Box integrals

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Abstract. By a "box integral" we mean here an expectation $\langle |\vec{r} - \vec{q}|^s \rangle$ where $\vec{r}$ runs over the unit n-cube, with $\vec{q}$ and s fixed, explicitly:

$$\int_0^1 \cdots \int_0^1 \left((r_1 - q_1)^2 + \cdots + (r_n - q_n)^2 \right)^{s/2} \, dr_1 \cdots dr_n.$$

The study of box integrals leads one naturally into several disparate fields of analysis. While previous studies have focused upon symbolic evaluation and asymptotic analysis of special cases (notably $s = 1$), we work herein more generally—in interdisciplinary fashion—developing results such as: (1) analytic continuation (in complex s), (2) relevant combinatorial identities, (3) rapidly converging series, (4) statistical inferences, (5) connections to mathematical physics, and (6) extreme-precision quadrature techniques appropriate for these integrals. These intuitions and results open up avenues of experimental mathematics, with a view to new conjectures and theorems on integrals of this type.

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1 Box integrals as expectations

Define a box integral for dimension n and parameters $\vec{q}, s$ as the expectation of distance, from a fixed point $\vec{q}$, of a point $\vec{r}$ chosen in equidistributed random fashion over the unit n-cube:

$$\begin{aligned} X_n(s,\vec{q}) &:= \langle |\vec{r}-\vec{q}|^s \rangle_{r\in[0,1]^n} \\ &= \int_{\vec{r}\in[0,1]^n} |\vec{r}-\vec{q}|^s \; \mathcal{D}\vec{r}, \\ &= \int_0^1 \cdots \int_0^1 \left((r_1-q_1)^2+\cdots+(r_n-q_n)^2\right)^{s/2} \; dr_1\cdots dr_n, \end{aligned} \tag{1}$$

where here and elsewhere $\mathcal{D}\vec{r} := dr_1 \cdots dr_n$ is the n-space volume element. We also shall denote simply by r the magnitude $|\vec{r}|$.

There are two classically important instances/functionals of the X-integrals, namely B_n and Δ_n defined:

$$\begin{aligned} B_n(s) &:= X_n(s,\vec{0}) = \int_{\vec{r}\in[0,1]^n} r^s \; \mathcal{D}\vec{r} \\ &= \int_0^1 \cdots \int_0^1 \left(r_1^2+\cdots+r_n^2\right)^{s/2} \; dr_1\cdots dr_n, \end{aligned} \tag{2}$$

$$\begin{aligned} \Delta_n(s) &:= \langle X_n(s,\vec{q})\rangle_{\vec{q}\in[0,1]^n} = \int_{\vec{r},\vec{q}\in[0,1]^n} |\vec{r}-\vec{q}|^s \; \mathcal{D}\vec{r}\,\mathcal{D}\vec{q} \\ &= \int_0^1 \cdots \int_0^1 \left((r_1-q_1)^2+\cdots+(r_n-q_n)^2\right)^{s/2} \; dr_1\cdots dr_n \; dq_1\cdots dq_n. \end{aligned} \tag{3}$$

Note that

1. $B_n(1)$ is the expected distance of a random point from any *vertex* of the n-cube
2. $\Delta_n(1)$ is the expected distance *between* two random points of said cube
3. $X_n(1,(1/2,1/2,\ldots 1/2)\,)$ is the expected distance of a random point from the *center* of said cube

and these are the most oft-discussed entities in the literature. There are many others such as the expected distance between points on distinct sides of a cube or hypercube investigated in [7, §1.7] or [5]. We remark that $B_3(1)$ is also known as the *Robbins constant*, after [9]. Note that the third entity here is not genuinely different, because for general s one has the expected distance from center as

$$X_n(s,(1/2,1/2,\ldots 1/2)\,) \;=\; \frac{1}{2^s} B_n(s), \tag{4}$$

as can be shown quickly from relations (1) by setting $\vec{q} = (1/2, 1/2, \ldots 1/2)$), changing to $p_i = (r_i - q_i)/2$, and observing how the integral has scaled. This is one of the various relations we shall develop that hold for all complex s; in particular, we shall address analytic continuation. It will turn out, interestingly, that $B_n(s)$ is always analytic except for a simple pole at $s = -n$.

There have been interesting modern treatments of the B_n and related integrals, as in [5], [7, p.208], [11], [10]. However, a pivotal, original treatment is the 1976 work of Anderssen et al, [1] who gave a large-n asymptotic series

$$B_n(1) \quad \sim \quad \sqrt{\frac{n}{3}}\left(1 + \frac{n}{10} + \ldots\right), \tag{5}$$

together with a convergent series development for $B_n(1)$ we cite later (and extend to general s), and a collection of bounds, derived via statistical theory, such as

$$\sqrt{\frac{n}{4}} \leq B_n(1) \leq \sqrt{\frac{n}{3}}.$$

This asymptotic is especially interesting when one realizes that the positive unit n-ball sector (the intersection of the n-ball with the cube $[0,1]^n$) has volume decaying superexponentially fast with n. Intuitively speaking, this discrepancy is due to the fact of "so many corners" of the n-cube, where integrable matter resides. We shall argue using statistical intuition that for general s,

$$B_n(s) \quad \sim \quad \left(\frac{n}{3}\right)^{s/2}. \tag{6}$$

A word here is relevant as to the importance of box integrals in other fields of research. It should be noted first that the Anderssen et al. work [1] was motivated by global-optimization study, which explains why the adroit use of statistical principles is apparent in that effort. Secondly, there are problems of lattice theory—such as derivation of what are called "jellium" potentials, that involve $B_n(s)$ for negative s. It is easy to imagine how potential theory for a periodic crystal can involve box integrals. We define and discuss later an n-dimensional jellium potential J_n as an expectation $\langle V_n \rangle$ where V_n is a potential relevant to the n-dimensional Laplace equation.

As we explain herein, it turns out that both $B_n(s), \Delta_n(s)$ even for large n can be numerically evaluated to extreme precision, in much the same way that Bailey et al. [4] resolved the Ising-class integrals C_n for dimension $n \sim 1000$ to hundreds of decimals. In that previous work, a modified-Bessel kernel was employed in a 1-dimensional representation suitable for numerical quadrature. In our present case, an error-function kernel is appropriate. These high-precision quadratures have motivated some conjectures and subsequent proofs of same.

