New results for Euler sums

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Abstract

We present a large number of analytic evaluations of Euler sums, namely sums such as

$$M(m, n_0, n_1, n_2, \ldots, n_t) = \sum_{k=1}^{\infty} \frac{H(k)^m}{k^{n_0}(k+1)^{n_1}(k+2)^{n_2}\cdots(k+t)^{n_t}},$$

for nonnegative integers m and (n_i), with $m \geq 1$ and $n_0 + n_1 + \cdots + n_t \geq 2$, where $H(k) = \sum_{j=1}^{k} 1/j$ is the harmonic function. These results were obtained either by algebraic manipulations, or else by very high-precision numerical evaluations combined with an integer relation algorithm to obtain the analytic formulas. We show how many of these results can be derived from a few basic facts, and that these techniques are applicable to Euler sums of even more general forms than the above cases. We then show that these results permit the calculation of constants for Euler sums resembling the Stieltjes γ constants arising in the theory of the Riemann zeta function, and we also present some preliminary results on the asymptotic behavior of these constants.[1]

1 Introduction

The investigations reported here had their origins in work [1] on the Keiper-Li criterion for the Riemann hypothesis [2, 3]. The Keiper-Li criterion involves positive valued coefficients a_n arising in expansions of the Riemann zeta function. The new representation of the a_n reported in [1] involved a combination of two sets of coefficients $C_{n,p}$ and Σ_p^{ξ}, again positive valued. This representation enabled the accurate calculation of the first 4000 coefficients a_n. The coefficients $C_{n,p}$ obeyed a recurrence relation, and had representations involving the classical Euler sums. A deeper understanding of the asymptotic behaviour of the $C_{n,p}$ as the two integer parameters n and p tended to infinity was sought, and naturally involved results from the extensive literature on Euler sums [4]-[11].

In this paper we address Euler sums of the form

$$M(m, n_0, n_1, n_2, \ldots, n_t) = \sum_{k=1}^{\infty} \frac{H(k)^m}{k^{n_0}(k+1)^{n_1}(k+2)^{n_2}\cdots(k+t)^{n_t}}, \tag{1}$$

for nonnegative integers m and (n_i), with $m \geq 1$ and $n_0+n_1+\cdots+n_t \geq 2$, where $H(k) = H_k = \sum_{j=1}^{k} 1/j$ is the harmonic function (we use both notations interchangeably below). However, the techniques presented below are applicable to Euler sums of even more general forms. We focus on Euler sums having a common order r, where $r = m + n_0 + n_1 + \cdots + n_t$. We combine results from the literature with many new results, in an effort to say as much as possible about systems of order r ranging from 3 to 12. The complexity of these analyses increases rapidly with r.

[1]The authors dedicate this paper to the memory of Jonathan and Peter Borwein, two giants of mathematical research who recently passed away. Jonathan in particular investigated Euler sums in some earlier studies that we reference.

Among the most striking results of this paper are the linkages between Euler sums and Stieltjes constants. The latter can be defined by:

$$\lim_{n\to\infty}\left[\sum_{k=1}^{n}\frac{(\log k)^p}{k}-\frac{(\log n)^{p+1}}{p+1}\right]=\gamma_p. \tag{2}$$

The Stieltjes constants γ_p have a sign which varies in a complicated way as p increases and their modulus increases. We define their equivalent for Euler sums as

$$\lim_{n\to\infty}\left[\sum_{k=1}^{n}\frac{H_k^{p-1}}{k}-\frac{1}{p}H_n^p\right]=\gamma_p^H. \tag{3}$$

Here the harmonic Stieltjes constants γ_p^H are all positive, and again increase rapidly as p increases. The harmonic Stieltjes constants are shown to be given by a sum over a set of primitive sums which form a basis for the Euler sums of order p.

2 Previous results on Euler Sums

We now present selected results from the literature on Euler sums, including relatively recent results due to the late Jonathan Borwein and collaborators [8, 9], which were obtained using the techniques of experimental mathematics to complement analysis. Two of the sets of sums they consider are:

$$s_h(m,n)=\sum_{k=1}^{\infty}\frac{H_k^m}{(k+1)^n} \tag{4}$$

$$\sigma_h(m,n)=\sum_{k=1}^{\infty}\frac{H_k^{(m)}}{(k+1)^n}, \tag{5}$$

where $H_k^{(m)}=\sum_{j=1}^{k}1/j^m$. We define slight modifications of these:

$$\mathcal{I}_h(m,n)=\sum_{k=1}^{\infty}\frac{H_k^m}{k^n},\qquad \mathcal{J}_h(m,n)=\sum_{k=1}^{\infty}\frac{H_k^{(m)}}{k^n}. \tag{6}$$

Then [8]:

$$\mathcal{J}_h(m,n)=\sum_{k=1}^{\infty}\frac{H_k^{(m)}}{k^n}=\sigma_h(m,n)+\zeta(m+n), \tag{7}$$

where $\zeta(p)=\sum_{k\geq 1}1/k^p$ is the Riemann zeta function. For $m=1$, this is

$$\mathcal{I}_h(1,n)-s_h(1,n)=\zeta(n+1). \tag{8}$$

For the special case $m=2$ under particular investigation in [8, 9],

$$\mathcal{I}_h(2,n)-s_h(2,n)=2\mathcal{I}_h(1,n+1)-\zeta(n+2)=2s_h(1,n+1)+\zeta(n+2). \tag{9}$$

For both these cases, the right-hand side tends down to unity as $n\to\infty$.

The sums $\mathcal{I}_h(n,p)$ can be represented in terms of the $s_h(m,n)$ as follows:

$$\mathcal{I}_h(n,p)=\sum_{k=1}^{\infty}\frac{H_k^n}{k^p}=\zeta(n+p)+\sum_{m=1}^{n}\binom{n}{m}s_h(m,p+n-m). \tag{10}$$

We can define the order of this expression to be $n+p$, i.e., the sum of the powers of H_k and k on the left-hand side. The sum of the arguments of s_h on the right-hand side is also $n+p$. The dual expression to equation (10) is

$$s_h(n,p) = \sum_{k=1}^{\infty} \frac{H_k^n}{(k+1)^p} = (-1)^n \zeta(n+p) + \sum_{m=1}^{n} \binom{n}{m} (-1)^{m-n} \mathcal{I}_h(m, p+n-m). \tag{11}$$

Euler provided the solution for $\sigma_h(1,m) = s_h(1,m)$ and thus for $\mathcal{J}_h(1,m) = \mathcal{I}_h(1,m)$ for all $m \geq 2$:

$$\sigma_h(1,m) = \frac{m}{2}\zeta(m+1) - \frac{1}{2}\sum_{k=1}^{m-2} \zeta(m-k)\zeta(k+1). \tag{12}$$

Another useful relationship is the reflection formula, valid for $m, n \geq 2$:

$$\sigma_h(m,n) + \sigma_h(n,m) = \zeta(m)\zeta(n) - \zeta(m+n), \tag{13}$$

or, written with different notation,

$$\mathcal{J}_h(m,n) + \mathcal{J}_h(n,m) = \zeta(m)\zeta(n) + \zeta(m+n), \tag{14}$$

so that $2\mathcal{J}_h(m,m) = \zeta(m)^2 + \zeta(2m)$ for $m \geq 2$.

Euler [9] was able to derive the following expansions in terms of zeta functions, for the particular case where the sum of parameters $s+t$ is odd, and $t > 1$. The first is for s odd, t even:

$$\sigma_h(s,t) = \frac{1}{2}\left[\binom{s+t}{s} - 1\right]\zeta(s+t) + \zeta(s)\zeta(t) - \sum_{j=2}^{(s+t-1)/2} \left[\binom{2j-2}{s-1} + \binom{2j-2}{t-1}\right] \zeta(2j-1)\zeta(s+t-2j+1). \tag{15}$$

For s even, t odd:

$$\sigma_h(s,t) = -\frac{1}{2}\left[\binom{s+t}{s} + 1\right]\zeta(s+t) + \sum_{j=2}^{(s+t-1)/2} \left[\binom{2j-2}{s-1} + \binom{2j-2}{t-1}\right] \zeta(2j-1)\zeta(s+t-2j+1). \tag{16}$$

A valuable result derived in [9] is:

$$s_h(2, 2n-1) = \frac{1}{6}(2n^2 - 7n - 3)\zeta(2n+1) + \zeta(2)\zeta(2n-1) - \frac{1}{2}\sum_{k=1}^{n-2}(2k-1)\zeta(2n-1-2k)\zeta(2k+2) + \frac{1}{3}\sum_{k=1}^{n-2}\zeta(2k+1)\sum_{j=1}^{n-2-k}\zeta(2j+1)\zeta(k+1-j)\zeta(2n-1-2k-2j). \tag{17}$$

Table 5 of [8] gives a list of sums s_h for which the authors were unable to find representations in terms of zeta functions or zeta functions complemented by powers of logarithms of integers and polylogarithms of argument $1/2$, using various search algorithms. These results highlight the difficulty of finding closed form representations of all the Euler-type sums arising in treatments of the sums $\mathcal{C}_{n,p}$ for p large.

The literature on Euler sums [5]-[11] concentrates on the sums $s_h(m,n)$, $\sigma_h(m,n)$, $\mathcal{I}_h(m,n)$ and $\mathcal{J}_h(m,n)$. Below we will analyze the mixed sums $M(m,n,p,q)$, which include the existing results for $s_h(m,n)$ (setting $n = q = 0$) and $\mathcal{I}_h(m,n)$ (setting $p = q = 0$) as special cases. The results we present in

Appendix 2 include literature results up to order 7, extend them to orders 8–11 and also include selected results for order 12, as well as adding many extra results of all orders (*inter alia* for $p, q \neq 0$).

Note that the extension to orders larger than 7 is not straightforward. Order 8 was painstakingly investigated by Bailey, Borwein and Girgensohn [8], and only the single analytic result for $\mathcal{I}_h(1,7)$ was found. By refining and extending the numerical methods used, and regarding $\mathcal{I}_h(2,6)$ as a known quantity, we have been able to obtain all the other sums for order eight in closed form, as reported in Appendix 2. Similar methods have been applied for orders 9, 10, 11 and 12, with the addition of the sums $\mathcal{I}_h(2,6)$, $\mathcal{I}_h(2,8)$, $\mathcal{I}_h(3,8)$, $\mathcal{I}_h(2,10)$, $\mathcal{I}_h(4,8)$ to the set of assumed constants. In Apppendix 1, the numerical values of these five assumed constants are given to 400 figures accuracy.

Some evaluations of $\mathcal{I}(m,n)$ are now presented [5, 8, 9, 4, 11], arranged according to the order $m+n$ (in the remainder of this section we will drop the h subscript on $\mathcal{I}_h$). For order 3, there is only one:

$$\mathcal{I}(1,2) = \sum_{k=1}^{\infty} \frac{H(k)}{k^2} = 2\zeta(3). \tag{18}$$

For order 4, there are two:

$$\mathcal{I}(1,3) = \sum_{k=1}^{\infty} \frac{H(k)}{k^3} = \frac{1}{4}\left(5\zeta(4)\right), \qquad \mathcal{I}(2,2) = \sum_{k=1}^{\infty} \frac{H(k)^2}{k^2} = \frac{1}{4}\left(17\zeta(4)\right). \tag{19}$$

For order 5, there are three:

$$\begin{aligned}
\mathcal{I}(1,4) &= \sum_{k=1}^{\infty} \frac{H(k)}{k^4} = 3\zeta(5) - \zeta(2)\zeta(3), \qquad \mathcal{I}(2,3) = \sum_{k=1}^{\infty} \frac{H(k)^2}{k^3} = \frac{1}{2}\left(7\zeta(5) - 2\zeta(2)\zeta(3)\right), \\
\mathcal{I}(3,2) &= \sum_{k=1}^{\infty} \frac{H(k)^3}{k^2} = 10\zeta(5) + \zeta(2)\zeta(3).
\end{aligned} \tag{20}$$

For order 6, there are four:

$$\begin{aligned}
&\sum_{k=1}^{\infty} \frac{H(k)}{k^5} = \frac{1}{4}\left(7\zeta(6) - 2\zeta(3)^2\right), \qquad \sum_{k=1}^{\infty} \frac{H(k)^2}{k^4} = \frac{1}{24}\left(97\zeta(6) - 48\zeta(3)^2\right), \\
&\sum_{k=1}^{\infty} \frac{H(k)^3}{k^3} = \frac{1}{16}\left(93\zeta(6) - 40\zeta(3)^2\right), \qquad \sum_{k=1}^{\infty} \frac{H(k)^4}{k^2} = \frac{1}{24}\left(979\zeta(6) + 72\zeta(3)^2\right).
\end{aligned} \tag{21}$$

