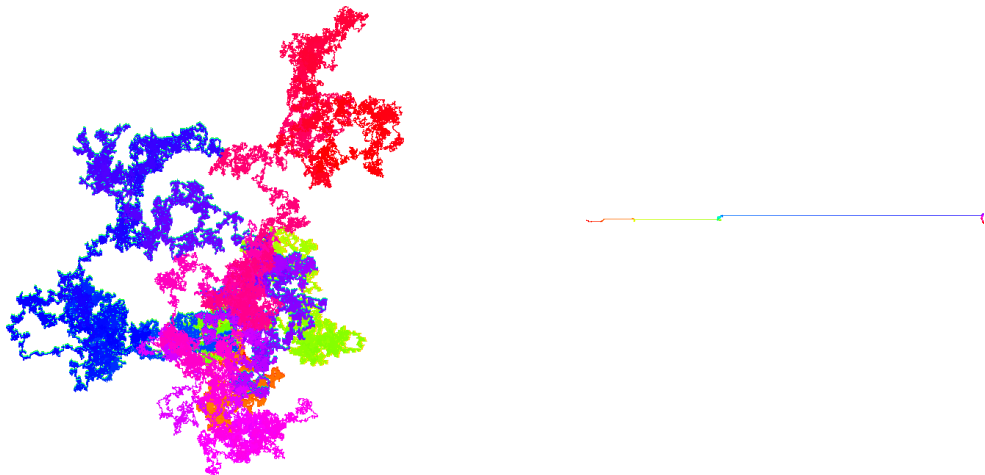
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Binary digits of $\alpha_{2,3}$ versus base-6 digits of $\alpha_{2,3}$: Random walks



- F. J. Aragon Artacho, D. H. Bailey, J. M. Borwein and P. B. Borwein, "Walking on real numbers," *Mathematical Intelligencer*, vol. 35 (2013), 42-60.

Summary

The study of π is a paradigm of experimental mathematics research:

- ▶ π is highly suitable for computational exploration.
- ▶ π can be explored visually as well as numerically.
- ▶ π has great public appeal, from grade school to serious scientific research.
- ▶ The question of the normality of π continues to fascinate (and frustrate!) research mathematicians.

Thanks! This talk is available at:

<http://www.davidhbailey.com/dhbpapers/dhb-pi-2017.pdf>