Walking on real numbers

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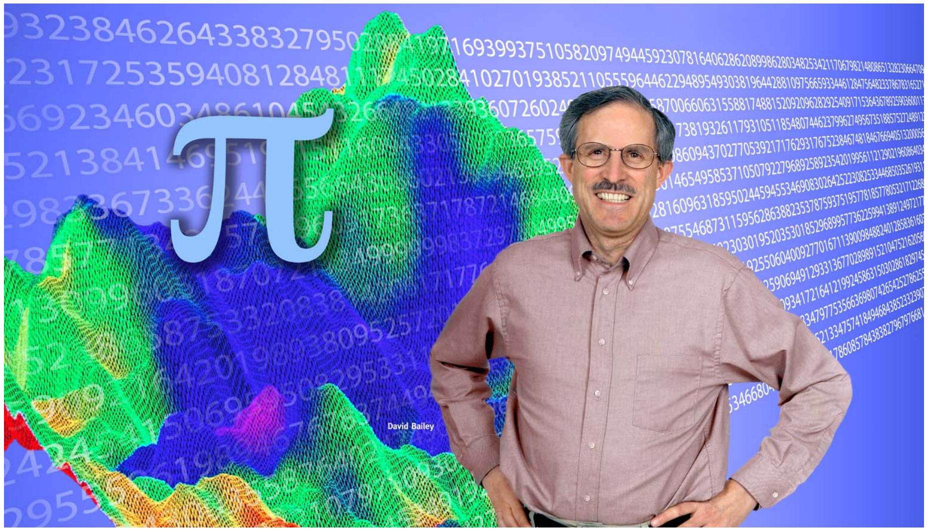
This talk is available at:

http://www.davidhbailey.com/dhbtalks/dhb-walking.pdf



Normal numbers

Given an integer $b > 1$, a real number x is ***b*-normal** (or "normal base *b*") if every m-long string of digits in the base-b expansion of x appears with precisely the expected limiting frequency b^{-m}.

Using measure theory, it is can be shown that almost all real numbers are *b*-normal for a given integer base *b*. In fact, almost all reals are *b*-normal for all integer bases $b > 1$ simultaneously (i.e., are "absolutely normal").

These are widely believed to be *b*-normal, for all integer bases $b > 1$:

π = 3.1415926535…

e = 2.7182818284…

sqrt(2) = 1.4142135623…

log(2) = 0.6931471805…

Every irrational algebraic number (this conjecture is due to Borel).

But there are no normality proofs for any of these constants in any base, nor are there any nonnormality results for any of these constants.

Until recently, normality proofs were known only for a few relatively contrived examples such as Champernowne's constant = 0.123456789101112131415… (which is 10-normal)

A result for algebraic numbers

If x is algebraic of degree $d > 1$, then its binary expansion through position n must have at least $C\, n^{1/d}$ 1-bits, for all sufficiently large n and some C that depends on x.

Example: The first n binary digits of sqrt(2) must have at least sqrt(n) ones.

However, note that these results are still a far cry from full normality, even in the single-digit sense.

DHB, J. M. Borwein, R. E. Crandall and C. Pomerance, “On the Binary Expansions of Algebraic Numbers,” *Journal of Number Theory Bordeaux*, vol. 16 (2004), pg. 487-518.

New computer-based approaches to the normality problem

In a recent paper to appear in the *Mathematical Intelligencer* (see below), we applied the following to this problem:

- Analyses of Stoneham numbers (with proofs or normality and non-normality), the Erdos-Borwein constants and other classes.
- High-resolution computer graphics.
- Representation of digits as a “random” walk.
- Statistical studies based on the random walk data.
- Fractal dimension analysis.
- “Strong” normality versus ordinary normality.
- Applying these techniques to analyses of genome sequences.