For order 7, there are five:

$$\begin{aligned}
\mathcal{I}(1,6) &= \sum_{k=1}^{\infty} \frac{H(k)}{k^6} = -\zeta(4)\zeta(3) - \zeta(2)\zeta(5) + 4\zeta(7), \\
\mathcal{I}(2,5) &= \sum_{k=1}^{\infty} \frac{H(k)^2}{k^5} = -\frac{5}{2}\zeta(4)\zeta(3) - \zeta(2)\zeta(5) + 6\zeta(7), \\
\mathcal{I}(3,4) &= \sum_{k=1}^{\infty} \frac{H(k)^3}{k^4} = \frac{693}{48}\zeta(7) + 2\zeta(5)\zeta(2) - \frac{51}{4}\zeta(4)\zeta(3), \\
\mathcal{I}(4,3) &= \sum_{k=1}^{\infty} \frac{H(k)^4}{k^3} = \frac{185}{8}\zeta(7) + 5\zeta(5)\zeta(2) - \frac{43}{2}\zeta(4)\zeta(3), \\
\mathcal{I}(5,2) &= \sum_{k=1}^{\infty} \frac{H(k)^5}{k^2} = \frac{2051}{16}\zeta(7) + \frac{57}{2}\zeta(5)\zeta(2) + 33\zeta(4)\zeta(3).
\end{aligned} \tag{22}$$

For order eight, there is a paucity of results in the literature. A careful study of this case was given by Bailey, Borwein and Girgensohn [8]. It employed an Euler-Maclaurin scheme for the high-precision evaluation of these sums, an enhanced version of which we describe in Section 5 below. The only analytic formula we can give in complete form is one studied by Euler:

$$\mathcal{I}(1,7) = \sum_{k=1}^{\infty} \frac{H(k)}{k^7} = \frac{9}{2}\zeta(8) - \zeta(6)\zeta(2) - \zeta(5)\zeta(3) - \frac{1}{2}\zeta(4)^2, \tag{23}$$

which can be simplified using the analytic expressions for $\zeta(2n)$ to

$$\mathcal{I}(1,7) = \frac{1}{4}\left(9\zeta(8) - 4\zeta(3)\zeta(5)\right). \tag{24}$$

We have been able to establish solutions for four additional $\mathcal{I}$ constants if we express them in terms of the set of constants $\zeta(8)$, $\zeta(3)\zeta(5)$ and $\zeta(2)\zeta(3)^2$, together with $\mathcal{I}(2,6)$:

$$\mathcal{I}(3,5) = \frac{1}{96}\left(595\zeta(8) + 120\zeta(2)\zeta(3)^2 - 576\zeta(3)\zeta(5) - 264\mathcal{I}(2,6)\right) \tag{25}$$

$$\mathcal{I}(4,4) = \frac{1}{144}\left(-14833\zeta(8) - 4032\zeta(2)\zeta(3)^2 + 16704\zeta(3)\zeta(5) + 3744\mathcal{I}(2,6)\right) \tag{26}$$

$$\mathcal{I}(5,3) = \frac{1}{288}\left(67811\zeta(8) + 19080\zeta(2)\zeta(3)^2 - 78768\zeta(3)\zeta(5) - 16920\mathcal{I}(2,6)\right) \tag{27}$$

$$\mathcal{I}(6,2) = \frac{1}{8}\left(5843\zeta(8) - 328\zeta(2)\zeta(3)^2 + 3896\zeta(3)\zeta(5) + 456\mathcal{I}(2,6)\right) \tag{28}$$

An earlier study [8] gives the expansions for all $s_h(m,n)$ with $m+n=9$, apart from those coming from (12) and (17). The basis of function values needed is $\zeta(9)$, $\zeta(2)\zeta(7)$, $\zeta(3)\zeta(6)$, $\zeta(4)\zeta(5)$ and $\zeta(3)^3$, the last coming from the double sum in (17). These may be used with (10) to produce the following evaluations of $\mathcal{I}(m,n)$ for order $m+n=9$:

$$\mathcal{I}(1,8) = 5\zeta(9) - \zeta(3)\zeta(6) - \zeta(4)\zeta(5) - \zeta(2)\zeta(7) \tag{29}$$

$$\mathcal{I}(2,7) = \frac{1}{6}\left(55\zeta(9) - 21\zeta(3)\zeta(6) - 15\zeta(4)\zeta(5) - 6\zeta(2)\zeta(7) + 2\zeta(3)^3\right) \tag{30}$$

$$\mathcal{I}(3,6) = \frac{1}{24}\left(521\zeta(9) - 291\zeta(3)\zeta(6) - 306\zeta(4)\zeta(5) + 72\zeta(2)\zeta(7) + 48\zeta(3)^3\right) \tag{31}$$

$$\mathcal{I}(4,5) = \frac{1}{12}\left(436\zeta(9) - 279\zeta(3)\zeta(6) - 258\zeta(4)\zeta(5) + 84\zeta(2)\zeta(7) + 40\zeta(3)^3\right) \tag{32}$$

$$\mathcal{I}(5,4) = \frac{1}{72}\left(9442\zeta(9) - 14685\zeta(3)\zeta(6) + 4752\zeta(4)\zeta(5) + 2385\zeta(2)\zeta(7) - 360\zeta(3)^3\right) \tag{33}$$

$$\mathcal{I}(6,3) = \frac{1}{24}\left(7474\zeta(9) - 13122\zeta(3)\zeta(6) + 6048\zeta(4)\zeta(5) + 1953\zeta(2)\zeta(7) - 544\zeta(3)^3\right) \tag{34}$$

$$\mathcal{I}(7,2) = \frac{1}{72}\left(276341\zeta(9) + 88665\zeta(3)\zeta(6) + 143163\zeta(4)\zeta(5) + 59166\zeta(2)\zeta(7) + 4032\zeta(3)^3\right) \tag{35}$$

The approximate numerical value of $\mathcal{I}(7,2)$ is 9043.54574728044; its integral estimate is 8976.6033415307.

We next present the first results we know of for order $m+n=10$. We originally obtained these results using the method described in Section 5. The eight basic sums $\mathcal{I}$ are obtained with two sums $\mathcal{I}(2,6)$ and $\mathcal{I}(2,8)$ assumed known:

$$\mathcal{I}(1,9) = \frac{1}{4}\left(11\zeta(10) - 4\zeta(3)\zeta(7) - 2\zeta(5)^2\right) \tag{36}$$

$$\mathcal{I}(3,7) = \frac{1}{160}\left(-1661\zeta(10) + 1280\zeta(3)\zeta(7) + 80\zeta(3)^2\zeta(4) - 560\zeta(2)\zeta(3)\zeta(5) + 720\zeta(5)^2 \right.$$
$$\left. +520\mathcal{I}(2,8)\right) \tag{37}$$

$$\begin{aligned}\mathcal{I}(4,6) = {} & \frac{1}{640}\big(-68823\zeta(10) + 60000\zeta(3)\zeta(7) + 1000\zeta(3)^2\zeta(4) - 21680\zeta(2)\zeta(3)\zeta(5) \\ & +23560\zeta(5)^2 + 12120\mathcal{I}(2,8) + 1280\zeta(2)\mathcal{I}(2,6)\big)\end{aligned} \tag{38}$$

$$\begin{aligned}\mathcal{I}(5,5) = {} & \frac{1}{256}\big(64433\zeta(10) - 57760\zeta(3)\zeta(7) + 360\zeta(3)^2\zeta(4) + 20560\zeta(2)\zeta(3)\zeta(5) \\ & -22648\zeta(5)^2 - 10920\mathcal{I}(2,8) - 1280\zeta(2)\mathcal{I}(2,6)\big)\end{aligned} \tag{39}$$

$$\begin{aligned}\mathcal{I}(6,4) = {} & \frac{1}{128}\big(-271367\zeta(10) + 176560\zeta(3)\zeta(7) - 84648\zeta(3)^2\zeta(4) - 400\zeta(2)\zeta(3)\zeta(5) \\ & +121688\zeta(5)^2 + 34376\mathcal{I}(2,8) + 15040\zeta(2)\mathcal{I}(2,6)\big)\end{aligned} \tag{40}$$

$$\begin{aligned}\mathcal{I}(7,3) = {} & \frac{1}{2560}\big(16614991\zeta(10) - 10315520\zeta(3)\zeta(7) + 5879160\zeta(3)^2\zeta(4) - 705040\zeta(2)\zeta(3)\zeta(5) \\ & -7710760\zeta(5)^2 - 2021880\mathcal{I}(2,8) - 1008000\zeta(2)\mathcal{I}(2,6)\big)\end{aligned} \tag{41}$$

$$\begin{aligned}\mathcal{I}(8,2) = {} & \frac{1}{480}\big(18741581\zeta(10) + 6689520\zeta(3)\zeta(7) - 524640\zeta(3)^2\zeta(4) + 1452480\zeta(2)\zeta(3)\zeta(5) \\ & +4247040\zeta(5)^2 + 485280\mathcal{I}(2,8) + 299520\zeta(2)\mathcal{I}(2,6)\big)\end{aligned} \tag{42}$$

We now present results for order $m+n=11$, which again are new in this study, and which again were originally found by us using the methods described below in Section 5. These formulas involve the two sums $\mathcal{I}(2,6)$ and $\mathcal{I}(3,8)$.

$$\mathcal{I}(1,10) = 6\zeta(11) - \zeta(2)\zeta(9) - \zeta(3)\zeta(8) - \zeta(4)\zeta(7) - \zeta(5)\zeta(6) \tag{43}$$

$$\mathcal{I}(2,9) = \frac{1}{2}\big(26\zeta(11) - 2\zeta(2)\zeta(9) - 9\zeta(3)\zeta(8) - 5\zeta(4)\zeta(7) - 7\zeta(5)\zeta(6) + 2\zeta(3)^2\zeta(5)\big) \tag{44}$$

$$\begin{aligned}\mathcal{I}(4,7) = {} & \frac{1}{48}\,(-2877\zeta(11) - 272\zeta(2)\zeta(9) + 1190\zeta(3)\zeta(8) + 1212\zeta(4)\zeta(7) + 1018\zeta(5)\zeta(6) \\ & +80\zeta(2)\zeta(3)^3 - 576\zeta(3)^2\zeta(5) + 176\mathcal{I}(3,8)\big)\end{aligned} \tag{45}$$

$$\begin{aligned}\mathcal{I}(5,6) = {} & \frac{1}{576}\,(-781671\zeta(11) - 88016\zeta(2)\zeta(9) + 296660\zeta(3)\zeta(8) + 411984\zeta(4)\zeta(7) \\ & +220080\zeta(5)\zeta(6) + 21120\zeta(2)\zeta(3)^3 - 141120\zeta(3)^2\zeta(5) + 8640\zeta(3)\mathcal{I}(2,6) \\ & +27840\mathcal{I}(3,8))\end{aligned} \tag{46}$$

$$\begin{aligned}\mathcal{I}(6,5) = {} & \frac{1}{192}\,(734643\zeta(11)83472\zeta(2)\zeta(9) - 271244\zeta(3)\zeta(8) - 395088\zeta(4)\zeta(7) \\ & -205424\zeta(5)\zeta(6) - 19360\zeta(2)\zeta(3)^3 + 130176\zeta(3)^2\zeta(5) - 9120\zeta(3)\mathcal{I}(2,6) \\ & -25600\mathcal{I}(3,8))\end{aligned} \tag{47}$$

$$\begin{aligned}\mathcal{I}(7,4) = {} & \frac{1}{1152}\,(16370805\zeta(11)1684144\zeta(2)\zeta(9) + 5889744\zeta(3)\zeta(8) - 10724760\zeta(4)\zeta(7) \\ & -10480104\zeta(5)\zeta(6) + 844032\zeta(2)\zeta(3)^3 - 2330496\zeta(3)^2\zeta(5) - 1431360\zeta(3)\mathcal{I}(2,6) \\ & -630336\mathcal{I}(3,8))\end{aligned} \tag{48}$$

$$\begin{aligned}\mathcal{I}(8,3) = {} & \frac{1}{72}\,(2824380\zeta(11)277304\zeta(2)\zeta(9) + 1926401\zeta(3)\zeta(8) - 1998972\zeta(4)\zeta(7) \\ & -2270310\zeta(5)\zeta(6) + 243648\zeta(2)\zeta(3)^3 - 803808\zeta(3)^2\zeta(5) - 341280\zeta(3)\mathcal{I}(2,6) \\ & -113760\mathcal{I}(3,8))\end{aligned} \tag{49}$$

$$\begin{aligned}\mathcal{I}(9,2) = {} & \frac{1}{64}\,(7739347\zeta(11) + 2048432\zeta(2)\zeta(9) + 5357920\zeta(3)\zeta(8) \\ & +8811792\zeta(4)\zeta(7) + 10526056\zeta(5)\zeta(6) - 294208\zeta(2)\zeta(3)^3 + 2064192\zeta(3)^2\zeta(5) \\ & +540096\zeta(3)\mathcal{I}(2,6) + 199936\mathcal{I}(3,8))\end{aligned} \tag{50}$$

Finally, we present results for order $m+n=12$, which as before are new to this study, having been originally obtained by us using the methods described in Section 5. These results involve the two sums $\mathcal{I}(2,10)$ and $\mathcal{I}(4,8)$:

$$\sum_{k=1}^{\infty}\frac{H(k)}{k^{11}} = \frac{1}{4}\left(13\zeta(12)-4\zeta(3)\zeta(9)-4\zeta(5)\zeta(7)\right) \tag{51}$$