2 Dimensional reduction via vector-field calculus

It turns out that a box integral $X_n(s, \vec{q})$ can be reduced to a suitable integral over the *faces* of a displaced n-cube, in some instances reducible yet further to edges, and so on. Let us write

$$X_n(s, \vec{q}) \;\; = \;\; \int_{\vec{r} \,\in\, \mathcal{C}} r^s \; \mathcal{D}\vec{r},$$

where $r := |\vec{r}|$ and the integration is over a translated cube

$$\mathcal{C} \;\; := \;\; [0, 1]^n - \vec{q}.$$

We may then invoke an elegant procedure from mathematical physics; namely, we attempt to write the (radially symmetric) integrand r^s as the Laplacian of a scalar field. That is, we seek a function Φ of position, such that

$$\nabla^2 \Phi(\vec{r}) \;\; = \;\; r^s.$$

A radially symmetric solution will satisfy the radial part of the Laplacian relation, as

$$\frac{1}{r^{n-1}} \frac{\partial}{\partial r} \left(r^{n-1} \frac{\partial}{\partial r} \right) \;\; = \;\; r^s,$$

whence there is a solution satisfying

$$\frac{\partial \Phi}{\partial r} \;\; = \;\; \frac{1}{n+s} r^{s+1}.$$

The point of these machinations is that we may now utilize the divergence theorem for vector fields, in the form[1]

$$\int_R \nabla \cdot \vec{\mathcal{F}} \; \mathcal{D}\vec{r} \;\; = \;\; \int_{\partial R} \vec{\mathcal{F}} \cdot \; \mathcal{D}\vec{a},$$

where $\vec{\mathcal{F}}(\vec{r})$ is a vector field, the left-hand integral is over the interior of a region R, the right integral is over the boundary, with $\mathcal{D}\vec{a}$ denoting an area element with vector direction always normal to the surface.

The next step is to consider the vector field defined $\vec{\mathcal{F}} := \nabla \Phi$. Using the above observations, we conclude

$$X_n(s, \vec{q}) \;\; = \;\; \frac{1}{n+s} \int_{\vec{r} \,\in\, \partial\mathcal{C}} r^s \; \vec{r} \cdot \hat{\mu} \; da \qquad (7)$$

[1]Known classically as the Gauss theorem for vector fields, this integral relation is ubiquitous in electrostatics and hydrodynamics.

where da is the surface element with normal unit vector $\hat{\mu}$. Note that we have hereby reduced the box integral to an integral over the *faces* of a certain, displaced unit cube. For the box integrals $B_n(s)$, so we have offset $\vec{q} = (1/2, \ldots, 1/2)$, we realize there are $2n$ symmetrically situated faces, and our results boils down to the dimensional-reduction relation

$$B_n(s) \quad = \quad \frac{n}{n+s} \int_{\vec{r} \,\in\, [0,1]^{n-1}} \left(r^2+1\right)^{s/2} \; \mathcal{D}\vec{r}. \tag{8}$$

So for example the 2-dimensional case reduces to a 1-dimensional integral and a final hypergeometric evaluation.

$$\begin{aligned} B_2(s) \quad &:= \quad \int_{\vec{r} \,\in\, [0,1]^2} r^s \; \mathcal{D}\vec{r} \quad = \quad \frac{2}{2+s} \int_0^1 (x^2+1)^{s/2} \; dx \\ &= \quad \frac{2}{2+s} \; {}_2F_1\left(\frac{1}{2}, -\frac{s}{2}; \frac{3}{2}; -1\right). \end{aligned} \tag{9}$$

This hypergeometric entity is rational when s is a nonnegative even integer, and evidently is always a surd plus the log of a surd for s a nonnegative odd integer (see Section 7 for some closed forms).

For the 3-dimensional case, we are able to reduce one further dimension by employing, after the first reduction step from (8), a 2-dimensional solution to

$$\nabla^2 \Phi \quad = \quad (r^2+1)^{s/2},$$

which solution has the property

$$r \frac{\partial \Phi}{\partial r} \quad = \quad \frac{(r^2+1)^{s/2} - 1}{s+2},$$

to get a 1-dimensional representation, like so:

$$\begin{aligned} B_3(s) \quad &= \quad \frac{3}{3+s} \int_{\vec{r} \,\in\, [0,1]^2} (r^2+1)^{s/2} \; \mathcal{D}\vec{r} \\ &= \quad \frac{3}{3+s} \frac{2}{2+s} \int_0^1 \frac{(y^2+2)^{s/2} - 1}{y^2+1} \; dy \\ &= \quad \frac{6}{(3+s)(2+s)} \left(-\frac{\pi}{4} + \int_0^{\pi/4} (1+\sec^2 t)^{s/2+1} \; dt \right). \end{aligned} \tag{10}$$

As with the cases $B_2(s)$, these $B_3(s)$ do enjoy some closed forms, as in Section 7.

3 Error-function formalism and combinatorics

We have seen that an n-dimensional box integral can be reduced by at least one dimension. It turns out that, for numerical quadrature applications, one may achieve a *one*-dimensional integral representation of either B_n or Δ_n. The procedure runs as follows.[2] We start with a certain representation of complex powers:

$$z^\rho \;=\; \frac{\rho}{\Gamma(1-\rho)}\int_0^\infty t^{-\rho-1}\left(1-e^{-tz}\right)\,dt, \tag{11}$$

valid for $\Re(z) > 0$ and $\Re(\rho) \in (0, 2)$. We next define two key functions

$$b(u) \;:=\; \int_0^1 e^{-u^2x^2}\,dx \;=\; \frac{\sqrt{\pi}\,\operatorname{erf}(u)}{2u}, \tag{12}$$

$$d(u) \;:=\; \int_0^1\int_0^1 e^{-u^2(x-y)^2}\,dx\,dy \;=\; \frac{-1+e^{-u^2}+\sqrt{\pi}\,u\,\operatorname{erf}(u)}{u^2}. \tag{13}$$

Now, the defining integrals (2) and (3), and the representation (11), lead to 1-dimensional integrals for each of B_n, Δ_n, like so:

$$B_n(s) \;=\; \frac{s}{\Gamma(1-s/2)}\int_0^\infty \frac{du}{u^{s+1}}\left(1-b(u)^n\right), \tag{14}$$

$$\Delta_n(s) \;=\; \frac{s}{\Gamma(1-s/2)}\int_0^\infty \frac{du}{u^{s+1}}\left(1-d(u)^n\right), \tag{15}$$

both of which being convergent integrals for $\Re(s) \in (0, 2)$. Incidentally, these integrals prove immediately that both B_n, Δ_n for any fixed real s are monotonic increasing in n.

We discuss the issue of numerical quadrature of these error-function representations later. For the moment, we give relevant series developments, as these, relevant to computations on B_N:

$$b(u) \;=\; \sum_{k=0}^\infty \frac{(-1)^k u^{2k}}{k!(2k+1)} \;=\; e^{-u^2}\sum_{k=0}^\infty \frac{2^k u^{2k}}{(2k+1)!!}, \tag{16}$$

and these for Δ_n work:

$$\begin{aligned} d(u) \;&=\; \sum_{k=0}^\infty \frac{(-1)^k u^{2k}}{(k+1)!(2k+1)} \\ &=\; e^{-u^2}\sum_{k=0}^\infty u^{2k}\left(\frac{2^{k+1}}{(2k+1)!!}-\frac{1}{(k+1)!}\right). \end{aligned} \tag{17}$$

[2] The present authors developed this technique for B_n, with a view to extreme-precision quadrature and subsequent experimental mathematics. We found later that M. Trott had previously applied a similar approach for the Δ_n [11]. In a sense, the present treatment is an attempt at unification of the ideas, for more general box integrals.