Francisco J. Aragon Artacho, David H. Bailey, Jonathan M. Borwein and Peter B. Borwein, “Walking on real numbers,” *Mathematical Intelligencer*, to appear (Jan 2013), available at **http://www.davidhbailey.com/dhbpapers/tools-walk.pdf**

A class of provably normal constants

DHB and Richard Crandall have shown that an infinite class of constants is *b*-normal (and thus b^m-normal for any positive integer *m*), for instance:

$$\begin{aligned}\alpha_{2,3} &= \sum_{n=1}^{\infty} \frac{1}{3^n 2^{3^n}} \\ &= 0.0418836808315029850712528986245716824260967\ldots_{10} \\ &= 0.0ab8e38f684bda12f684bf35ba781948b0fcd6e9e0\ldots_{16}\end{aligned}$$

This particular constant was proven 2-normal by Stoneham in 1971. We extended this to the case where (2,3) are any pair (*b*,*c*) of relatively prime integers > 1, and also to an uncountable class (here r_n is *n*-th bit of *r* in [0,1)):

$$\alpha_{2,3}(r) = \sum_{n=1}^{\infty} \frac{1}{3^n 2^{3^n + r_n}}$$

More recently, DHB and Michal Misiurewich established the $\alpha_{2,3}$ result more simply by means of a "hot spot" lemma proved using ergodic theory.

DHB and M. Misiurewicz, "A Strong Hot Spot Theorem," *Proceedings of the American Mathematical Society*, vol. 134 (2006), no. 9, pg. 2495-2501.
DHB and R. E. Crandall, "Random Generators and Normal Numbers," *Experimental Mathematics*, vol. 11, no. 4 (2002), pg. 527-546.

A nonnormality result

Although $\alpha_{2,3}$ is provably 2-normal, surprisingly enough it is NOT 6-normal. Note that we can write

$$6^n \alpha_{2,3} \bmod 1 = \left(\sum_{m=1}^{\lfloor \log_3 n \rfloor} 3^{n-m} 2^{n-3^m} \right) \bmod 1 + \sum_{m=\lfloor \log_3 n \rfloor + 1}^{\infty} 3^{n-m} 2^{n-3^m}$$

The first portion of this expression is zero, since all of the terms in the summation are integers. When $n = 3^m$, the second portion is accurately approximated by the first term of the series. Thus,

$$6^{3^m} \alpha_{2,3} \bmod 1 \approx \frac{\left(\frac{3}{4}\right)^{3^m}}{3^{m+1}}$$

Because this is so small for large m, this means the base-6 expansion of $\alpha_{2,3}$ has long stretches of zeroes beginning at positions $3^m + 1$. This observation can be fashioned into a rigorous proof of nonnormality.

DHB and J. M. Borwein, "Normal Numbers and Pseudorandom Generators," to appear in Heinz Bauschke, ed., *Proceedings of the Workshop on Computational and Analytical Mathematics in Honour of Jonathan Borwein's 60th Birthday*, Springer, 2011, http://crd.lbl.gov/~dhbailey/dhbpapers/normal-pseudo.pdf.

$\alpha_{2,3}$ in base-6 digits

0.
013014043000333425113050213000000124355504543223301150024352532055135234354101
043000000000000000051411300540405554553031442504334351012413452351125142125134
505503545015053522052044340452151505102411552500425130051124454001044131150032
420303213000000000000000000000000000000000000000000142120343111214520135254453
421134122402205253010542044235524110554150155204350414555400310145303033532002
534340401301240104453254343502142020432415025555101004043300045544114501031331
451151014451412344334234124005513133350454235305531511535015334524354502500555
214530542343421530350125024205404135451231323245353031534552304115020154242121
145201542222534340340450530123325534440443103332445332141415014233454542412432
031253400501341502455144043000000000000000000000000000000000000000000000000000
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244443111105553414105201454021341231300142433313311551144002552443344143003224
335300102353515310402031320020533130021221200323413302222122521000035254543002
545040035245112205252324300200332523104022522341033520112253412034105440205323
451011020023525150015212545502421545343134553235001545540400023411521342323335
352445254035541452214132514501442052123331155522222331511535031203434310450421
303512422055300241415240113013242224123415203330120530530041004541155232331540 4
124335430412302302244135255011522313421543013205100505502430344422115032003312
043000355240351155014141310244412135431134045402354431451021050554450430513450
125125412054431444230051425001123154412512211541355432123414214030443515204024
235524524554300502531314011015041242313530200005202433402032025411301324022320
213143235351510550435543534053314401324313010031034432330333020541411524554102
024511252554525045245533541442124051204242044021002315121441215553532044450502
551452010241043242055023411434513124341231325032130205521143454221303254322525
001003215023424404353350251050434103131435231504421350241010405301043142100315
114422021202242031254124153534242220443313314424101440231132320424442150031332
450441322425500055412431400314402120525313133331045531400355154135352044001245
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213000000000000000000000000000000000000000000000000000000000000000000000000000
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General nonnormality results for the Stoneham alpha constants