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{H(k)^3}{k^{9}} = \frac{1}{22112}&(355355\zeta(12)-221120\zeta(3)\zeta(9)-265344\zeta(5)\zeta(7)\\ &-33168\zeta(3)^2\zeta(6)+5528\zeta(3)^4+49752\zeta(2)\zeta(5)^2+99504\zeta(2)\zeta(3)\zeta(7)\\ &-82920\mathcal{I}(2,10))\end{aligned} \tag{52}$$

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{H(k)^5}{k^{7}} = \frac{1}{265344}&(3612841\zeta(12)-884480\zeta(3)\zeta(9)-597024\zeta(5)\zeta(7)\\ &+364848\zeta(3)^2\zeta(6)+221120\zeta(3)^4+364848\zeta(2)\zeta(5)^2+729696\zeta(2)\zeta(3)\zeta(7)\\ &-3250464\zeta(3)\zeta(4)\zeta(5)-1028208\mathcal{I}(2,10)+663360\mathcal{I}(4,8))\end{aligned} \tag{53}$$

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{H(k)^6}{k^{6}} = \frac{1}{530688}&(-4262917573\zeta(12)+2820739392\zeta(3)\zeta(9)+2446737024\zeta(5)\zeta(7)\\ &+112663404\zeta(3)^2\zeta(6)-41128320\zeta(3)^4-402626352\zeta(2)\zeta(5)^2\\ &-741769152\zeta(2)\zeta(3)\zeta(7)-205077744\zeta(3)\zeta(4)\zeta(5)+52538112\zeta(4)\mathcal{I}(2,6)\\ &+84213552\zeta(2)\mathcal{I}(2,8)+519676224\mathcal{I}(2,10)-22554240\mathcal{I}(4,8))\end{aligned} \tag{54}$$

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{H(k)^7}{k^{5}} = \frac{1}{1061376}&(-29991036967\zeta(12)+19798731008\zeta(3)\zeta(9)+17219233536\zeta(5)\zeta(7)\\ &+722473668\zeta(3)^2\zeta(6)-292232192\zeta(3)^4-2832315024\zeta(2)\zeta(5)^2\\ &-5220245184\zeta(2)\zeta(3)\zeta(7)-1329671952\zeta(3)\zeta(4)\zeta(5)+381697344\zeta(4)\mathcal{I}(2,6)\\ &+589494864\zeta(2)\mathcal{I}(2,8)+3662808576\mathcal{I}(2,10)-167166720\mathcal{I}(4,8))\end{aligned} \tag{55}$$

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{H(k)^8}{k^{4}} = \frac{1}{199008}&(-6469168763\zeta(12)-4417645920\zeta(3)\zeta(9)+2316436536\zeta(5)\zeta(7)\\ &-7185432600\zeta(3)^2\zeta(6)+210815808\zeta(3)^4+2292190728\zeta(2)\zeta(5)^2\\ &+3705761136\zeta(2)\zeta(3)\zeta(7)+4396086720\zeta(3)\zeta(4)\zeta(5)+2171077776\zeta(4)\mathcal{I}(2,6)\\ &+241230864\zeta(2)\mathcal{I}(2,8)-842782296\mathcal{I}(2,10)+116552352\mathcal{I}(4,8))\end{aligned} \tag{56}$$

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{H(k)^9}{k^{3}} = \frac{1}{176896}&(4340755723\zeta(12)-37498812096\zeta(3)\zeta(9)-8003239392\zeta(5)\zeta(7)\\ &-29417337684\zeta(3)^2\zeta(6)+1136645248\zeta(3)^4+12010630320\zeta(2)\zeta(5)^2\\ &+20062394496\zeta(2)\zeta(3)\zeta(7)+18880585488\zeta(3)\zeta(4)\zeta(5)+8292663360\zeta(4)\mathcal{I}(2,6)\\ &+375428592\zeta(2)\mathcal{I}(2,8)-7045878240\mathcal{I}(2,10)+635233536\mathcal{I}(4,8))\end{aligned} \tag{57}$$

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{H(k)^{10}}{k^{2}} = \frac{1}{176896}&(702828643635\zeta(12)+39514453568\zeta(3)\zeta(9)+93510608736\zeta(5)\zeta(7)\\ &-23538514220\zeta(3)^2\zeta(6)+2706951040\zeta(3)^4+35094519056\zeta(2)\zeta(5)^2\\ &+62104868800\zeta(2)\zeta(3)\zeta(7)+96955381936\zeta(3)\zeta(4)\zeta(5)+16400028160\zeta(4)\mathcal{I}(2,6)\\ &+954077520\zeta(2)\mathcal{I}(2,8)-12973442080\mathcal{I}(2,10)+1115329280\mathcal{I}(4,8))\end{aligned} \tag{58}$$

m	$\mathcal{I}(m,2)$	Formula (60)	ratio
1	2.4041138063	1.5772156649	0.656048
2	4.5998737432	3.4876092536	0.758196
3	12.346581901	10.655143277	0.863003
4	45.833941465	42.731580639	0.932313
5	220.80305576	213.72197848	0.967930
6	1302.2827194	1282.3688561	0.984708
7	9043.5457472	8976.6033415	0.992597
8	72074.045293	71812.839054	0.996375
9	647472.79308	646315.55860	0.998212

Table 1: The sums $\mathcal{I}(m,2)$ are compared with their integral approximation (60), together with ratios.

2.1 Asymptotics of the sum $\mathcal{I}(m,2)$

The summands of the Euler sums $\mathcal{I}(m,n)$ are always positive, and increase as m increases, while decreasing as n increases. The values of sums depending on the $\mathcal{I}(m,n)$ discussed in this paper tend to be dominated by the lowest sum $\mathcal{I}(m,2)$ for large values of m, and so it is valuable to have asymptotic approximations for it. The value of the sum can be well estimated by an integral, given that the maximum of the truncated summand $(\log k+\gamma)^m/k^2$ occurs for $k=\exp(m/2)$, large enough for the discrete sum to be well approximated by the corresponding integral. For a general positive integer q, the result follows from the recursion

$$\mathcal{N}_{q+1}=\int_1^\infty \frac{(\log k+\gamma)^{q+1}\,\mathrm{d}k}{k^2}=\gamma^{q+1}+(q+1)\mathcal{N}_q, \qquad \mathcal{N}_1=1+\gamma, \tag{59}$$

where $\gamma=0.5772156649\ldots$ is Euler's constant. This recurrence can be solved exactly, giving

$$\int_1^\infty \frac{(\log k+\gamma)^m}{k^2}\mathrm{d}k=m!\,[e^\gamma]_m\,. \tag{60}$$

Here we have introduced the notation for the truncated exponential:

$$[e^\gamma]_m=1+\sum_{q=1}^m\frac{\gamma^q}{q!}. \tag{61}$$

Note that for large m, $\mathcal{N}_m/m!\to\exp(\gamma)$.

Although the integral in equation (60) is exactly evaluated, its use in approximating the sums $\mathcal{I}(m,2)$ for m large depends on two approximations: the sum is well approximated by an integral, and the two-term asymptotic series of the harmonic number function gives a sufficiently accurate representation for the integrand. These approximations are tested in Table 1, which shows that the integral approximation gains relative accuracy rapidly as m increases, until at $m=9$ it is accurate to two parts in 1000.

3 Mixed Euler sums

In the previous sections, we have focused on the $\mathcal{I}$ sums, whose denominators are powers of k, and on the s_h sums, which have powers of $k+1$. But one is immediately led to consider more general denominators, which have not been previously studied in the literature in any detail. To that end we now consider "mixed Euler sums," namely sums such as

$$M(m,n_0,n_1,n_2,\ldots,n_t)=\sum_{k=1}^\infty\frac{H(k)^m}{k^{n_0}(k+1)^{n_1}(k+2)^{n_2}\cdots(k+t)^{n_t}}, \tag{62}$$

for nonnegative integers m and (n_i), with $m \geq 1$ and $n_0+n_1+\cdots+n_t \geq 2$, where $H(k) = H_k = \sum_{j=1}^{k} 1/j$ is the harmonic function as before, and where $r = m + n_0 + n_1 + \cdots + n_t$ is the order. It is clear that the s_h and $\mathcal{I}$ sums are merely special cases: $s_h(m,n) = M(m,0,n)$ and $\mathcal{I}(m,n) = M(m,n)$, so hereafter we will use the M notation. We first demonstrate, by means of examples, why Euler sums with more complicated denominators can be reduced to the basic $M(m,n)$ cases.

Theorem 1. *If the order of a mixed Euler sum of the form* (62) *is* 12 *or less, then it is expressible as a rational linear sum of terms chosen from the following list, depending on the order as shown (constants for a given order include all those of smaller orders, plus the listed "additional constants"):*

Constants for order 3: 1, $\zeta(2)$, $\zeta(3)$
Additional constant for order 4: $\zeta(4)$
Additional constants for order 5: $\zeta(5)$, $\zeta(2)\zeta(3)$
Additional constants for order 6: $\zeta(6)$, $\zeta(3)^2$
Additional constants for order 7: $\zeta(7)$, $\zeta(2)\zeta(5)$, $\zeta(3)\zeta(4)$
Additional constants for order 8: $\zeta(8)$, $\zeta(2)\zeta(3)^2$, $\zeta(3)\zeta(5)$, $M(2,6)$
Additional constants for order 9: $\zeta(9)$, $\zeta(2)\zeta(7)$, $\zeta(3)\zeta(6)$, $\zeta(4)\zeta(5)$, $\zeta(3)^3$
Additional constants for order 10: $\zeta(10)$, $\zeta(3)\zeta(7)$, $\zeta(3)^2\zeta(4)$, $\zeta(2)\zeta(3)\zeta(5)$, $\zeta(5)^2$, $\zeta(2)M(2,6)$, $M(2,8)$
Additional constants for order 11: $\zeta(11)$, $\zeta(2)\zeta(9)$, $\zeta(3)\zeta(8)$, $\zeta(4)\zeta(7)$, $\zeta(5)\zeta(6)$, $\zeta(2)\zeta(3)^3$, $\zeta(5)\zeta(3)^2$, $\zeta(3)M(2,6)$, $M(3,8)$
Additional constants for order 12: $\zeta(12)$, $\zeta(3)\zeta(9)$, $\zeta(5)\zeta(7)$, $\zeta(2)\zeta(5)^2$, $\zeta(2)\zeta(3)\zeta(7)$, $\zeta(3)\zeta(4)\zeta(5)$, $\zeta(3)^2\zeta(6)$, $\zeta(3)^4$, $\zeta(4)M(2,6)$, $\zeta(2)M(2,8)$, $M(2,10)$, $M(4,8)$

Note: We conjecture that the representation of a order-12 or less mixed Euler sum as a rational linear combination of the constants in the list in Theorem 1 above is unique, since integer relation computations on this set rule out any relations with reasonable-sized coefficients (see next paragraph for details), but we have no proof of this. We also conjecture that a result similar to Theorem 1 applies for all higher orders: most likely it only remains to identify the appropriate "atoms," akin to the list in Theorem 1.

We should also clarify that Theorem 1 relies in part on some results in the previous section that were obtained using the computational techniques described below in Section 5.

Note that the above list includes the constants $M(2,6)$, $M(2,8)$, $M(3,8)$, $M(2,10)$ and $M(4,8)$. These constants appear to be linearly independent from the rest of the set, as indicated by the fact that a multipair PSLQ computer run (see Section 5) with the full set of order 8 constants shown above finds no integer relation with Euclidean norm less than $5.88 \cdot 10^{22}$; the full set of order 10 constants above produces no integer relation with Euclidean norm less than $1.28 \cdot 10^{13}$; and the full set of order 12 constants produces no integer relation with Euclidean norm less than $2.13 \cdot 10^6$. Nevertheless, the question of whether $M(2,6)$, $M(2,8)$, $M(3,8)$, $M(2,10)$ and $M(4,8)$, singly or collectively, can be expressed analytically in terms of zetas or other well-known mathematical constants remains open. As an aid to further research, we include 400-digit values of these constants in Appendix 1 (Section 7).

Sketch of proof: We first observe (see Section 2 above) that *each* of the basic Euler sums $M(m,n) = \mathcal{I}(m,n)$ with order $m+n \leq 12$ is reducible to a rational linear sum of the above-listed "atomic" constants. We now argue that any general mixed Euler sum (62) of order 12 or less can be reduced to a rational linear combination of the basic Euler sums $M(m,n)$ of the same order or less, and thus to a rational linear combination of the constants in Theorem 1, by the application (possibly repeated) of these two algebraic techniques:

1. Changing sums with expressions involving $(k+1)$, $(k+2)$ or $(k+w)$ for any integer $w > 0$ to sums involving only k, by means of a process akin to "completing the square" of elementary algebra.

2. Applying a partial fraction decomposition: Recall that any rational function can be written uniquely as the sum of terms based on the factorization of the denominator polynomial, as in the example

$$\frac{1}{(k+1)(k+2)^2} = \frac{1}{k+1} - \frac{1}{k+2} - \frac{1}{(k+2)^2}. \tag{63}$$

This can be produced in *Wolfram Mathematica* by the command: `Apart[1/((k+1)*(k+2)^2)]`.