We discuss in Section 8 how these series play key roles in numerical quadrature. For the moment, we analyze properties of the 1-dimensional integral representations (14), and (15). Important relations along these lines will be these two, where coefficients β_{nk}, δ_{nk} are implicitly defined

$$\left(b\left(\sqrt{t/2}\right)e^{t/2}\right)^n =: \sum_{k\geq 0}\beta_{nk}t^k, \tag{18}$$

$$\left(d\left(\sqrt{t/2}\right)e^{t/2}\right)^n =: \sum_{k\geq 0}\delta_{nk}t^k. \tag{19}$$

Inserting these into (14, 15) we obtain two formal series:

$$B_n(s) = n^{s/2}\sum_{k=0}^{\infty}\left(\frac{2}{n}\right)^k(-s/2)_k\beta_{nk}, \tag{20}$$

$$\Delta_n(s) = n^{s/2}\sum_{k=0}^{\infty}\left(\frac{2}{n}\right)^k(-s/2)_k\delta_{nk}, \tag{21}$$

where $(z)_k$ is the *Pochhammer symbol*[3]. Though developed formally, with regard for convergence issues, it can be shown that each series here converges absolutely whenever $\Re(s) + n > 0$.

To sketch the convergence argument, we initially focus on combinatorial relations for the β_{nk} (the analysis for the Δ_n series is similar). An elementary observation is in order. First, for $s = 2m$ with m a nonnegative integer, the box integral $B_n(2m)$ can always be written as a finite combinatorial sum of rational components, via simple expansion of the defining integrand. Equivalently, series (20) devolves for $s = 2m$ an even integer, into a finite sum

$$\begin{aligned} B_n(2m) &:= \int_{\vec{r}\,\in\,[0,1]^n}\left(r_1^2+\ldots r_n^2\right)^m\ \mathcal{D}\vec{r} \\ &= n^m\sum_{k=0}^{m}(-m)_k\left(\frac{2}{n}\right)^k\beta_{nk}. \end{aligned} \tag{22}$$

This representation of B_n at even integers will prove quite useful in further analysis. Next, stemming from the implicit definition (18) one can derive various relations, the first of which being a beautiful reciprocal relation with the finite sums $B_n(2m)$:

$$\beta_{nk} = \frac{n^k}{2^k k!}\sum_{j=0}^{k}\binom{k}{j}\left(\frac{-1}{n}\right)^j B_n(2j). \tag{23}$$

[3]The Pochhammer symbol $(z)_k := z(z+1)\cdots(z+k-1)$ is extended, for z not a positive integer, by $(z)_k := \Gamma(m-z)/\Gamma(1-z)$, and for all z we define $(z)_0 := 1$.

Other derivable relations are

$$\beta_{nk} = \frac{n^k}{2^k k!} \int_{\vec{r} \,\in\, [0,1]^n} \left(1 - \frac{r^2}{n}\right)^k \mathcal{D}\vec{r}, \tag{24}$$

$$\beta_{nk} = \sum_{k_1+\ldots k_n=k} \frac{1}{(2k_1+1)!!} \cdots \frac{1}{(2k_n+1)!!},$$

$$\beta_{nk} = \sum_{j=0}^{k} \frac{1}{(2k+1)!!} \beta_{n-1,k-j}. \tag{25}$$

In checking all of these combinatorial relations, it is convenient to know some "starting cases." We define $\beta_{n0} := 1$ if $n = 0$, else 0, and note

$$\beta_{1k} = \frac{1}{(2k+1)!!}, \quad \beta_{n1} = \frac{n}{3}, \quad \beta_{n2} = \frac{n^2}{18} + \frac{n}{90},$$

$$B_n(0) = 1, \quad B_n(2) = \frac{n}{3}, \quad B_n(4) = \frac{n^2}{9} + \frac{4n}{45},$$

and so on. Now to the convergence issue for the general expansion (20). From relation (24) one can show

$$\beta_{nk} \le \frac{n^k}{2^k k!} \max\left(1, (n/k)^{n/2}\right),$$

and one has for the relevant Pochhammer symbol

$$|(-s/2)_k| = O\left(k! k^{-1-\Re(s/2)}\right).$$

Thus the k-th summand in (20) is $O(1/k^{n+\Re(s)+1})$ and absolute convergence obtains whenever $n + \Re(s) > 0$.

Using the above analysis for the general series (20)—and after a similar analysis for (21)—we see that several results accrue. We obtain convenient expansions for the even-argument $B_n(2m), \Delta_{2m}$, and an analytic continuation at least for $n + \Re(s) > 0$. There are various additional inferences we may pursue, such as asymptotic behavior (see Section 6), but first we shall describe a more powerful analytic continuation—and more rapidly converging general series—for the B_n in particular.

4 Analytic continuations

Remarkably, and perhaps surprisingly, the relation (8) actually leads to a rapidly (linearly) converging general series for s, and a subsequent analytic continuation to *all* complex s.

Indeed, using the ideas behind (14) we can infer from the dimensional-reduction formula (8) that

$$B_n(s) \;=\; \frac{n}{n+s}\frac{s}{\Gamma(1-s/2)}\int_0^\infty \frac{du}{u^{s+1}}\left(1-e^{-u^2}b(u)^{n-1}\right), \tag{26}$$

leading, after term-by-term integration as before, to an efficient general expansion

$$B_n(s) \;=\; n^{s/2}\frac{n}{n+s}\sum_{k=0}^{\infty}\left(\frac{2}{n}\right)^k(-s/2)_k\beta_{n-1,k}. \tag{27}$$

The rather innocent-looking modifications here over the generally slower series (20) give a much more efficient series. Indeed, since

$$\left(\frac{2}{n}\right)^k(-s/2)_k\beta_{n-1,k} \;=\; O\left((1-1/n)^k k^{-1-\Re(s/2)-n/2}\right)$$

the sum in the general series (27) is linearly convergent for fixed n. Thus, (27) *provides an analytic continuation of B_n to all complex s except for a simple pole at $s=-n$.*

It is not hard to see how analytic continuation works for the box integrals B_n. Take the trivial scenario of $n=1$ dimension. Then, formally, $B_1(s)=\int_0^1 x^s\,dx = 1/(s+1)$ and though the integral diverges for $s=-1$, the analytic continuation of B_1 is the function $1/(s+1)$. The same kind of thinking reveals that in n dimensions, the integrand r^s does diverge for $s=-n$; yet, there is an analytic continuation to finite B_n values at any other s. An example of a continued value—when the literal integral of r^s is infinite—is

$$B_4(-5) \;=\; -0.96120393268995345712165978002474521286412992715\ldots,$$

which could well have a closed form but we do not know it; this approximate value was obtained from the series (27). Note in this regard that our previous, hypergeometric-like reductions (9, 10) for $B_2(s), B_3(s)$ respectively are *already* in analytic continuation form.