Given co-prime integers $b > 1$ and $c > 1$, and integers $p, q, r > 0$, with neither b nor c dividing r, let $B = b^p c^q r$, and assume this condition holds:

$$D = c^{q/p} r^{1/p} / b^{c-1} < 1$$

Then the constant

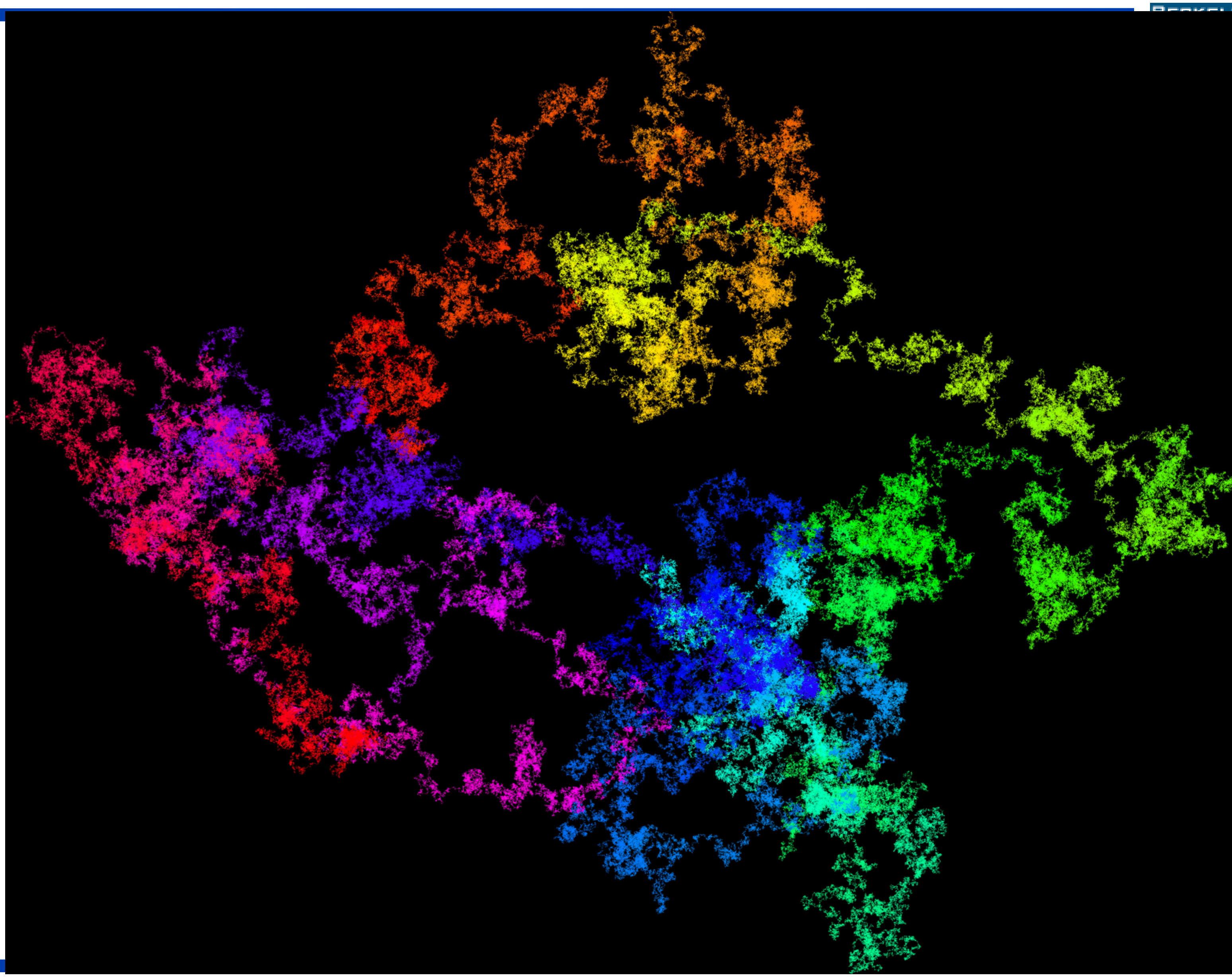
$$\alpha_{b,c} = \sum_{n=1}^{\infty} \frac{1}{c^n b^{c^n}}$$

is not B-normal. Thus, for example, $\alpha_{b,c}$ is b-normal but not bc-normal.