To illustrate these techniques, note that one can write $M(2,0,2) = \sum_{k=1}^{\infty} H(k)^2/(k+1)^2$ as

$$\begin{aligned}
M(2,0,2) &= \sum_{k=1}^{\infty} \frac{H(k)^2}{(k+1)^2} \\
&= \frac{1}{2^2} + \frac{(1+1/2)^2}{3^2} + \frac{(1+1/2+1/3)^2}{4^2} + \frac{(1+1/2+1/3+1/4)^2}{5^2} + \cdots \\
&= \left(\frac{(1+1/2)^2}{2^2} - \frac{2/2}{2^2} - \frac{1/4}{2^2}\right) + \left(\frac{(1+1/2+1/3)^2}{3^2} - \frac{2/3(1+1/2)}{3^2} - \frac{1/9}{3^2}\right) \\
&\quad + \left(\frac{(1+1/2+1/3+1/4)^2}{4^2} - \frac{2/4(1+1/2+1/3)}{4^2} - \frac{1/16}{4^2}\right) + \cdots \\
&= \left(\frac{(1+1/2)^2}{2^2} + \frac{(1+1/2+1/3)^2}{3^2} + \cdots\right) - 2\left(\frac{1}{2^3} + \frac{(1+1/2)}{3^3} \cdots\right) - \left(\frac{1}{2^4} + \frac{1}{3^4} + \cdots\right) \\
&= \left(\sum_{k=1}^{\infty} \frac{H(k)^2}{k^2} - 1\right) - 2\sum_{k=1}^{\infty} \frac{H(k)}{(k+1)^3} - (\zeta(4) - 1) \\
&= M(2,2) - 2\,M(1,0,3) - \zeta(4). \qquad (64)
\end{aligned}$$

Note, crucially, that this manipulation rewrites the mixed Euler sum $M(2,0,2)$ (of order 4) to an expression involving $M(2,2)$ (of order 4), the mixed sum $M(1,0,3) = \sum_{k=1}^{\infty} H(k)/(k+1)^3$, (also of order 4), and the constant $\zeta(4)$ (again of order 4). A similar manipulation can now be performed on $M(1,0,3)$:

$$\begin{aligned}
M(1,0,3) = \sum_{k=1}^{\infty} \frac{H(k)}{(k+1)^3} \;&=\; \frac{1}{2^3} + \frac{(1+1/2)}{3^3} + \frac{(1+1/2+1/3)}{4^3} + \cdots \\
&= \left(\frac{(1+1/2)}{2^3} - \frac{1/2}{2^3}\right) + \left(\frac{(1+1/2+1/3)}{3^3} - \frac{1/3}{3^3}\right) + \cdots \\
&= \left(\sum_{k=1}^{\infty} \frac{H(k)}{k^3} - 1\right) - (\zeta(4) - 1) \\
&= M(1,3) - \zeta(4) \;=\; 5/4\,\zeta(4) - \zeta(4) \;=\; 1/4\,\zeta(4), \qquad (65)
\end{aligned}$$

so that $M(2,0,2) = M(2,2) - 2\,M(1,0,3) - \zeta(4) = 17/4\,\zeta(4) - 1/2\,\zeta(4) - \zeta(4) = 11/4\,\zeta(4)$, which is of order 4. Note that none of these algebraic manipulations increased the order.

A second example of this technique is $M(2,1,1) = \sum_{k\geq 1} H(k)^2/(k(k+1))$. Note that by employing a manipulation similar to that used above in (64) and (65), combined with the partial fraction decomposition

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}, \qquad (66)$$

this can be written

$$
\begin{aligned}
M(2,1,1) = \sum_{k=1}^{\infty} \frac{H(k)^2}{k(k+1)} \;&=\; \sum_{k=1}^{\infty} \left(\frac{H(k)^2}{k} - \frac{H(k)^2}{k+1} \right) \\
&= \left(\frac{1}{1} + \frac{(1+1/2)^2}{2} + \frac{(1+1/2+1/3)^2}{3} + \cdots \right) - \left(\frac{1}{2} + \frac{(1+1/2)^2}{3} + \frac{(1+1/2+1/3)^2}{4} + \cdots \right) \\
&= \left(\frac{1}{1} + \frac{(1+1/2)^2}{2} + \frac{(1+1/2+1/3)^2}{3} + \cdots \right) \\
&\quad - \left[\left(\frac{(1+1/2)^2}{2} - \frac{2/2}{2} - \frac{1/4}{2} \right) + \left(\frac{(1+1/2+1/3)^2}{3} - \frac{2/3\,(1+1/2)}{3} - \frac{1/9}{3} \right) \right. \\
&\quad \left. + \left(\frac{(1+1/2+1/3+1/4)^2}{4} - \frac{2/4\,(1+1/2+1/3)}{4} - \frac{1/16}{4} \right) + \cdots \right] \\
&= 1 + 2 \sum_{k=1}^{\infty} \frac{H(k)}{(k+1)^2} + (\zeta(3) - 1) \;=\; 2M(1,2) + \zeta(3) \;=\; 3\,\zeta(3). \qquad (67)
\end{aligned}
$$

Note again that none of these operations increased the order; in fact, in this case the order of the final result, namely 3, is less than the order of the original problem, namely 4.

Consider now a more complicated sum such as $M(2,2,2) = \sum_{k \geq 1} H(k)^2/(k^2(k+1)^2$. Sums like this can be readily reduced by means of a partial fraction decomposition, which in this case is:

$$
\frac{1}{k^2(k+1)^2} = -2\left(\frac{1}{k} - \frac{1}{k+1} \right) + \frac{1}{k^2} + \frac{1}{(k+1)^2}, \qquad (68)
$$

so that

$$
\begin{aligned}
M(2,2,2) &= \sum_{k=1}^{\infty} \frac{H(k)^2}{k^2(k+1)^2} \\
&= -2 \sum_{k=1}^{\infty} H(k)^2 \left(\frac{1}{k} - \frac{1}{k+1} \right) + \sum_{k=1}^{\infty} \frac{H(k)^2}{k^2} + \sum_{k=1}^{\infty} \frac{H(k)^2}{(k+1)^2} \\
&= -6\zeta(3) + 17/4\,\zeta(4) + 11/4\,\zeta(4) \\
&= 7\zeta(4) - 6\zeta(3), \qquad (69)
\end{aligned}
$$

where we have employed results from (64), (65) and (67) above.

One example involving $(k+2)$ is $M(2,0,0,2) = \sum_{k \geq 1} H(k)^2/(k+2)^2$. This can be reduced as follows (omitting details of some intermediate evaluations using the above techniques):

$$
\begin{aligned}
M(2,0,0,2) &= \sum_{k=1}^{\infty} \frac{H(k)^2}{(k+2)^2} \\
&= \frac{1^2}{3^2} + \frac{(1+1/2)^2}{4^2} + \frac{(1+1/2+1/3)^2}{5^2} + \frac{(1+1/2+1/3+1/4)^2}{6^2} + \cdots \\
&= \left(\frac{(1+1/2+1/3)^2}{3^2} - \frac{2(1/2+1/3)}{3^2} - \frac{(1/2+1/3)^2}{3^2} \right) \\
&\quad + \left(\frac{(1/2+1/3+1/4)^2}{4^2} - \frac{2(1+1/2)(1/3+1/4)}{4^2} - \frac{(1/3+1/4)^2}{4^2} \right) \\
&\quad + \left(\frac{1/2+1/3+1/4+1/5)^2}{5^2} - \frac{2(1+1/2+1/3)(1/4+1/5)}{5^2} - \frac{(1/4+1/5)^2}{5^2} \right) + \cdots
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{k=3}^{\infty} \frac{H(k)^2}{k^2} - 2\sum_{k=1}^{\infty} \frac{H(k)}{(k+1)(k+2)^2} - 2\sum_{k=1}^{\infty} \frac{H(k)}{(k+2)^3} - \sum_{k=1}^{\infty} \frac{1}{(k+1)^2(k+2)^2} \\
&\quad - 2\sum_{k=1}^{\infty} \frac{1}{(k+1)(k+2)^3} - \sum_{k=1}^{\infty} \frac{1}{(k+2)^4} \\
&= \left(-\frac{25}{16} + \frac{17}{4}\,\zeta(4)\right) - 2\left(3 - \zeta(2) - \zeta(3)\right) - 2\left(-3 + \zeta(2) + \zeta(3) + \frac{1}{4}\zeta(4)\right) \\
&\quad - \left(-\frac{13}{4} + 2\zeta(2)\right) - 2\left(\frac{23}{8} - \zeta(2) - \zeta(3)\right) - \left(-\frac{17}{16} + \zeta(4)\right) \\
&= -3 + 2\zeta(3) + \frac{11}{4}\zeta(4) \qquad (70)
\end{aligned}
$$

The same techniques work for denominators involving $(k+w)$ for any integer $w > 2$. For example, after rather laborious effort one can deduce that

$$
M(2,0,0,0,0,2) = \sum_{k=1}^{\infty} \frac{H(k)^2}{(k+4)^2} \;=\; \frac{1}{3456}\left(-1045 + 288\,\zeta(2) + 648\,\zeta(3)\right). \qquad (71)
$$

We should note, however, that the algebraic manipulations required in these evaluations grow very sharply in complexity with increasing powers of $H(k)$ in the numerator and increasing terms in the denominator. Thus we have found, quite frankly, that in most cases these analytic formulas are more easily obtained by the computational methods we describe below in Section 5.

4 Euler sum analogues to the Stieltjes constants

As an application of these techniques, we address a finite Euler sum result from Choi and Srivastava[4]:

$$
\sum_{k=1}^{n-1} \frac{H_k}{k} = \frac{1}{2}H_{n-1}^2 + \frac{1}{2}H_{n-1}^{(2)}. \qquad (72)
$$

(Note that in this section it is more convenient to use the subscript notation for the harmonic function: $H_k = H(k)$.) We note also the following result from [7]:

$$
\sum_{k=1}^{n} \frac{H_k^{(2)}}{k} = H_n^{(2)} H_n - \sum_{k=1}^{n} \frac{H_k}{k^2} + H_n^{(3)}. \qquad (73)
$$

This is easily generalised to an arbitrary harmonic number of order p:

$$
\sum_{k=1}^{n} \frac{H_k^{(p)}}{k} = H_n^{(p)} H_n - \sum_{k=1}^{n} \frac{H_k}{k^p} + H_n^{(p+1)}. \qquad (74)
$$

We continue with the sums over a finite range, commencing with a result given in [6]:

$$
\sum_{k=1}^{n} \frac{H_k^2}{k} = \frac{1}{3}H_n^3 - \frac{1}{3}H_n^{(3)} + \sum_{k=1}^{n} \frac{H_k}{k^2}. \qquad (75)
$$

The method used to derive (75) employs Abel's summation formula and may easily be generalised to higher values of p. When this was done, a pattern emerged for all the sums:

$$
\sum_{k=1}^{n} \frac{H_k^{p-1}}{k} = \frac{1}{p}H_n^p + \mathcal{D}_{p,1} H_n^{(p)} + \sum_{q=2}^{p-1} \mathcal{D}_{p,q} \sum_{k=1}^{n} \frac{H_k^{q-1}}{k^{p-q+1}}. \qquad (76)
$$

p	$\mathcal{D}_{p,q}$
2	1/2
3	-1/3, 1
4	1/4, -1, 3/2
5	-1/5, 1, -2, 2
6	1/6, -1, 5/2, -10/3, 5/2
7	-1/7, 1, -3, 5, -5, 3
8	1/8, -1, 7/2, -7, 35/4, -7, 7/2
9	-1/9, 1, -4, 28/3, -14, 14, -28/3, 4
10	1/10, -1, 9/2, -12, 21, -126/5, 21, -12, 9/2
11	-1/11, 1, -5, 15, -30, 42, -42, 30, -15, 5
12	1/12, -1, 11/2, -55/3, 165/4, -66, 77, -66, 165/4, -55/3, 11/2
13	-1/13, 1, -6, 22, -55, 99, -132, 132, -99, 55, -22, 6
14	1/14, -1, 13/2, -26, 143/2, -143, 429/2, -1716/7, 429/2, -143, 143/2, -26, 13/2
15	-1/15, 1, -7, 91/3, -91, 1001/5, -1001/3, 429, -429, 1001/3, -1001/5, 91, -91/3, 7
16	1/16, -1, 15/2, -35, 455/4, -273, 1001/2, -715, 6435/8, -715, 1001/2, -273, 455/4, -35, 15/2
17	-1/17, 1, -8, 40, -140, 364, -728, 1144, -1430, 1430, -1144, 728, -364, 140, -40, 8
18	1/18, -1, 17/2, -136/3, 170, -476, 3094/3, -1768, 2431, -1768, 3094/3, -476, 170, -136/3, 17/2
19	1/19, 1, -9, 51, -204, 612, -1428, 2652, -3978, 4862, -4862, 3978, -2652, 1428, -612, 204, -51, 9
20	1/20, -1, 19/2, -57, 969/4, -3876/5, 1938, -3876, 12597/2, -8398, 9237, -8398, 12597/2, -3876, 1938, -3876/5, 969/4, -57, 19/2

Table 2: The quantities $\mathcal{D}_{p,q}$ in equation (76) and (78) for various values of p.