There is another way to obtain an efficient series and subsequent continuation, which is foreshadowed by the statistical work in [1] working exclusively with $B_n(1)$. Within our present formalism we can generalize to arbitrary s by contemplating the expectation

$$\langle r^s\rangle_{\vec r\in[0,1]^n} \;=\; \frac{n}{n+s}\left\langle\left((1+(n-1)/2)+(r^2-(n-1)/2)\right)^{s/2}\right\rangle_{\vec r\in[0,1]^{n-1}},$$

where we have written $1+r^2$ in an intentionally cumbersome way in order to invoke the binomial theorem for power $s/2$. After manipulation, we obtain a very efficient series

$$B_n(s) \;=\; \left(\frac{n+1}{2}\right)^{s/2}\frac{n}{n+s}\sum_{k=0}^{\infty}\binom{s/2}{k}\frac{\alpha_{n-1,k}}{(n+1)^k}, \tag{28}$$

where the new α-coefficients are defined

$$\alpha_{nk} := 2^k \sum_{j=0}^{k} \binom{k}{j} \left(\frac{1-n}{2}\right)^{k-j} B_n(2j).$$

Incidentally, in computations involving the series (28) it is useful to know, as a simple consequence of (8), that

$$B_n(2m) \;=\; \frac{n}{n+2m} \sum_{k=0}^{m} \binom{m}{k} B_{n-1}(2k).$$

In this way, numerical evaluation of (28) becomes an exercise in the use of recursion relations.

Again we have a convergent series for all complex s except for the pole at $s = -n$; indeed (28) appears to be the fastest converging series we have, although (27) has certain practical features, such as the appearance of the β-terms which in turn can be evaluated via fast convolution from (25).

5 "Jellium" physics and box integrals

Given an n-cube of uniformly charged jelly of total charge $+1$, what is the electrostatic potential of an electron (having charge (-1)) at the cube center? This question cannot be answered until we settle on suitable potentials in n dimensions. One possibility—which we hereby adopt—is to take the radial potential at distance r from the electron-center as $V_n(r)$, where

$$V_1(r) \;:=\; r - 1/2, \tag{29}$$

$$V_2(r) \;:=\; \log(2r), \tag{30}$$

$$V_n(r) \;:=\; 2^{n-2} - \left(\frac{1}{r}\right)^{n-2}, \quad n > 2. \tag{31}$$

These potentials a) satisfy the Laplace equation in n dimensions, and b) vanish on any face-center. (We are free in electrostatic theory to give any potential a constant offset.)

Let us then define the n-th *jellium potential* as

$$J_n \;:=\; \langle V_n(r) \rangle_{\vec{r} \in [-1/2,1/2]^n}.$$

Interestingly, every J_n except J_2 is essentially—up to offset—a box integral. We can dispense with exact evaluations for $n = 1, 2$ (see Section 7), and observe that

$$J_n \;=\; 2^{n-2}(1 - B_n(2-n)), \quad n > 2. \tag{32}$$

Because the general jellium potential involves negative parameter $s = 2 - n$ for $n > 2$, we are moved to use one of (20, 27, 28) for evaluation. However, reminiscent of relations (14), (15) one may derive an additional error-function representation

$$B_n(s) = \frac{2}{\Gamma(-s/2)} \int_0^\infty \frac{du}{u^{s+1}} b(u)^n, \tag{33}$$

valid for $\Re(s)$ in the negative interval $(-n, 0)$.

6 Intuition via the central limit theorem

The *central limit theorem* of classical probability theory tells us that in some appropriate sense, the distribution of the random variable $\chi := \vec{r} \cdot \vec{r}$ over $\vec{r} \in [0,1]^n$ is Gaussian-normal, with mean and variance

$$\begin{aligned} \langle \chi \rangle &= \frac{n}{3}, \\ \langle (\chi - \langle \chi \rangle)^2 \rangle &= \langle (r^2 - n/3)^2 \rangle = \frac{4n}{45}. \end{aligned}$$

Heuristically speaking, then, we should have

$$B_n(s) \sim_n \frac{1}{\sqrt{2\pi v}} \int_{-\infty}^\infty (n/3 + x)^s e^{-x^2/(2v)}\, dx,$$

where $v := 4n/45$. Interestingly, even though error terms in central-limit expansions can be problematic, binomial development of the integrand gives

$$B_n(s) \sim_n \left(\frac{n}{3}\right)^{s/2} \left(1 + \frac{s(s-2)}{10n} + \dots\right).$$

This agrees, at least through the first two parenthetical terms, with the proven asymptotic of Anderssen et al. [1] for their case $s = 1$.

Still thinking statistically and heuristically, there is another intriguing way to infer that $B_n(s) \sim_n (n/3)^{s/2}(1 + c/n)$ for constant c, which is to rewrite (24) in the form

$$\beta_{nk} = \frac{n^k}{3^k k!} \int_{\vec{r} \in [0,1]^n} \left(1 - \frac{3r^2 - n}{2n}\right)^k \mathcal{D}\vec{r}, \tag{34}$$

where we note that there is now a 3^k in the denominator. Evidently, then, the β-coefficient is seen to depend on moments $\mu_m := \langle (r^2 - n/3)^m \rangle$; additionally we have $\mu_0 = 1, \mu_1 = 0, \mu_2 = 4n/45$, and generally $\mu_m = O(1 + m^2/n)$. Now these estimates for μ, when inserted into the converging series (20), can be seen to give the desired asymptotic. We have not made this argument rigorous; however, relation (34) is promising in this regard.

7 Various closed forms

We next state some known closed and nearly closed forms. The nearly-closed forms engage four unresolved integrals which are given numerically to high precision in Appendix 1.

1. Box integrals as expectations of distance—or inverse-distance-from-vertex:

$$
\begin{aligned}
B_2(-1) &= \log\left(3+2\sqrt{2}\right), \\
B_3(-1) &= -\frac{\pi}{4}-\frac{1}{2}\log 2+\log(5+3\sqrt{3}), \\
B_1(1) &= \frac{1}{2}, \\
B_2(1) &= \frac{\sqrt{2}}{3}+\frac{1}{3}\log\left(\sqrt{2}+1\right), \\
B_3(1) &= \frac{\sqrt{3}}{4}+\frac{1}{2}\log\left(2+\sqrt{3}\right)-\frac{\pi}{24}, \\
B_4(1) &= \frac{2}{5}+\frac{7}{20}\pi\sqrt{2}-\frac{1}{20}\pi\log\left(1+\sqrt{2}\right)+\log(3)-\frac{7}{5}\sqrt{2}\arctan\left(\sqrt{2}\right)+\frac{1}{10}\mathcal{K}_0,
\end{aligned}
$$

where the one unresolved term, namely

$$
\mathcal{K}_0 := \int_0^1 \frac{\log(1+\sqrt{3+y^2})-\log(-1+\sqrt{3+y^2})}{1+y^2}\,dy = 2\int_0^1 \frac{\operatorname{arctanh}\left(\frac{1}{\sqrt{3+y^2}}\right)}{1+y^2}\,dy, \tag{35}
$$

is a dilogarithm-like entity that can be evaluated reasonably rapidly, via the 2-dimensional sum

$$
\mathcal{K}_0 = \sum_{m,k\geq 0}\frac{2^{k+1}}{2m+1}\frac{I_{m+k}}{3^{m+k+1}} = \frac{2}{3}\sum_{p=0}^{\infty} I_p\left(\frac{2}{3}\right)^p\sum_{n=0}^{p}\frac{1}{2^n(2n+1)},
$$

where $I_0 := 1/2$ and

$$
I_m = \frac{1}{2m+1}\left\{2mI_{m-1}+\left(\frac{3}{4}\right)^m\frac{1}{2}\right\}.
$$