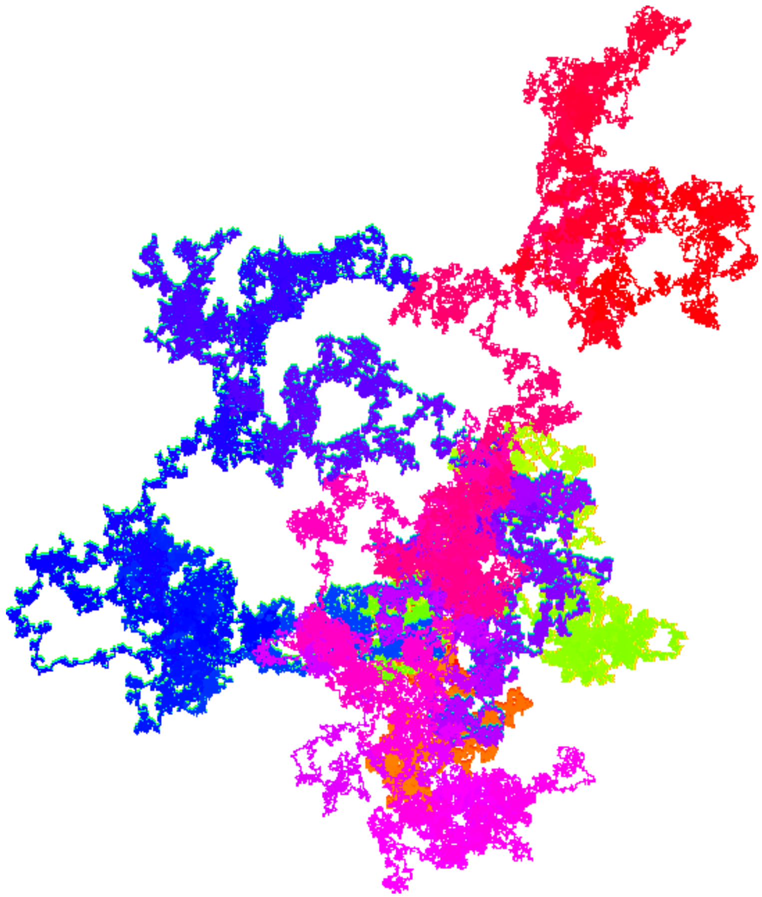
If $\alpha_{b,c}$ and $\alpha_{d,e}$ are two Stoneham constants both nonnormal to base B, as given in the previous result, with c and e multiplicatively independent, then the sum is also nonnormal base B.

Ref: DHB and J. M. Borwein, "Nonnormality of the Stoneham constants, *Ramanujan Journal*, to appear, available at http://crd.lbl.gov/~dhbailey/dhbpapers/nonnormality.pdf.