Employing this pattern, it is easy to evaluate the coefficients $\mathcal{D}_{p,q}$ by choosing the same number of values of n as the number of unknowns, and solving linear equations for the $p-1$ unknowns. The values obtained can easily be checked for other values of n. Note that the coefficients in the linear equations are exactly known, and the values for the $\mathcal{D}_{p,q}$ are also exact. Some values are given in Table 4.

The lists of coefficients in Table 4 have some evident properties. The sum of the $\mathcal{D}_{p,q}$ over q when combined with $1/p$ from the first term on the right-hand side in equation (76) is required to be unity, so that the results for $n = 1$ on both sides of equation (76) match. For p even, $\mathcal{D}_{p,1} = 1/p$ and later coefficients show an even symmetry. For p odd, later coefficients show an odd symmetry. For p odd, the second coefficient in Table 4 is 1, while for p even it is -1. There are $p - 1$ coefficients $\mathcal{D}$, with the first two being $1/p, \pm 1$. The rest of the $\mathcal{D}$'s fall into $(p - 3)/2$ pairs which combine subtractively for p odd, or $(p - 4)/2$ additive pairs and a central element for p even.

We take the limit as $n \to \infty$ in equation (76) to define a set of harmonic sum Stieltjes constants γ_p^H, where:

$$\lim_{n\to\infty}\left[\sum_{k=1}^{n}\frac{H_k^{p-1}}{k}-\frac{1}{p}H_n^p\right]=\gamma_p^H, \tag{77}$$

and

$$\begin{aligned}\gamma_p^H &= \mathcal{D}_{p,1}\zeta(p)+\sum_{q=2}^{p-1}\mathcal{D}_{p,q}\sum_{k=1}^{\infty}\frac{H_k^{q-1}}{k^{p-q+1}}\\ &= \mathcal{D}_{p,1}\zeta(p)+\sum_{q=2}^{p-1}\mathcal{D}_{p,q}M_{q-1,p-q+1}.\end{aligned} \tag{78}$$

The most slowly convergent by direct summation of the terms in (78) occurs for $q = p - 1$, where the summand goes to zero as $\log(k)^{p-2}/k^2$.

p	γ_p^H	Formula (89)	Ratio
3	2.0034281719	1.5772156649	0.787258
4	5.8174873811	5.2314138804	0.899256
5	22.315371582	21.310286555	0.954959
6	109.08138223	106.82895159	0.979350
7	647.55020378	641.16593544	0.990140
8	4510.0214667	4488.2909965	0.995181
9	35992.013221	35906.413366	0.997621
10	323539.34424	323157.77574	0.998820
11	3233473.9305	3231577.7930	0.999413

Table 3: The values γ_p^H are compared with their integral approximations (89), together with ratios.

By applying formula (78), together with the computational techniques described below in Section 5, we were able to obtain these results:

$$\gamma_2^H = \frac{1}{2}\left(\zeta(2)\right) \tag{79}$$

$$\gamma_3^H = \frac{1}{3}\left(5\zeta(3)\right) \tag{80}$$

$$\gamma_4^H = \frac{1}{8}\left(43\zeta(4)\right) \tag{81}$$

$$\gamma_5^H = \frac{1}{5}\left(79\zeta(5) + 15\zeta(2)\zeta(3)\right) \tag{82}$$

$$\gamma_6^H = \frac{1}{24}\left(2187\zeta(6) + 272\zeta(3)^2\right) \tag{83}$$

$$\gamma_7^H = \frac{1}{56}\left(18311\zeta(7) + 4060\zeta(2)\zeta(5) + 8358\zeta(3)\zeta(4)\right) \tag{84}$$

$$\gamma_8^H = \frac{1}{576}\left(1926401\zeta(8) + 48384\zeta(2)\zeta(3)^2 + 440064\zeta(3)\zeta(5)\right) \tag{85}$$

$$\gamma_9^H = \frac{1}{36}\left(501978\zeta(9) + 266355\zeta(3)\zeta(6) + 241794\zeta(4)\zeta(5) + 105273\zeta(2)\zeta(7) + 12104\zeta(3)^3\right) \tag{86}$$

$$\gamma_{10}^H = \frac{1}{80}\big(17061619\zeta(10) + 3161210\zeta(3)\zeta(7) + 705180\zeta(3)^2\zeta(4) + 928080\zeta(2)\zeta(3)\zeta(5) + 1770112\zeta(5)^2 + 37320\zeta(2)M(2,6)\big) \tag{87}$$

$$\gamma_{11}^H = \frac{1}{264}\big(230253219\zeta(11) + 49094276\zeta(2)\zeta(9) + 165822855\zeta(3)\zeta(8) + 130449891\zeta(4)\zeta(7) + 156493260\zeta(5)\zeta(6) + 805200\zeta(2)\zeta(3)^3 + 19281504\zeta(3)^2\zeta(5) + 1849320\zeta(3)M(2,6) + 1232880M(3,8)\big) \tag{88}$$

Using the integral estimate $\mathcal{N}_n \to [\exp(\gamma)]_n n!$ and replacing n by $p-2$, we multiply this by $(p-1)/2$, the final $\mathcal{D}_{p,q}$ in each line of Table 2. This gives the estimate for γ_p^H:

$$\gamma_p^H \approx [\exp(\gamma)]_{p-2}\frac{(p-1)!}{2}. \tag{89}$$

The approximation (89) is compared with the γ_p^H for p ranging from 3 to 11 in Table 4. The trend is clearly for the relative accuracy to improve as p increases.

We turn now to equivalent expressions for the case when the denominator in the basic sum is $k+1$

rather than k. From [4],

$$\sum_{k=1}^{n-1} \frac{H_k}{k+1} = \frac{1}{2} H_n^2 - \frac{1}{2} H_n^{(2)}. \tag{90}$$

The equivalent of equation (75) is

$$\sum_{k=1}^{n-1} \frac{H_k^2}{k+1} = \frac{1}{3} H_n^3 - \frac{1}{3} H_n^{(3)} - \sum_{k=1}^{n} \frac{H_k}{(k+1)^2}. \tag{91}$$

Once again, this may be extended to higher powers of H_k in the numerator, giving results of the following form:

$$\sum_{k=1}^{n-1} \frac{H_k^{p-1}}{k+1} = \frac{1}{p} H_n^p + \mathcal{E}_{p,1} H_n^{(p)} + \sum_{q=2}^{p-1} \mathcal{E}_{p,q} \sum_{k=1}^{n} \frac{H_k^{q-1}}{(k+1)^{p-q+1}}. \tag{92}$$

The sums on the right-hand side are the quantities s_h studied inter alia in [8, 9], and connected with the $\mathcal{I}$ by (11). The coefficients $\mathcal{E}_{p,q}$ are given for p up to 21 in Table 4. Note that all the $\mathcal{E}_{p,q}$ are negative, unlike the alternating sign behaviour of the $\mathcal{D}_{p,q}$.

We can define an alternate set of Stieltjes-like constants from equation (92):

$$\lim_{n\to\infty} \left[\sum_{k=1}^{n-1} \frac{H_k^{p-1}}{k+1} - \frac{1}{p} H_n^p \right] = \gamma_p^h, \tag{93}$$

where

$$\gamma_p^h = \mathcal{E}_{p,1} \zeta(p) + \sum_{q=2}^{p-1} \mathcal{E}_{p,q} s_h(q-1, p-q+1). \tag{94}$$

The γ_p^h would then all be negative, and by virtue of (9) with a modulus well approximated asymptotically by (89).

5 Computational techniques

We initially obtained many of the formulas presented above and in Appendix 2 (Section 8) by a computational procedure that utilizes advanced techniques to produce a very high-precision numerical value of the sum, then employs an integer relation algorithm to identify the numerical value as a rational linear sum of constants from the list in Theorem 1. We present here a brief summary of these techniques, which are based in part on schemes described in [8, 15].

In this way, we have found formulas for the much more numerous set of mixed Euler cases

$$M(m, n, p, q) = \sum_{k=1}^{\infty} \frac{H_k^m}{k^n (k+1)^p (k+2)^q} \tag{95}$$

for orders $r = m + n + p + q = 3$ through 11, and also a selection of results of order 12. As noted above, this class includes the cases $s_h(m, n)$ and $\mathcal{I}(m, n)$ as subsets. We present the full collection of these formulas in Appendix 2 (Section 8).

5.1 Computing Euler sums to high precision

One key tool for these computations is the Euler-Maclaurin summation formula [12, pg. 285], which approximates a summation as an integral with high-order corrections (here $f(t)$ is assumed to have $(2s+2)$-th order derivatives on $[a, b]$):

$$\sum_{j=a}^{b} f(j) = \int_a^b f(t)\,\mathrm{d}t \; + \; \frac{1}{2}\left(f(a) + f(b)\right) \; + \; \sum_{j=1}^{s} \frac{B_{2j}\left(D^{2j-1} f(b) - D^{2j-1} f(a)\right)}{(2j)!} \; + \; R_s(a, b), \tag{96}$$

p	$\mathcal{E}_{p,q}$
2	-1/2
3	-1/3, -1
4	-1/4, -1, -3/2
5	-1/5, -1, -2, -2
6	-1/6, -1, -5/2, -10/3, -5/2
7	-1/7, -1, -3, -5, -5, -3
8	-1/8, -1, -7/2, -7, -35/4, -7, -7/2
9	-1/9, -1, -4, -28/3, -14, -14, -28/3, -4
10	-1/10, -1, -9/2, -12, -21, -126/5, -21, -12, -9/2
11	-1/11, -1, -5, -15, -30, -42, -42, -30, -15, -5
12	-1/12,-1, -(11/2), -(55/3), -(165/4), -66, -77, -66, -(165/4), -(55/3), -(11/2)
13	-1/13,-1, -6, -22, -55, -99, -132, -132, -99, -55, -22, -6
14	-1/14,-1, -(13/2), -26, -(143/2), -143, -(429/2), -(1716/7), -(429/2), -143, -(143/2), -26, -(13/2)
15	-1/15,-1, -7, -(91/3), -91, -(1001/5), -(1001/3), -429, -429, -(1001/3), -(1001/5), -91, -(91/3), -7
16	-1/16,-1, -(15/2), -35, -(455/4), -273, -(1001/2), -715, -(6435/8), -715, -(1001/2), -273, -(455/4), -35, -(15/2)
17	-1/17,-1, -8, -40, -140, -364, -728, -1144, -1430, -1430, -1144, -728, -364, -140, -40, -8
18	-1/18,-1, -(17/2), -(136/3), -170, -476, -(3094/3), -1768, -2431, -(24310/9), -2431, -1768, -(3094/3), -476, -170, -(136/3), -(17/2)
19	-1/19, -1, -9, -51, -204, -612, -1428, -2652, -3978, -4862, -4862, -3978, -2652, -1428, -612, -204, -51, -9
20	-1/20, -1, -(19/2), -57, -(969/4), -(3876/5), -1938, -3876, -(12597/2), -8398, -(46189/5), -8398, -(12597/2), -3876, -1938, -(3876/5), -(969/4), -57, -(19/2)
21	-1/21,-1, -10, -(190/3), -285, -969, -2584, -(38760/7), -9690, -(41990/3), -16796, -16796, -(41990/3), -9690, -(38760/7), -2584, -969, -285, -(190/3), -10

Table 4: The coefficients $\mathcal{E}_{p,q}$ in equation (76) for various values of p.

where B_k is the k-th Bernoulli number [10], $D^k f(a)$ is the k-th derivative of $f(t)$ evaluated at $t = a$, and

$$R_s(a,b) = \frac{-1}{(2s+2)!} \int_a^b B_{2s+2}(t - [t]) D^{2s+2} f(t)\, \mathrm{d}t, \tag{97}$$

where $[\cdot]$ denotes greatest integer and $B_k(\cdot)$ is the k-th Bernoulli polynomial [10] (note $B_k = B_k(0)$).

Applying the Euler-Maclaurin summation formula to the harmonic function $H(t) = \sum_{j=1}^{t} 1/j$ yields

$$H(t) = \gamma + \log(t) + \frac{1}{2t} + \sum_{j=1}^{s} \frac{B_{2j}}{2j t^{2j}} + R_s(t), \tag{98}$$

where $\gamma = 0.5772156649\ldots$ is Euler's constant and $|R_s(t)| \le |B_{2s+2}|/((2s+2)t^{2s+2})$; see [8] for full details. In the computations for the present study, we set $s = 21$, so that $H(t)$ is approximated by

$$\begin{aligned}\hat{H}(t) = \gamma &+ \log(t) + \frac{1}{2t} - \frac{1}{12t^2} + \frac{1}{120t^4} - \frac{1}{252t^6} + \frac{1}{240t^8} - \frac{1}{132t^{10}} + \frac{691}{32760t^{12}} - \frac{1}{12t^{14}} \\ &+ \frac{3617}{8160t^{16}} - \frac{43867}{14364t^{18}} + \frac{174611}{6600t^{20}} - \frac{77683}{276t^{22}} + \frac{236364091}{65520t^{24}} - \frac{657931}{12t^{26}} + \frac{3392780147}{3480t^{28}} \\ &- \frac{1723168255201}{85932t^{30}} + \frac{7709321041217}{16320t^{32}} - \frac{151628697551}{12t^{34}} + \frac{26315271553053477373}{69090840t^{36}} \\ &- \frac{154210205991661}{12t^{38}} + \frac{261082718496449122051}{541200t^{40}} - \frac{1520097643918070802691}{75852t^{42}},\end{aligned} \tag{99}$$

which approximates $H(t)$ to within roughly t^{-44} for large t. The expression (99) can be obtained using *Wolfram Mathematica* with the command `Series[HarmonicNumber[t],{t,Infinity,42}]`.