Now $\mathcal{K}_0$ can also be recast from this sum in a form revealing more obviously a linear convergence (essentially, by powers of (2/3)):

$$
\mathcal{K}_0 = \sum_{n=1}^{\infty}\beta\left(\frac{1}{2},n\right)\kappa_n\left(\frac{2}{3}\right)^n-\frac{1}{2}\sum_{n=1}^{\infty}{}_2\mathrm{F}_1\left(1,n+\frac{1}{2},n+1,\frac{3}{4}\right)\kappa_n\left(\frac{1}{2}\right)^n
$$

where

$$\kappa_n \quad := \quad \sum_{m=1}^{n} \frac{1}{(2\,m-1)\,2^m}.$$

The expression for $B_4(1)$ results from two dimension reductions followed by substantial symbolic computation with the remaining two-dimensional integrals, all of which ultimately resolved—via dilogarithms—except for $\mathcal{K}_0$.

2. Average distance—or inverse-distance between two points:

$$\begin{aligned}
\Delta_2(-1) &= \frac{2}{3} - \frac{4}{3}\sqrt{2} + 4\log(1+\sqrt{2}),\\
\Delta_1(1) &= \frac{1}{3},\\
\Delta_2(1) &= \frac{1}{15}\left(2+\sqrt{2}+5\log(1+\sqrt{2})\right),\\
\Delta_3(1) &= \frac{4}{105} + \frac{17}{105}\sqrt{2} - \frac{2}{35}\sqrt{3} + \frac{1}{5}\log(1+\sqrt{2}) + \frac{2}{5}\log(2+\sqrt{3}) - \frac{1}{15}\pi,\\
\Delta_4(1) &= \frac{26}{15}\,\mathrm{G} - \frac{34}{105}\,\pi\,\sqrt{2} - \frac{16}{315}\,\pi + \frac{197}{420}\,\log(3) + \frac{52}{105}\,\log\left(2+\sqrt{3}\right)\\
&+ \frac{1}{14}\,\log\left(1+\sqrt{2}\right) + \frac{8}{105}\,\sqrt{3} + \frac{73}{630}\,\sqrt{2} - \frac{23}{135} + \frac{136}{105}\,\sqrt{2}\arctan\left(\frac{1}{\sqrt{2}}\right)\\
&- \frac{1}{5}\,\pi\,\log\left(1+\sqrt{2}\right) + \frac{4}{5}\,\alpha\,\log\left(1+\sqrt{2}\right) - \frac{4}{5}\,\mathrm{Cl}_2\left(\alpha\right) - \frac{4}{5}\,\mathrm{Cl}_2\left(\alpha+\frac{\pi}{2}\right).
\end{aligned}$$

$$\begin{aligned}
\Delta_5(1) &= \frac{65}{42}\,\mathrm{G} - \frac{380}{6237}\,\sqrt{5} + \frac{568}{3465}\,\sqrt{3} - \frac{4}{189}\,\pi - \frac{449}{3465} - \frac{73}{63}\,\sqrt{2}\arctan\left(\frac{\sqrt{2}}{4}\right) - \frac{184}{189}\,\log(2)\\
&+ \frac{64}{189}\,\log\left(\sqrt{5}+1\right) + \frac{1}{54}\,\log\left(1+\sqrt{2}\right) + \frac{40}{63}\,\log\left(\sqrt{2}+\sqrt{6}\right) - \frac{5}{28}\,\pi\,\log\left(1+\sqrt{2}\right)\\
&+ \frac{52}{63}\,\pi\,\log(2) + \frac{295}{252}\,\log(3) + \frac{4}{315}\,\pi^2 + \frac{3239}{62370}\,\sqrt{2} - \frac{8}{21}\sqrt{3}\,\arctan\left(\frac{1}{\sqrt{15}}\right)\\
&- \frac{52}{63}\,\pi\,\log\left(\sqrt{2}+\sqrt{6}\right) + \frac{5}{7}\alpha\,\log\left(1+\sqrt{2}\right) - \frac{5}{7}\,\mathrm{Cl}_2\left(\alpha\right) - \frac{5}{7}\,\mathrm{Cl}_2\left(\alpha+\frac{\pi}{2}\right) + \frac{52}{63}\,\mathcal{K}_1,
\end{aligned}$$

where the unresolved quantity is the integral is

$$\mathcal{K}_1 \quad := \quad \int_3^4 \frac{\mathrm{arcsec}\,(x)}{\sqrt{x^2-4\,x+3}}\,dx, \tag{36}$$

and where $\alpha := \arcsin\left(2/3 - 1/6\sqrt{2}\right)$, G is the *Catalan constant*, Cl_2 is the order-2 *Clausen function*, Ψ_n is the order-n *polygamma* function, and Li_n is the *polylogarithm* function—see [7] or [8] for details.

The evaluations for $\Delta_4(1)$ and $\Delta_5(1)$ come from taking those given in [11] and then (i) carefully eliminating dependent terms (which often entails reexpressing logarithms and polylogarithms) and (ii) using *Kummer's formula* [8, eq. (5.5)] to express the remaining polylogarithms as Clausen functions.