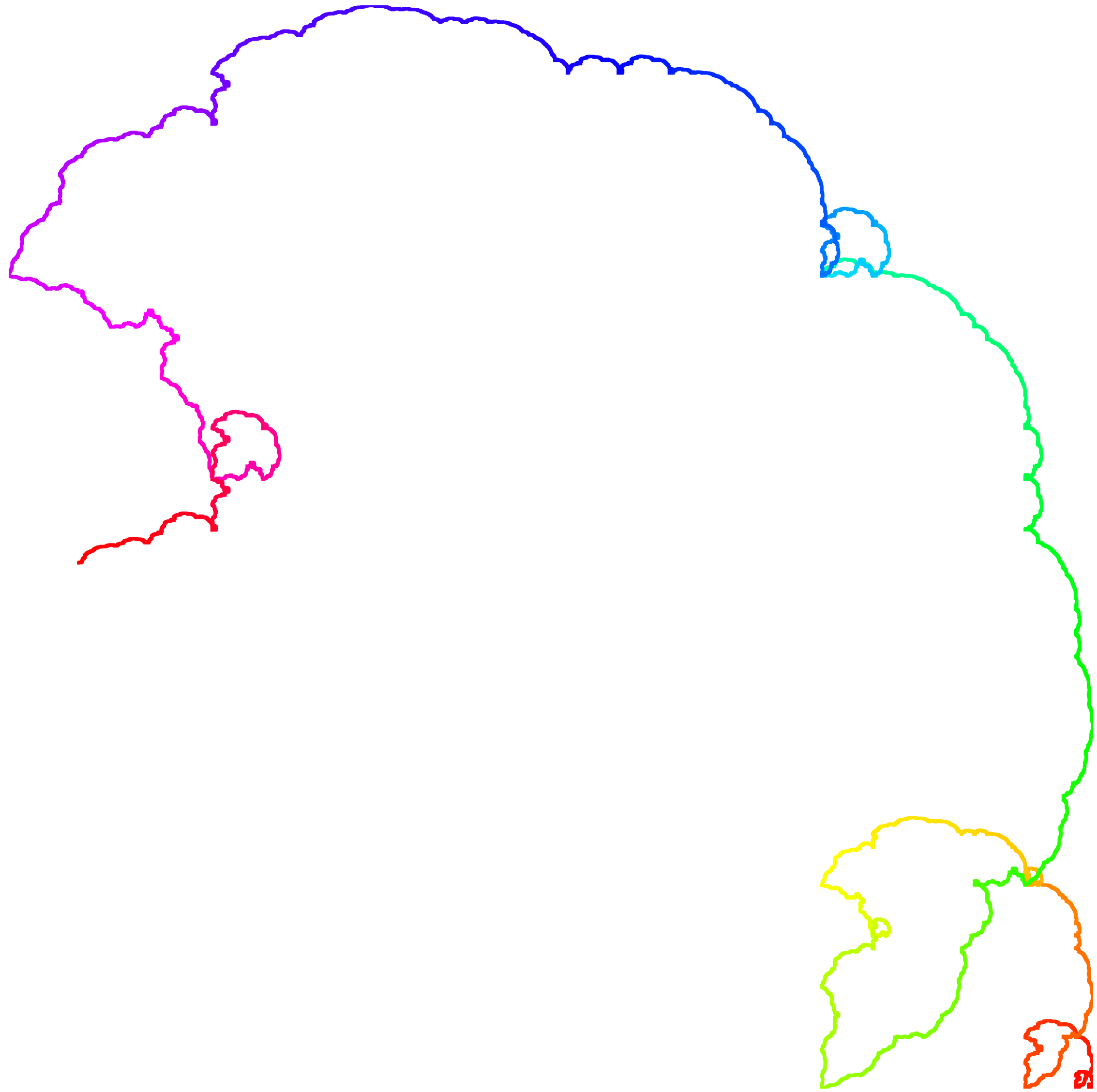
Walk on the first 100 billion base-4 digits of pi
(0: up; 1: right; 2: down; 3: left)



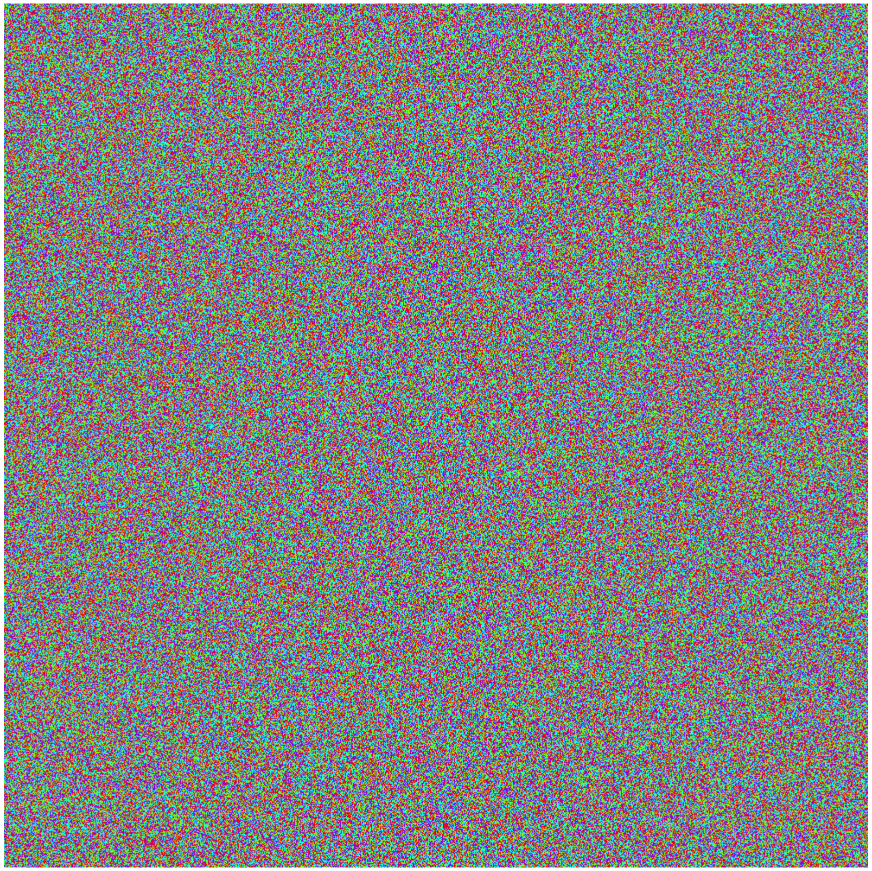
$\alpha_{2,3}$ as a “random” walk: base 2 (normal) vs base 6 (nonnormal)