Given $M(m,n,p,q)$, denote $\hat{G}(t) = \hat{H}(t)^m/(t^n(t+1)^p(t+2)^q)$. Using the Euler-Maclaurin summation formula (96) once again, one can write

$$\begin{aligned}M(m,n,p,q) &= \sum_{j=1}^{k} \frac{H(j)^m}{j^n(j+1)^p(j+2)^q} + \sum_{j=k+1}^{\infty} \frac{H(j)^m}{j^n(j+1)^p(j+2)^q} \approx \sum_{j=1}^{k} \frac{H(j)^m}{j^n(j+1)^p(j+2)^q} + \sum_{j=k+1}^{\infty} \hat{G}(j) \\ &\approx \sum_{j=1}^{k} \frac{H(j)^m}{j^n(j+1)^p(j+2)^q} + \int_{k+1}^{\infty} \hat{G}(t)\,\mathrm{d}t + \frac{1}{2}\hat{G}(k+1) - \sum_{j=1}^{s} \frac{B_{2j} D^{2j-1}\hat{G}(k+1)}{(2j)!},\end{aligned} \tag{100}$$

where $s = 21$, which is accurate to within roughly k^{-44}. Initially we set $k = 10^8 = 100{,}000{,}000$, so the approximation in the second line of (100) is correct to within roughly 10^{-354}, which was sufficient for our early investigations. For larger cases, and for all runs listed in Appendix 2, we set $k = 10^9 = 1{,}000{,}000{,}000$, so this approximation is correct to within roughly 10^{-396}.

We evaluated the first term of (100) (the explicit summation) using an arbitrary precision package [13]. Using $k = 10^8$ and a working precision of 360 digits (producing roughly 350 good digits) required 5–9 minutes CPU time per case on a 2024 Apple Mac Studio system with an M4 processor; using $k = 10^9$ and a working precision of 420 digits (producing roughly 400 good digits) required 50–90 minutes per case. For our 400-digit computations, we evaluated the second term (the integral) using the exp-sinh quadrature algorithm [13, 14], with the arbitrary precision software set to 400 digits; this required only 3–4 seconds per case (we first tried to evaluate these integrals using *Wolfram Mathematica* version 14.2, but this failed for larger m). The third term is straightforward. The fourth term, which involves the symbolic expansion and numerical evaluation to 400-digit accuracy of high-order derivatives of the approximation function $\hat{G}(t) = \hat{H}(t)^m/(t^n(t+1)^p(t+2)^q)$, where $\hat{H}(t)$ is given by the expression (99), was computed using *Wolfram Mathematica*; this required up to 400 seconds CPU time per case for larger m.

5.2 Using an integer relation algorithm to find formulas

Once a 400-digit value for a given mixed Euler constant was obtained, we employed the multipair PSLQ algorithm to search for integer relations with known constants [15, 16, 17]. Given an v-long vector

$x = (x_0, x_1, \cdots, x_{v-1})$ of high-precision floating-point reals, the multipair PSLQ algorithm searches for integers $(a_0, a_1, \cdots, a_{v-1})$ such that $a_0 x_0 + a_1 x_1 + \cdots + a_{v-1} x_{v-1} = 0$ to within available precision, or else establishes that there is no such integer relation within a given bound. The algorithm operates by generating an iterative sequence of $v \times v$ integer matrices B, so that the entries of the vector $y = B \cdot x$ become progressively smaller, until one entry of y is numerically zero, at which iteration the algorithm halts, with the relation given by the row of B corresponding to the zero entry of y. In the application here, we set x_0 to the 400-digit value of $M(m, n, p, q)$. For the other entries of the input x vector, we specified 400-digit values of constants listed in Theorem 1, depending on the order r.

Integer relation detection by any algorithm requires very high precision (at least $v \cdot \max_i \log_{10} |a_i|$ digits) to produce numerically reliable results, since otherwise the real relation, if any, will be lost in a sea of numerical artifacts. An effective check of numerical reliability with the multipair PSLQ algorithm is to note the dynamic range of the entries of the y vector at the iteration of detection. In the computer runs for results presented above and in Appendix 8, this dynamic range always exceeded 10^{63}, and in most cases exceeded 10^{300}. In other words, each of these relations holds to at least 63 digits (and in most cases to more than 300 digits) beyond the level required to discover the relation. However, these results should not be regarded as formally proven by these computations.

Figure 1 illustrates the process of finding a relation using the multipair PSLQ algorithm and assessing the numerical reliability of the result. This shows the base-10 logarithm of the minimum absolute value of the y vector (vertical axis), plotted against the iteration number (horizontal axis), in the multipair PSLQ computer run that the present authors employed to discover the order-10 formula

$$M(1,3,6,0) = \sum_{k=1}^{\infty} \frac{H(k)}{k^3(k+1)^6} = \frac{1}{4}\left(84\zeta(2) - 108\zeta(3) - 5\zeta(4) - 48\zeta(5) + 24\zeta(2)\zeta(3) - 9\zeta(6) + 6\zeta(3)^2 \right.$$
$$\left. -12\zeta(7) + 4\zeta(2)\zeta(5) + 4\zeta(3)\zeta(4)\right). \tag{101}$$

Note that as the algorithm proceeds, the minimum absolute value of the y vector slowly decreases, from approximately 10^{-3} to approximately 10^{-65}, but at iteration 311 abruptly drops to approximately 10^{-405}, a drop of 340 orders of magnitude. Note that since we are using 400-digit precision, 10^{-405} is effectively zero, so the algorithm terminates here with the relation $(4, -84, 108, 5, 48, -24, 9, -6, 12, -4, -4)$. In other words, formula (101) holds to roughly 340 digits beyond the precision level required to discover it. This dynamic range at the iteration of detection can thus be considered a "confidence level" of the result's numerical reliability.

5.3 Computational results

The process described above succeeded in finding relations for *each* of the cases of form (95) with orders between 3 and 11, a total of 960 cases, plus an additional 54 selected cases of order 12. See Appendix 2 (Section 8) for a complete listing of these formulas. As noted below, in order to minimize the possibility of transcription errors, *the LaTeX code for each section of results was generated automatically by a computer program from the output computer files, and this LaTeX code is included here without any alteration.*

The techniques described in this section are applicable to more general classes of Euler sums, including Euler sums of orders higher than 12 and Euler sums with more complicated polynomial denominators. However, the computational cost increases with the precision required and the number of selected right-hand-side constants. The principal challenges here are the first and fourth term of (100), namely (a) the cost of explicitly computing and summing to high precision a large number of terms of the mixed Euler sum series, and (b) the symbolic expansion and numerical evaluation of high-order derivatives of the function $\hat{G}(t)$. Perhaps further investigation into the underlying theory of Euler sums will yield computational schemes that are more efficient for large problems.

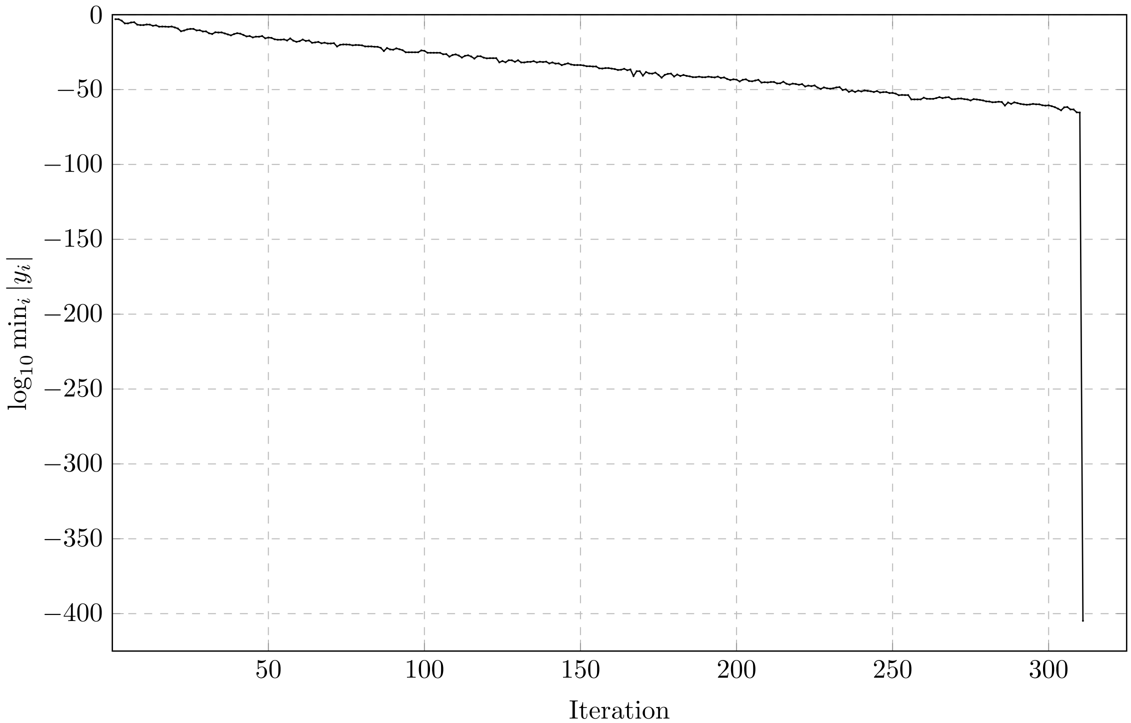

Figure 1: Plot of $\log_{10} \min_i |y_i|$ in the multipair PSLQ computer run for $M(1, 3, 6, 0)$, showing the detection of the relation at iteration 311.

6 Conclusions

We have presented techniques, both algebraic and computational, for finding analytic evaluations of a significantly larger class of Euler sums than studied previously. We believe that most of these formulas are new to the literature. Along this line, we have found that *Wolfram Mathematica* (version 14.2) can evaluate many of the basic cases, but a large majority are not evaluated by this software.

These methods appear to be applicable to even more general Euler sums. For example, by applying the methods described above, we have obtained these intriguing computational results, among others:

$$\sum_{k=1}^{\infty} \frac{H(k)}{k(2k+1)} = 2\log(2)^2 \tag{102}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^2(2k+1)} = 2\zeta(3) - 4\log(2)^2 \tag{103}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{(2k+1)^2} = \frac{1}{4}\left(7\zeta(3) - 6\log(2)\zeta(2)\right) \tag{104}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^2(2k+1)^2} = 9\zeta(3) - 6\log(2)\zeta(2) - 8\log(2)^2 \tag{105}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{(2k+1)^4} = \frac{1}{16}\left(62\zeta(5) - 21\zeta(2)\zeta(3) - 30\log(2)\zeta(4)\right) \tag{106}$$

Note the appearance of log(2) in these formulas. Each of these formulas holds to nearly 400-digit accuracy (approximately 350 digits beyond the level required to discover them), but at present we do not yet know how they can be rigorously proven.

The new results presented in this study also highlight the benefits of attempting to solve for all Euler sums of a given order. The results given for sums of mixed type may be of use in indicating the zeta function values likely to arise in attempts to numerically solve for recalcitrant sums like those for order eight and higher. It is hoped that the asymptotic form inferred for the constants γ_p^H can be deduced rigorously, as it may well prove useful in other applications of high-order Euler sums.

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7 Appendix 1: 400-digit values of three key constants

We present here 400-digit approximations of $M(2,6)$, $M(2,8)$, $M(3,8)$, $M(2,10)$, $M(4,8)$:

1.041413395855265060833934370636480151499859280096830090748511645153773087302971
78483751544719684852509976852158376374740737268847953695380222383595172532123654
63963612795034976112760332996361625685218808108323018034356756036322549570832977
08604139265652530043836463078378465035583569011375448218307043216126923803712749
23988797094981204968396475470138806138535478550733612009250215922048841374239723
645442685850

1.009386471889869832518544227219279156409372942639652641202049549364385367847079
49180863769095027121905627225975982985135460410529740749826141104503536876835470
18469301862442802589875242849768895787689895958104331283788277223328457927340866
40158920385626435450329285165922784555461987108701748322359094830741802548831985
88668354450902612335818964472409228594433865464246509588184931824643738162119198
662316661058

1.014305290895216264339827024366251554326370696089068947073583456867667637693523
94671117513355550815252027023563642862142136424802329417381850641867359561102369
07708608852232885420834448581394420559852401108798464519014241848466439384418357
64122407964525143823389069592803884034573487533288088530610292952331243167418134
67198428033583320677784291408584463648948157254030603048103031772735772545074505
256977622009

1.002258993186511461546882204200782204716716526446955625961726703382258341612187
43078937691748374313286427718378216976785972900132909275418926271525587620211343
23828486038109593077991669275749305259201098402321866062725818826804223344328667
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11238941907596547797317455957463173904843228565114429853394147887119147441919740
167418480233

1.021889991239632409955119439812528407213142628943801388819377166083386890422403
37811800890914656945480609918216413770587578233999208809687116705691954898386671
55265235208787528310809835281206739252549491701207237871226480178257316430518840
84028017686753324378706573579491021902617968550914371718501262794713100873318573
13123823952255996488552702615920587505801228078818063210192756692513776767867839
561290224768

8 Appendix 2: Formulas for orders 3 through 12

We present here the full set of results for $M(m,n,p,q)$ for orders 3 through 11, a total of 960 cases, plus 54 additional selected cases of order 12. The algorithms employed by our computer programs to generate these formulas are described above in Section 5. Each of these formulas holds to at least 380-digit precision, which is at least 63 digits (and in most cases more than 300 digits) beyond the level required to discover the relation. However, these formulas should not be regarded as formally proven solely by these computations.