3. Jellium potentials vs. dimension:

$$\begin{aligned}
J_1 &= -\frac{1}{4},\\
J_2 &= -\frac{3}{2} + \frac{\pi}{4} + \frac{1}{2}\log 2,\\
J_3 &= 2 + \frac{1}{2}\pi - 3\log\left(2+\sqrt{3}\right),\\
J_4 &= 4 + \pi^2 + 8\,\mathrm{G} - \frac{4}{3}\ln\left(2+\sqrt{3}\right)\pi - 4\,\mathrm{Cl}_2\left(\frac{1}{6}\pi\right) - 4\,\mathrm{Cl}_2\left(\frac{5}{6}\pi\right) - 16\,\mathcal{K}_2\\
J_5 &= 8 - \frac{5}{3}\pi^2 - 20\ln\left(2+\sqrt{3}\right)\pi + 80\,\mathcal{K}_3.
\end{aligned}$$

The unresolved quantities in the above are given by

$$\begin{aligned}
\mathcal{K}_2 &:= \int_0^{\pi/4}\sqrt{1+\sec^2(a)}\,\arctan\left(\frac{1}{\sqrt{1+\sec^2(a)}}\right)da\\
&= \frac{\pi^2}{16} - \sum_{m=1}^{\infty}\frac{(-1)^m}{2^m}\sum_{k=1}^{m-1}\frac{\binom{-(m-k)}{k}}{2(m-k)+1}\sum_{j=0}^{k-1}\frac{(-1)^j}{2j+1}\\
&= 1 - \frac{\pi}{4} + \frac{\pi^2}{16} - \sum_{N=1}^{\infty}\left(\frac{1}{2}\right)^N\sum_{n=1}^{N}\binom{N-1}{n-1}\sum_{m=1}^{N-1}\frac{(-1)^{n+m}}{(2m+1)(2n+1)}\\
&+ \sum_{N=1}^{\infty}\left(\frac{1}{2}\right)^N\sum_{n=1}^{N}\binom{N-1}{n-1}\sum_{m=1}^{n}\frac{(-1)^{n+N-m}}{(2N-2m+1)(2n+1)}
\end{aligned} \tag{37}$$

and

$$\begin{aligned}
\mathcal{K}_3 &:= \int_0^{\pi/4}\int_0^{\pi/4}\sqrt{1+\sec^2(a)+\sec^2(b)}\,da\,db \qquad (38)\\
&= -\sqrt{3}\sum_{n=0}^{\infty}\frac{\binom{2n}{n}/(2n-1)}{12^n}\sum_{k=0}^{n}\binom{n}{k}\left(\frac{\pi}{4} - \sum_{j=0}^{k-1}\frac{(-1)^j}{2j+1}\right)\left(\frac{\pi}{4} - \sum_{j=0}^{n-k-1}\frac{(-1)^j}{2j+1}\right).
\end{aligned}$$

The value of J_4 was obtained from (8). The even-more partial expansion of J_5 was likewise obtained from two polar transformations.

Note again that the dimensions $n = 1, 2$ have special status, as per (29), (30) and (31).

8 Extreme-precision quadrature

Using the 1-dimensional integral representations (14), (15) and (33), we were able to generate extreme-precision values of $B_n := B_n(1)$, $\Delta_n := \Delta_n(1)$ and $B_n(2-n)$, respectively, for a selection of n. Note that J_n can be readily and accurately evaluated from $B_n(2-n)$ by using (32). These numerical values are given explicitly in Appendix 1, together with values for the unresolved integrals $\mathcal{K}_n$ for $n = 0, 1, 2$ and 3, which we computed using (35), (36), (37) and (38), respectively.

These integrals were computed using the tanh-sinh quadrature scheme. Tanh-sinh quadrature is remarkably effective in evaluating integrals to very high precision, even in cases where the integrand function has an infinite derivative or blow-up singularity at one or both endpoints. It is well-suited for highly parallel evaluation [2], and is also amenable to computation of provable bounds on the error [3]. It is based on the transformation $x = g(t)$, where $g(t) = \tanh[\pi/2 \cdot \sinh(t)]$. In a straightforward implementation of the tanh-sinh scheme, one first calculates a set of *abscissas* x_k and *weights* w_k

$$\begin{aligned} x_j &:= \tanh[\pi/2 \cdot \sinh(jh)] \\ w_j &:= \frac{\pi/2 \cdot \cosh(jh)}{\cosh^2[\pi/2 \cdot \sinh(jh)]}, \end{aligned}$$

where h is the interval of integration. Then the integral of the function $f(t)$ on $[-1, 1]$ is performed as

$$\int_{-1}^{1} f(x)\, dx \;=\; \int_{-\infty}^{\infty} f(g(t))g'(t)\, dt \;\approx\; \sum_{-N}^{N} w_j f(x_j)$$

where N is chosen so that the terms $w_j f(x_j)$ are sufficiently small that they can be ignored for $j > N$. Full details of a robust implementation are given in [6].

Computing B_n using (14) requires one to perform two integrals, one with the integrand function $f(u) = (1 - (\sqrt{\pi}/(2u) \cdot \mathrm{erf} u)^n/u^{s+1}$, from 0 to 1, and a second integral of $f(1/u)/u^2$, from 0 to 1. Adding the two together gives the integral from 0 to ∞. Computing these integrals is complicated by the fact that in tanh-sinh quadrature, the integrand function must be evaluated to high precision very close to the endpoints. In this case, it is not sufficient just to compute erf to high relative precision near zero; because of the subtractions here, one must use a Taylor series expansion for the integrand function when the argument is within say 10^{-10} of zero. Computing these Taylor series coefficients (which we did using *Mathematica*) turned out to be the most expensive part of the entire computation. Once highly accurate integrand functions were available, the quadrature evaluations for B_n were completed in less than one minute each.

Computing Δ_n using (15) also required Taylor series expansions, at least for the first of the two integrals to be performed. Again, obtaining these Taylor series coefficients turned out to be the most expensive aspect of the computation. Computing the $B_n(2-n)$ integrals required no Taylor series expansions and was completely straightforward.

Computing $\mathcal{K}_0$, $\mathcal{K}_1$ and $\mathcal{K}_2$ using (35), (36) and (37) was relatively straightforward. However, computing $\mathcal{K}_3$ using (38) requires 2-dimensional quadrature. We were able to do this by a straightforward extension of the 1-dimensional tanh-sinh scheme to two dimensions. However, because many times more function evaluations are required, the run time was correspondingly longer—four hours, as opposed to a few seconds for the others. We also computed $\mathcal{K}_3$ using the nested infinite series given just following (38), but this required even more run time. The two numerical values, however, agreed.

9 Open problems

- Can the jellium potential J_3 be generalized for different offset vectors (but still in a 3 dimensional setting), to yield via summation the true jellium potential due to an *infinite* cube of charged jelly?

  This leads to the intriguing research area of obtaining other Madelung and Wigner lattice sums via box integrals with changing offset. Note the fixed point $\vec{q}$ in the very definition of X_n does *not* have to be within the unit cube.

- What is the precise asymptotic behavior of $\Delta_n(s)$?

- The authors of the original treatment [1] pointed out that a series of the type (28) does not seem to be available even for their parameter case of $s = 1$. Nor do we know presently how to convert (27) for the Δ_n problems. We do have (21) which converges, albeit slowly. So, what is a rapidly converging series for Δ_n?

- How can (7) be used to reduce the dimension—in a convenient way—for some specific Δ_n? We say "convenient" because the many symmetries of the B_n cases allowed us to make practical use of (8).

- How can (34) be used to establish a precise asymptotic expansion for $B_n(s)$? The original reference [1] perhaps contains sufficient clues.

- Which of the $\mathcal{K}_i$ integrals can be further or completely resolved?