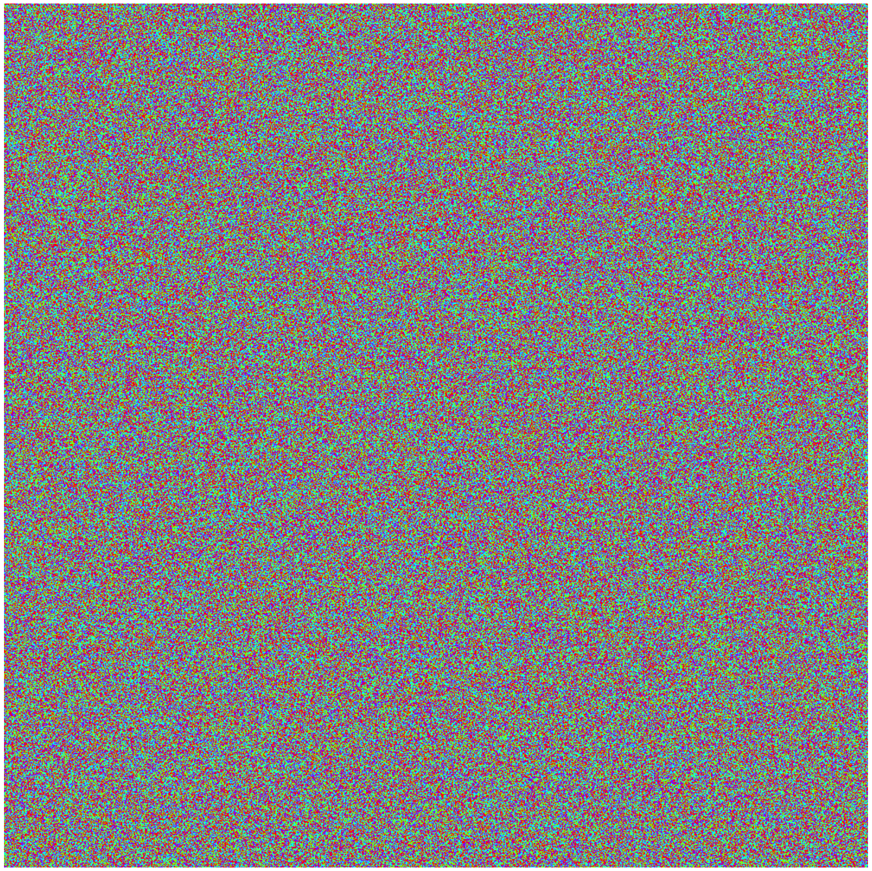


Champernowne's number C_4 (normal base 4)