To minimize the possibility of transcription errors, in each section below the formulas were produced by a computer program that parses the computer run output files, extracts the formulas, sorts them lexiographically and then generates LaTeX code (including all spacing, line breaks and page breaks). We have included this LaTeX code here without any alteration.

Formulas for order $r = m + n + p + q = 3$:

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^2} = 2\zeta(3) \tag{107}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k(k+1)} = \zeta(2) \tag{108}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{(k+1)^2} = \zeta(3) \tag{109}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k(k+2)} = \frac{1}{2}\left(1 + \zeta(2)\right) \tag{110}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{(k+1)(k+2)} = 1 \tag{111}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{(k+2)^2} = 2 - \zeta(2) - \zeta(3) \tag{112}$$

Formulas for order $r = m + n + p + q = 4$:

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^3} = \frac{1}{4}\left(5\zeta(4)\right) \tag{113}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^2(k+1)} = -\zeta(2) + 2\zeta(3) \tag{114}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k(k+1)^2} = \zeta(2) - \zeta(3) \tag{115}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{(k+1)^3} = \frac{1}{4}\left(\zeta(4)\right) \tag{116}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^2(k+2)} = \frac{1}{4}\left(-1 - \zeta(2) + 4\zeta(3)\right) \tag{117}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k(k+1)(k+2)} = \frac{1}{2}\left(1 - \zeta(2)\right) \tag{118}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{(k+1)^2(k+2)} = -1 + \zeta(3) \tag{119}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k(k+2)^2} = \frac{1}{4}\left(5 - \zeta(2) - 2\zeta(3)\right) \tag{120}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{(k+1)(k+2)^2} = 3 - \zeta(2) - \zeta(3) \tag{121}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{(k+2)^3} = \frac{1}{4}\left(-12 + 4\zeta(2) + 4\zeta(3) + \zeta(4)\right) \tag{122}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k^2} = \frac{1}{4}\left(17\zeta(4)\right) \tag{123}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k(k+1)} = 3\zeta(3) \tag{124}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{(k+1)^2} = \frac{1}{4}\left(11\zeta(4)\right) \tag{125}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k(k+2)} = \frac{1}{2}\left(1 + \zeta(2) + 3\zeta(3)\right) \tag{126}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{(k+1)(k+2)} = 1 + \zeta(2) \tag{127}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{(k+2)^2} = \frac{1}{4}\left(-12 + 8\zeta(3) + 11\zeta(4)\right) \tag{128}$$

Formulas for order $r = m + n + p + q = 5$:

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^4} = 3\zeta(5) - \zeta(2)\zeta(3) \tag{129}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^3(k+1)} = \frac{1}{4}\left(4\zeta(2) - 8\zeta(3) + 5\zeta(4)\right) \tag{130}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^2(k+1)^2} = 2\zeta(2) - 3\zeta(3) \tag{131}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k(k+1)^3} = \frac{1}{4}\left(4\zeta(2) - 4\zeta(3) - \zeta(4)\right) \tag{132}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{(k+1)^4} = 2\zeta(5) - \zeta(2)\zeta(3) \tag{133}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^3(k+2)} = \frac{1}{8}\left(1 + \zeta(2) - 4\zeta(3) + 5\zeta(4)\right) \tag{134}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^2(k+1)(k+2)} = \frac{1}{4}\left(1 - 3\zeta(2) + 4\zeta(3)\right) \tag{135}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k(k+1)^2(k+2)} = \frac{1}{2}\left(1 + \zeta(2) - 2\zeta(3)\right) \tag{136}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{(k+1)^3(k+2)} = \frac{1}{4}\left(4 - 4\zeta(3) + \zeta(4)\right) \tag{137}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^2(k+2)^2} = \frac{1}{4}\left(3 - 3\zeta(3)\right) \tag{138}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k(k+1)(k+2)^2} = \frac{1}{4}\left(7 - 3\zeta(2) - 2\zeta(3)\right) \tag{139}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{(k+1)^2(k+2)^2} = 4 - \zeta(2) - 2\zeta(3) \tag{140}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k(k+2)^3} = \frac{1}{8}\left(17 - 5\zeta(2) - 6\zeta(3) - \zeta(4)\right) \tag{141}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{(k+1)(k+2)^3} = \frac{1}{4}\left(24 - 8\zeta(2) - 8\zeta(3) - \zeta(4)\right) \tag{142}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{(k+2)^4} = -4 + \zeta(2) + \zeta(3) + \zeta(4) + 2\zeta(5) - \zeta(2)\zeta(3) \tag{143}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k^3} = \frac{1}{2}\left(7\zeta(5) - 2\zeta(2)\zeta(3)\right) \tag{144}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k^2(k+1)} = \frac{1}{4}\left(12\zeta(3) - 17\zeta(4)\right) \tag{145}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k(k+1)^2} = \frac{1}{4}\left(12\zeta(3) - 11\zeta(4)\right) \tag{146}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{(k+1)^3} = \frac{1}{2}\left(-3\zeta(5) + 2\zeta(2)\zeta(3)\right) \tag{147}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k^2(k+2)} = \frac{1}{8}\left(-2 - 2\zeta(2) - 6\zeta(3) + 17\zeta(4)\right) \tag{148}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k(k+1)(k+2)} = \frac{1}{2}\left(1 + \zeta(2) - 3\zeta(3)\right) \tag{149}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{(k+1)^2(k+2)} = \frac{1}{4}\left(-4 - 4\zeta(2) + 11\zeta(4)\right) \tag{150}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k(k+2)^2} = \frac{1}{8}\left(14 + 2\zeta(2) - 2\zeta(3) - 11\zeta(4)\right) \tag{151}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{(k+1)(k+2)^2} = \frac{1}{4}\left(16 + 4\zeta(2) - 8\zeta(3) - 11\zeta(4)\right) \tag{152}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{(k+2)^3} = \frac{1}{2}\left(12 - 2\zeta(2) - 6\zeta(3) - \zeta(4) + 3\zeta(5) - 2\zeta(2)\zeta(3)\right) \tag{153}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^3}{k^2} = 10\zeta(5) + \zeta(2)\zeta(3) \tag{154}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^3}{k(k+1)} = 10\zeta(4) \tag{155}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^3}{(k+1)^2} = \frac{1}{2}\left(15\zeta(5) + 2\zeta(2)\zeta(3)\right) \tag{156}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^3}{k(k+2)} = \frac{1}{2}\left(1 + 2\zeta(2) + 4\zeta(3) + 10\zeta(4)\right) \tag{157}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^3}{(k+1)(k+2)} = 1 + 2\zeta(2) + 4\zeta(3) \tag{158}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^3}{(k+2)^2} = \frac{1}{4}\left(16 + 12\zeta(2) + 4\zeta(3) - 33\zeta(4) - 30\zeta(5) - 4\zeta(2)\zeta(3)\right) \tag{159}$$

Formulas for order $r = m + n + p + q = 6$:

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^5} = \frac{1}{4}\left(7\zeta(6) - 2\zeta(3)^2\right) \tag{160}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^4(k+1)} = \frac{1}{4}\left(4\zeta(2) - 8\zeta(3) + 5\zeta(4) - 12\zeta(5) + 4\zeta(2)\zeta(3)\right) \tag{161}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^3(k+1)^2} = \frac{1}{4}\left(12\zeta(2) - 20\zeta(3) + 5\zeta(4)\right) \tag{162}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^2(k+1)^3} = \frac{1}{4}\left(-12\zeta(2) + 16\zeta(3) + \zeta(4)\right) \tag{163}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k(k+1)^4} = \frac{1}{4}\left(4\zeta(2) - 4\zeta(3) - \zeta(4) - 8\zeta(5) + 4\zeta(2)\zeta(3)\right) \tag{164}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{(k+1)^5} = \frac{1}{4}\left(3\zeta(6) - 2\zeta(3)^2\right) \tag{165}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^4(k+2)} = \frac{1}{16}\left(1 + \zeta(2) - 4\zeta(3) + 5\zeta(4) - 24\zeta(5) + 8\zeta(2)\zeta(3)\right) \tag{166}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^3(k+1)(k+2)} = \frac{1}{8}\left(-1 + 7\zeta(2) - 12\zeta(3) + 5\zeta(4)\right) \tag{167}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^2(k+1)^2(k+2)} = \frac{1}{4}\left(-1 - 5\zeta(2) + 8\zeta(3)\right) \tag{168}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k(k+1)^3(k+2)} = \frac{1}{4}\left(-2 + 2\zeta(2) - \zeta(4)\right) \tag{169}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{(k+1)^4(k+2)} = \frac{1}{4}\left(4 - 4\zeta(3) + \zeta(4) - 8\zeta(5) + 4\zeta(2)\zeta(3)\right) \tag{170}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^3(k+2)^2} = \frac{1}{16}\left(7 + \zeta(2) - 10\zeta(3) + 5\zeta(4)\right) \tag{171}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^2(k+1)(k+2)^2} = \frac{1}{4}\left(4 - 3\zeta(2) + \zeta(3)\right) \tag{172}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k(k+1)^2(k+2)^2} = \frac{1}{4}\left(9 - \zeta(2) - 6\zeta(3)\right) \tag{173}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{(k+1)^3(k+2)^2} = \frac{1}{4}\left(20 - 4\zeta(2) - 12\zeta(3) + \zeta(4)\right) \tag{174}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^2(k+2)^3} = \frac{1}{16}\left(-23 + 5\zeta(2) + 12\zeta(3) + \zeta(4)\right) \tag{175}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k(k+1)(k+2)^3} = \frac{1}{8}\left(-31 + 11\zeta(2) + 10\zeta(3) + \zeta(4)\right) \tag{176}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{(k+1)^2(k+2)^3} = \frac{1}{4}\left(-40 + 12\zeta(2) + 16\zeta(3) + \zeta(4)\right) \tag{177}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k(k+2)^4} = \frac{1}{16}\left(49 - 13\zeta(2) - 14\zeta(3) - 9\zeta(4) - 16\zeta(5) + 8\zeta(2)\zeta(3)\right) \tag{178}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{(k+1)(k+2)^4} = \frac{1}{4}\left(40 - 12\zeta(2) - 12\zeta(3) - 5\zeta(4) - 8\zeta(5) + 4\zeta(2)\zeta(3)\right) \tag{179}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{(k+2)^5} = \frac{1}{4}\left(-20 + 4\zeta(2) + 4\zeta(3) + 4\zeta(4) + 4\zeta(5) + 3\zeta(6) - 2\zeta(3)^2\right) \tag{180}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k^4} = \frac{1}{24}\left(97\zeta(6) - 48\zeta(3)^2\right) \tag{181}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k^3(k+1)} = \frac{1}{4}\left(12\zeta(3) - 17\zeta(4) + 14\zeta(5) - 4\zeta(2)\zeta(3)\right) \tag{182}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k^2(k+1)^2} = 6\zeta(3) - 7\zeta(4) \tag{183}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k(k+1)^3} = \frac{1}{4}\left(12\zeta(3) - 11\zeta(4) + 6\zeta(5) - 4\zeta(2)\zeta(3)\right) \tag{184}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{(k+1)^4} = \frac{1}{24}\left(37\zeta(6) - 24\zeta(3)^2\right) \tag{185}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k^3(k+2)} = \frac{1}{16}\left(2 + 2\zeta(2) + 6\zeta(3) - 17\zeta(4) + 28\zeta(5) - 8\zeta(2)\zeta(3)\right) \tag{186}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k^2(k+1)(k+2)} = \frac{1}{8}\left(2 + 2\zeta(2) - 18\zeta(3) + 17\zeta(4)\right) \tag{187}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k(k+1)^2(k+2)} = \frac{1}{4}\left(2 + 2\zeta(2) + 6\zeta(3) - 11\zeta(4)\right) \tag{188}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{(k+1)^3(k+2)} = \frac{1}{4}\left(4 + 4\zeta(2) - 11\zeta(4) - 6\zeta(5) + 4\zeta(2)\zeta(3)\right) \tag{189}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k^2(k+2)^2} = \frac{1}{4}\left(4 + \zeta(2) + \zeta(3) - 7\zeta(4)\right) \tag{190}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k(k+1)(k+2)^2} = \frac{1}{8}\left(18 + 6\zeta(2) - 14\zeta(3) - 11\zeta(4)\right) \tag{191}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{(k+1)^2(k+2)^2} = \frac{1}{2}\left(-10 - 4\zeta(2) + 4\zeta(3) + 11\zeta(4)\right) \tag{192}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k(k+2)^3} = \frac{1}{16}\left(62 - 6\zeta(2) - 26\zeta(3) - 15\zeta(4) + 12\zeta(5) - 8\zeta(2)\zeta(3)\right) \tag{193}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{(k+1)(k+2)^3} = \frac{1}{4}\left(40 - 20\zeta(3) - 13\zeta(4) + 6\zeta(5) - 4\zeta(2)\zeta(3)\right) \tag{194}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{(k+2)^4} = \frac{1}{24}\left(240 - 48\zeta(2) - 96\zeta(3) - 36\zeta(4) - 96\zeta(5) + 48\zeta(2)\zeta(3) - 37\zeta(6) + 24\zeta(3)^2\right) \tag{195}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^3}{k^3} = \frac{1}{16}\left(93\zeta(6) - 40\zeta(3)^2\right) \tag{196}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^3}{k^2(k+1)} = 10\zeta(4) - 10\zeta(5) - \zeta(2)\zeta(3) \tag{197}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^3}{k(k+1)^2} = \frac{1}{2}\left(20\zeta(4) - 15\zeta(5) - 2\zeta(2)\zeta(3)\right) \tag{198}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^3}{(k+1)^3} = \frac{1}{16}\left(-33\zeta(6) + 32\zeta(3)^2\right) \tag{199}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^3}{k^2(k+2)} = \frac{1}{4}\left(1 + 2\zeta(2) + 4\zeta(3) + 10\zeta(4) - 20\zeta(5) - 2\zeta(2)\zeta(3)\right) \tag{200}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^3}{k(k+1)(k+2)} = \frac{1}{2}\left(1 + 2\zeta(2) + 4\zeta(3) - 10\zeta(4)\right) \tag{201}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^3}{(k+1)^2(k+2)} = \frac{1}{2}\left(2 + 4\zeta(2) + 8\zeta(3) - 15\zeta(5) - 2\zeta(2)\zeta(3)\right) \tag{202}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^3}{k(k+2)^2} = \frac{1}{8}\left(18 + 16\zeta(2) + 12\zeta(3) - 13\zeta(4) - 30\zeta(5) - 4\zeta(2)\zeta(3)\right) \tag{203}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^3}{(k+1)(k+2)^2} = \frac{1}{4}\left(20 + 20\zeta(2) + 20\zeta(3) - 33\zeta(4) - 30\zeta(5) - 4\zeta(2)\zeta(3)\right) \tag{204}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^3}{(k+2)^3} = \frac{1}{16}\left(160 + 48\zeta(2) - 48\zeta(3) - 144\zeta(4) + 72\zeta(5) - 48\zeta(2)\zeta(3) + 33\zeta(6) - 32\zeta(3)^2\right) \tag{205}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^4}{k^2} = \frac{1}{24}\left(979\zeta(6) + 72\zeta(3)^2\right) \tag{206}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^4}{k(k+1)} = 30\zeta(5) + 6\zeta(2)\zeta(3) \tag{207}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^4}{(k+1)^2} = \frac{1}{24}\left(859\zeta(6) + 72\zeta(3)^2\right) \tag{208}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^4}{k(k+2)} = \frac{1}{4}\left(2 + 6\zeta(2) + 22\zeta(3) + 37\zeta(4) + 60\zeta(5) + 12\zeta(2)\zeta(3)\right) \tag{209}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^4}{(k+1)(k+2)} = \frac{1}{2}\left(2 + 6\zeta(2) + 22\zeta(3) + 37\zeta(4)\right) \tag{210}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^4}{(k+2)^2} = \frac{1}{24}\left(-120 - 192\zeta(2) - 432\zeta(3) - 48\zeta(4) + 720\zeta(5) + 96\zeta(2)\zeta(3)\right.$$
$$\left.+859\zeta(6) + 72\zeta(3)^2\right) \tag{211}$$