10 Appendix 1: Numerical values

We give here some numerical values, to 400-digit precision, for B_n, Δ_n and $B_n(2-n)$, using formulas (14), (15) and (33), respectively, and via methodology described in Section 8. Note that J_n can be readily and accurately calculated from $B_n(2-n)$ – see (32). These constants are followed by 400-digit values for $\mathcal{K}_0$, $\mathcal{K}_1$, $\mathcal{K}_2$, $\mathcal{K}_3$. Note that closed forms have already been presented in Section 7 for several instances from each class.

B_3:

0.9605919564550529594251079513938063602409769075457239876908985153103876633401632890312279356917748245312164166986007359737684845127383162930777389088732003706911770347511241028894972718344134849125281638669503920734964634783372014360360629072371339141477787290367440166360494296109519496758260098146943820205650037755329676853665979934600683189270145391844972059784188539292526013795172127

B_4:

1.1218996187158609773516151755675427092008079564395458308367924669164035486069153490246731455786376449763403760041726801510115007963136975718767737785523598132759365746125169205517189790894487626923594342121547934391865401675773455401521775713691478521517176414143131711306778574325608937643218519398649169489909522948451157118892878523150581712760768255084802396618535520861420213653932593

B_5:

1.2624066350362518542778971151593382082288086573637362914268783135965228864710809323680157231830034604534204448310384862710340195052011729749887345427570065082501592539413681262968331488923311762549603401740793121456379069820698694973075038600753175819644310767381624580201045649818667252811032715595551399163489446756444073252195468470480314321829137342131214283170243933102113352962316806

B_6:

1.3885740844573478425302540730307888159109450887822070297589331397626378969376828857918435778645157478203781995703617965136256197518532843932099952105974084999316093873716806095766615617270764496948791352915099606840664925798852391235786066874469900136908824049172113121733139469824376583463293684849179984891522368344310820087517017632357629765266938986058767300542248274361017926206913592

B_8:

1.61127355610152924819375450716268500176099203660643217232566632401648443785049629465725195226534964383152654167674186294471132518443915414673338785993466770665158941374868709818315119464415096312555223604133732543900168933939873826511382292244259867993717103648583798726351597191384821216795984755606711335615215100056496138785454851984879009456520115101872216250264779767705574683701428346

B_16:

2.2945271609776747896246757146222629228303704963642261093084540978512791308021437762549837382632049769176853471487876775346557608751869986721449421077926436

1012551609542807249242692380878762925961927271632088691053609918362896428014664012720496276482295624621487098902918319570622527715998920621878447196337069157101551724052688371140995361291950150868372453409107870497610857498671363361 10

B_32:

3.2556283475032900815009371018742820445128021157075055493354242041453691108454
7637939562022789547686491709927842759167895111347894822443913683773237241624181963617895385095535621266956101265367523894715909250920676461416595384454705459630047094294552137436297721314457535025917522954017061845823120097711477774079616224782521522741221417441407309066039461533352902228776959923642601028102 63

B_64:

4.611532276700324612830552509711293301447281715977460131304593777509178923244974735920269631241875009005034439028804471238741080338879997269719612827534357139262597082030961750508165713962648970642360510679898631887373506893764539050864799458602920169844288626046465260305858083001037173262882704582261983216188775088997848329291369100873752714555205544263095959925114313098850349039827791 3

B_128:

6.52685092055981722005086041713350348778980345092952744947986933295536638010088178951505432002757335119808076983149990962085324678032127959179854700589260835713858611603597285364344031166157242605499262182082052004027778165915051913300290957454685496968633280996701961358903764163015986329674623953786384748190556357519985414258750374567335841000381583964060710967052306429863943667029004 31

B_256:

9.2339893036017979821813299560317564867759184854981270468525451313188138520846645147515212921977581551287520595738011766244530522680719248983179885564954351678524446709178250314847122442711343388381616728693944871815358430832564799333069267074687407255092374804523896145120970615296607468123763911942145394213127209730398544886001708341630756123541378173396548339451661490297352140299836668

B_512:

13.061391425743765374699689988825238836400628480849388076753408486929842136376312123831816040251049846788764704571430912641718909199764443175573104436620534740992015621847637895932628830925229110454520981940403611568915233145813780927437286664144716873378067302512765155025944025247165529560783697907304746045962261115313028133859985746883221884608151788554136622452316355350723098731990395

B_1024:

18.473403575815224768394892605266856038204337522823578150467252017100071635212
124293170103398091079082263366569845177540455130598719868405480868501124703919
306722343865527853613544616020301541142229423057526514493758459077280030869091
989606908946900162941152217494957040830364963218638677469330051513767345888894
705197416642970965243018829790657940322084776757106515814924850571987373958832

Delta_3:

0.6617071822671762351558311332484135817464001357909536048089442294795846461385
976313066524807681071201517097753107594109724786805816437216874533242072298244
423276409229206078600086480533266938951526942028215425692085403456100394606163
834472771107263924054689743459232206969510457176785303874823891119488713091981
047559429531205455891503267539401643933207902944734734790101329001545166600642

Delta_4:

0.7776656535862671153379340946178195099628827244171330580234459648650573531592
654011461516568931681884655034447831370951051591390267441809140438428998517624
210232396857024987342411951696562691690582806984694082633284285876412443029864
236559482249802453834733379102480251274813941020682776748739186655210948229736
258759623285258102287734298695709044584888772916586825699031845023601414880863

Delta_5:

0.8785309152674802771789167963362247495118969910897683251376162909611903799327
949972543463494842970110990898249207706783047263811919449405451224631840451809
715167996974471022016195484264140487939336031273898799813162066312813605600588
534362709232451407189347366930101386684205359156108266436021195147574508781297
948585440545276532589325909472353724500198989411976100590886941736128344150490

Delta_6:

0.9689420830367467728060658946668998649945711532713998177258046433716722812520
783658332238668906844993303437015022027291282267370210858131364266978886399883
565536536487074414207107813430509386417558465440695243868933310873069164192705
224174401309497623500010734590819535081373100362474067523492597730107954858514
787605971380025005236671922834828496545558514833526001322020728950261679637232

Delta_8:

1.1281653402412599846612009394532321854563251231539804097940714333981430537331
000164158380650316945397064397038462788265030487689322652826650585865724660279

586798780716236652033742272779804200228289773702015019662201245169436721483503115379382642242930804186256869191401242132987772484464478700954069253424104663027590955571621766420091665288614036614629755969645709283880859098711181164994

Delta_16:

1.6146680158164524396585398092175603932999328397589143275304908277836544921047242112058902770868433426559163698862715874270432620530347314879415808260145815701821516180561108002203979182180621324988737049341612243216518884863909531807811624528053216201875642128824787218216892462543183805528242146112004504799071913063489404914006433166423335632609903796024777305437835420883234020300878316

Delta_32:

2.2966071387598634257091215228252877603275252461721809960501397417063958849102582680426664392340469815440508294297647158103265149363544074310728653513740596926145008546179815629717021943361252852446382451340265804886547908173602840869865445656430566957370483134174185065952444501831234088035970952691130820061342878702339508768418648293965733914939882526222315829093283975719264922522230702