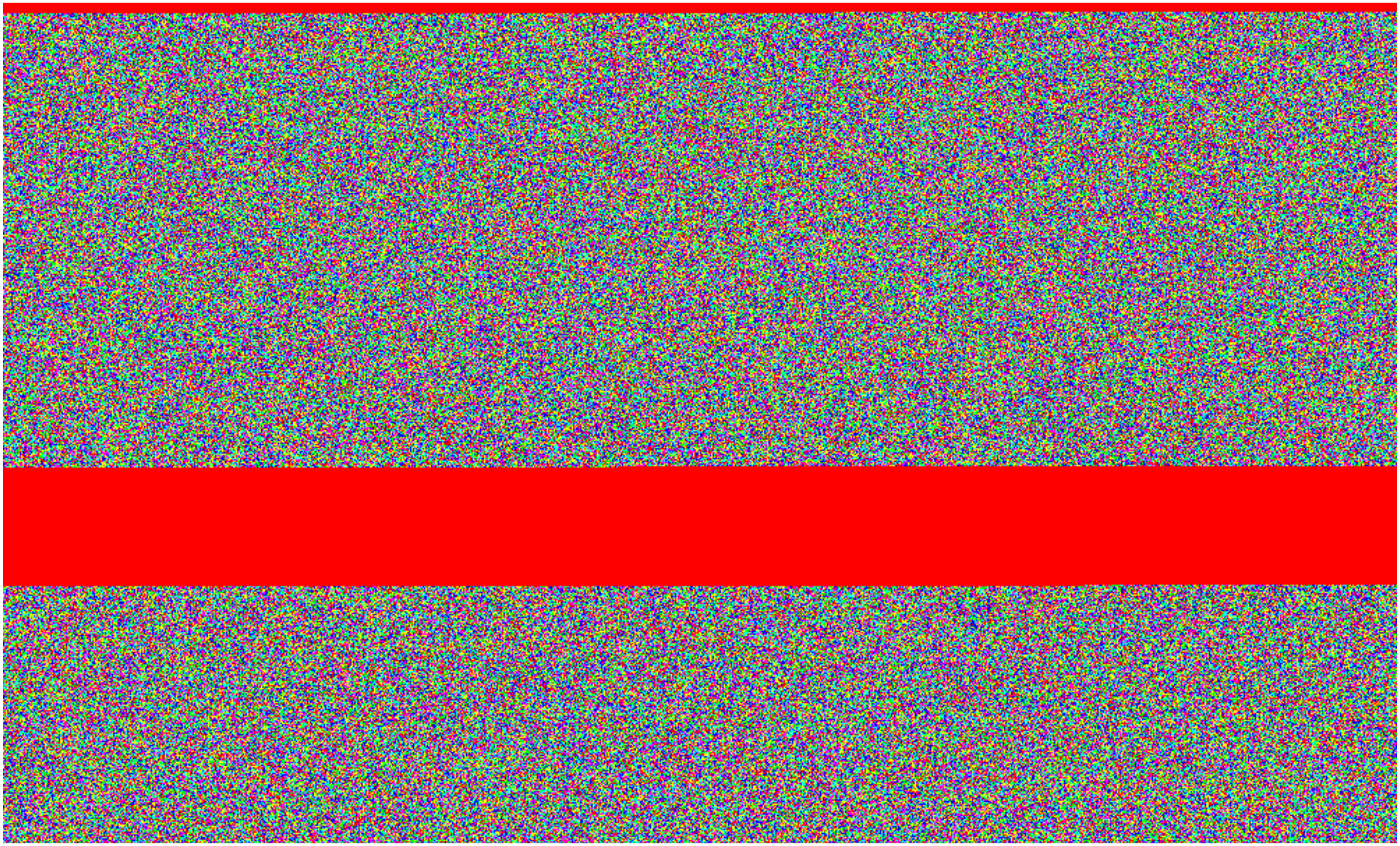


Horizontal color maps of the digits of pi versus pseudorandom digits





Color map of α_{23} base 6 (nonnormal)



For additional details

This talk is available at

http://www.davidhbailey.com/dhbtalks/dhb-walking.pdf

For full details, see:

1. Francisco J. Aragon Artacho, David H. Bailey, Jonathan M. Borwein and Peter B. Borwein, “Walking on real numbers,” *Mathematical Intelligencer*, to appear (Jan 2013), available at **http://www.davidhbailey.com/dhbpapers/tools-walk.pdf**
2. David H. Bailey and Jonathan M. Borwein, “Nonnormality of Stoneham constants,” *Ramanujan Journal*, to appear, available at **http://www.davidhbailey.com/dhbpapers/nonnormality.pdf**
3. David H. Bailey, Jonathan M. Borwein, Cristian S. Calude, Michael J. Dinneen, Monica Dumitrescu and Alex Yee, “An empirical approach to the normality of pi,” *Experimental Mathematics*, vol. 21 (2012), pg. 375-384, available at **http://www.davidhbailey.com/dhbpapers/normality.pdf**