Delta_64:

3.2569978703061582629113130570458302026732231791059539553462576793146704933110950379696597426638423150522793798218971061034534906534685159282691952539849076638012991744455493369451096222952081156129432426331297785649545487370191368689602403236455057021370205833619207381560789026838556247766772858364632190089314159705707376991264744467586166967623739728073489003152648164804263768099402476

Delta_128:

4.6124668995266662261327146890574713789476754267630572531665825113264923712008775751924127308239003707236195631370210588456717190034834428027897921547812234262103051475209516174198235912692454375348126019560488485290755004148746839797903426336886311580713828814189969498498368024109989437719583960037566311604138445589233412009351762157737887854134426028558427911248620077892739397409719239

Delta_256:

6.5275001948905222214786015823845510522953886103465739150365673291750121339601533574360517181319698297205610548945697713557067034260424212678994736833277734997555196634258491928424771895203515110586051483835893144497750311964600371373686491293743382444908121654329130136231929294863991927209344165856809401983117863547768072347508776719185657627973945516326755471528687654583856938527045317

Delta_512:

9.2344443652328348589084547872948597116730832495663187457237277247418740772571
760903663327232671211644576829604443233960679766812133386147916951221402709516
113233961926839299796907636386892743075688603018733523665769503599028670660625
923168989356968874523683285358483090810651130235500756133359970800947883255058
430768757550123746853458514617479226692811872559381497794859044914030353340161

B_3(-1):

1.1900386819897767533219086751420769449911806073574982644089722373037361765531
137144543198138396234083391611332200868923179412641407719117411588051665327607
359961756674889008069793157448495583291788379845629923540847555446809479572771
934623476322161550548542200537852188419384631072870740888098683693858630245580
883325746704942692861162141888231736531968185754545526237096212805578129604969

B_4(-2):

1.0717135155754604706708086778443375210172779320832659847301510617601797742918
025928479805177071789996585886840849426361343769802899675126794784919625780854
210333837425873688425508214010025545714063045391348224294643168231697825080225
435282475698015416250011874216927303965529638404182110457986792008233316685244
513238791005088281028544803215513071529445581445132214591977201405126716834985

B_5(-3):

8.0012321133838410594859790093424520754091697421787928390283028883874828334915
527911032446925213431866540122957252759629118145540072237559291499710396102833
850950668087714103367221644783746150624235057271320941217304847330608442452002
704421057032394368783248220382940304917501048562247134687225815446166814618720
718616800490203110901044614877824149877226921950990355789999399140860844039385e-1

B_6(-4):

5.1922068443349364720879982234783151043232263181242380771895160697611320027476
050225434029434473410871685977389818688348366546211459575361138470884749121069
448389643684872060064064510517468237817560349906131906641798818995802239824409
967335617826962319786561710220145838407740088635635710634629343335546091273601
980026329270202782401297689824491531047592107626188080572169970364630206892e-1

B_8(-6):

1.5953780295854604168166455621857823551584293826352304839592865693680317588179
884953536596814947773932557478441231873099732665411110190303180662932479426213

926593279193799140729367878638269592698582626441041895117682886800522808866564
808077073991918446486735563591093167578452211242079094750794462239391216496502
556724535724291487864577332850093536980397940583465609964981365958726338568e-1

B_16(-14):

1.09883109421558587566302294019454581398383308836157279977360943219308669801105
750878353755889372508456049563895222915645920364314687434139463609694218162488
037319178287784924970395955238553599562273056425130078592165445655669439732608
855150377505905112524377065156554977938617786974773146509050262639906127983995
795501125811421439342086214922710867782214870784569566786842653399962525738e-4

B_32(-30):

9.7046001415525909361604981009971061122931328238789176398236263636346557572500
476038357558248166910798376364722034996142541924602216899067452196217240240603
706332512008366517960190083600368804801758296884027661051538308204071970419842
349760366073130108621385565950710421011748826921807462892893823325264144048172
60517622071767843973251017872032579544956420719087223700447388089168620245e-14

B_64(-62):

5.2912118345129707451288801969572510798557499233220704679746395572024722841683
302494313684987388920809741972596854486445926026311421895783770738322034554293
611647210069108553770408421573187557920294069987892998116901522986601868740666
746899079921121744124974492649694973248723594478550040674057069230070034238498
53240962642528517172345031405880358961178660966838577191047552487926167523e-37

B_128(-126):

1.6200746170781777504605426516026963940780798365909962583516816452325965643001
016818318151880638577091006847622812979685586882697357373665882526069393996495
353366642805968444162543310432651191959824303989649762894340613207805296127040
453702802640951800184450548004351225095336200327115280696222507602381355449641
43283392443781560168737237017787474130033326281105208552574665827074590323e-93

B_256(-254):

3.5350805720914227767089798687883314710396249627140676295904382269230853726478
154558011233885009930466983366183538201823581813190254210827269008933585063842
661534842038515790774375506907990914040258956292913258551548165885667154534979
643968316374061999900537555017804444703610235647116037989753088917945545749807
2066740263602708373733644665053339029843514259454436776150289160536401348e-226

B_512(-510):

2.0714779154945833548195268340089879693285608528781642062369216657137708286908
5676916864855859795744891576257166833512909164894007531294116808686911113915262
962429703905115257245781386437981167271382027924697148177162824312538711027203
484663189986180880756577933597065676826661163134395645548238434651661268054409
8744309872683443225200186123587219842735439730877208370759699996856999303e-530

K_0:

0.9815454579166903183254345064067652807638466955171693124852520241562726831097
573999425144360715791046684112604523265251137555157109158423061788829413678603
497635541446270319117899059918624337253292459941632757677351380232610157093735
135802137246172630900866337076465453049725921279000748256456180655955381933657
683531900278173497487605599543210786979876048268349094015034359315261874663010

K_1:

1.6619078747381233774065816861630594973488686732512589183415081943423549310930
452066938483805687234510381894905136878662909045609227412670328732394609024403
620972625584993222612064376149581805846498737804601038156457864420832794816980
986297511979120643170000822509758940449434954440813415143898294962059870207692
949825313946737277789333826151704801556903907967063865651098248564855419302661

K_2:

1.4933085037789944562935729344558833604589128885764255629365031757311429406189
228852008436255405245028991566979365055943001890238695703641363742859085508361
903676483223286809226261534973297686656884274625467933409426721652718721269375
430189450876305556535315589839744434271664966420886150143520272583986723716894
141342989881707266783031732661253882074983844213478856535905829697544396518115

K_3:

1.5990143092929271524112167925260383396726642062569657287773576892900137332067
577671174042459829254506767901634197792296705483567942209952566800231683117732
919239465529828717306222684173003079500611954182943247026156122135513246038397
836463154374084504973401423693647756065999161150202292250219347643066548305023
425106559607837511279164308097740680098218462439207715368000480687311324768491

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