Formulas for order $r = m + n + p + q = 7$:

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^6} = 4\zeta(7) - \zeta(2)\zeta(5) - \zeta(3)\zeta(4) \tag{212}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^5(k+1)} = \frac{1}{4}\left(4\zeta(2) - 8\zeta(3) + 5\zeta(4) - 12\zeta(5) + 4\zeta(2)\zeta(3) + 7\zeta(6) - 2\zeta(3)^2\right) \tag{213}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^4(k+1)^2} = \frac{1}{2}\left(-8\zeta(2) + 14\zeta(3) - 5\zeta(4) + 6\zeta(5) - 2\zeta(2)\zeta(3)\right) \tag{214}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^3(k+1)^3} = 6\zeta(2) - 9\zeta(3) + \zeta(4) \tag{215}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^2(k+1)^4} = \frac{1}{2}\left(8\zeta(2) - 10\zeta(3) - \zeta(4) - 4\zeta(5) + 2\zeta(2)\zeta(3)\right) \tag{216}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k(k+1)^5} = \frac{1}{4}\left(4\zeta(2) - 4\zeta(3) - \zeta(4) - 8\zeta(5) + 4\zeta(2)\zeta(3) - 3\zeta(6) + 2\zeta(3)^2\right) \tag{217}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{(k+1)^6} = 3\zeta(7) - \zeta(2)\zeta(5) - \zeta(3)\zeta(4) \tag{218}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^5(k+2)} = \frac{1}{32}\left(1 + \zeta(2) - 4\zeta(3) + 5\zeta(4) - 24\zeta(5) + 8\zeta(2)\zeta(3) + 28\zeta(6) - 8\zeta(3)^2\right) \tag{219}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^4(k+1)(k+2)} = \frac{1}{16}\left(1 - 15\zeta(2) + 28\zeta(3) - 15\zeta(4) + 24\zeta(5) - 8\zeta(2)\zeta(3)\right) \tag{220}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^3(k+1)^2(k+2)} = \frac{1}{8}\left(1 + 17\zeta(2) - 28\zeta(3) + 5\zeta(4)\right) \tag{221}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^2(k+1)^3(k+2)} = \frac{1}{4}\left(1 - 7\zeta(2) + 8\zeta(3) + \zeta(4)\right) \tag{222}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k(k+1)^4(k+2)} = \frac{1}{2}\left(1 + \zeta(2) - 2\zeta(3) - 4\zeta(5) + 2\zeta(2)\zeta(3)\right) \tag{223}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{(k+1)^5(k+2)} = \frac{1}{4}\left(4 - 4\zeta(3) + \zeta(4) - 8\zeta(5) + 4\zeta(2)\zeta(3) + 3\zeta(6) - 2\zeta(3)^2\right) \tag{224}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^4(k+2)^2} = \frac{1}{16}\left(-4 - \zeta(2) + 7\zeta(3) - 5\zeta(4) + 12\zeta(5) - 4\zeta(2)\zeta(3)\right) \tag{225}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^3(k+1)(k+2)^2} = \frac{1}{16}\left(9 - 13\zeta(2) + 14\zeta(3) - 5\zeta(4)\right) \tag{226}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^2(k+1)^2(k+2)^2} = \frac{1}{4}\left(-5 - 2\zeta(2) + 7\zeta(3)\right) \tag{227}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k(k+1)^3(k+2)^2} = \frac{1}{4}\left(11 - 3\zeta(2) - 6\zeta(3) + \zeta(4)\right) \tag{228}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{(k+1)^4(k+2)^2} = \frac{1}{2}\left(12 - 2\zeta(2) - 8\zeta(3) + \zeta(4) - 4\zeta(5) + 2\zeta(2)\zeta(3)\right) \tag{229}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^3(k+2)^3} = \frac{1}{16}\left(15 - 2\zeta(2) - 11\zeta(3) + 2\zeta(4)\right) \tag{230}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^2(k+1)(k+2)^3} = \frac{1}{16}\left(39 - 17\zeta(2) - 8\zeta(3) - \zeta(4)\right) \tag{231}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k(k+1)^2(k+2)^3} = \frac{1}{8}\left(49 - 13\zeta(2) - 22\zeta(3) - \zeta(4)\right) \tag{232}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{(k+1)^3(k+2)^3} = 15 - 4\zeta(2) - 7\zeta(3) \tag{233}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^2(k+2)^4} = \frac{1}{16}\left(36 - 9\zeta(2) - 13\zeta(3) - 5\zeta(4) - 8\zeta(5) + 4\zeta(2)\zeta(3)\right) \tag{234}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k(k+1)(k+2)^4} = \frac{1}{16}\left(111 - 35\zeta(2) - 34\zeta(3) - 11\zeta(4) - 16\zeta(5) + 8\zeta(2)\zeta(3)\right) \tag{235}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{(k+1)^2(k+2)^4} = \frac{1}{2}\left(-40 + 12\zeta(2) + 14\zeta(3) + 3\zeta(4) + 4\zeta(5) - 2\zeta(2)\zeta(3)\right) \tag{236}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k(k+2)^5} = \frac{1}{32}\left(129 - 29\zeta(2) - 30\zeta(3) - 25\zeta(4) - 32\zeta(5) + 8\zeta(2)\zeta(3) - 12\zeta(6) + 8\zeta(3)^2\right) \tag{237}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{(k+1)(k+2)^5} = \frac{1}{4}\left(60 - 16\zeta(2) - 16\zeta(3) - 9\zeta(4) - 12\zeta(5) + 4\zeta(2)\zeta(3) - 3\zeta(6) + 2\zeta(3)^2\right) \tag{238}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{(k+2)^6} = 6 - \zeta(2) - \zeta(3) - \zeta(4) - \zeta(5) - \zeta(6) - 3\zeta(7) + \zeta(2)\zeta(5) + \zeta(3)\zeta(4) \tag{239}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k^5} = \frac{1}{2}\left(12\zeta(7) - 2\zeta(2)\zeta(5) - 5\zeta(3)\zeta(4)\right) \tag{240}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k^4(k+1)} = \frac{1}{24}\left(-72\zeta(3) + 102\zeta(4) - 84\zeta(5) + 24\zeta(2)\zeta(3) + 97\zeta(6) - 48\zeta(3)^2\right) \tag{241}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k^3(k+1)^2} = \frac{1}{4}\left(36\zeta(3) - 45\zeta(4) + 14\zeta(5) - 4\zeta(2)\zeta(3)\right) \tag{242}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k^2(k+1)^3} = \frac{1}{4}\left(-36\zeta(3) + 39\zeta(4) - 6\zeta(5) + 4\zeta(2)\zeta(3)\right) \tag{243}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k(k+1)^4} = \frac{1}{24}\left(72\zeta(3) - 66\zeta(4) + 36\zeta(5) - 24\zeta(2)\zeta(3) - 37\zeta(6) \right.$$
$$\left. +24\zeta(3)^2\right) \tag{244}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{(k+1)^5} = \frac{1}{2}\left(2\zeta(7) - 2\zeta(2)\zeta(5) + \zeta(3)\zeta(4)\right) \tag{245}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k^4(k+2)} = \frac{1}{96}\left(6 + 6\zeta(2) + 18\zeta(3) - 51\zeta(4) + 84\zeta(5) - 24\zeta(2)\zeta(3) - 194\zeta(6) \right.$$
$$\left. +96\zeta(3)^2\right) \tag{246}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k^3(k+1)(k+2)} = \frac{1}{16}\left(-2 - 2\zeta(2) + 42\zeta(3) - 51\zeta(4) + 28\zeta(5) - 8\zeta(2)\zeta(3)\right) \tag{247}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k^2(k+1)^2(k+2)} = \frac{1}{8}\left(-2 - 2\zeta(2) - 30\zeta(3) + 39\zeta(4)\right) \tag{248}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k(k+1)^3(k+2)} = \frac{1}{2}\left(-1 - \zeta(2) + 3\zeta(3) + 3\zeta(5) - 2\zeta(2)\zeta(3)\right) \tag{249}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{(k+1)^4(k+2)} = \frac{1}{24}\left(-24 - 24\zeta(2) + 66\zeta(4) + 36\zeta(5) - 24\zeta(2)\zeta(3) + 37\zeta(6) \right.$$
$$\left. -24\zeta(3)^2\right) \tag{250}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k^3(k+2)^2} = \frac{1}{32}\left(18 + 6\zeta(2) + 10\zeta(3) - 45\zeta(4) + 28\zeta(5) - 8\zeta(2)\zeta(3)\right) \tag{251}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k^2(k+1)(k+2)^2} = \frac{1}{8}\left(10 + 4\zeta(2) - 16\zeta(3) + 3\zeta(4)\right) \tag{252}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k(k+1)^2(k+2)^2} = \frac{1}{8}\left(22 + 10\zeta(2) - 2\zeta(3) - 33\zeta(4)\right) \tag{253}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{(k+1)^3(k+2)^2} = \frac{1}{4}\left(24 + 12\zeta(2) - 8\zeta(3) - 33\zeta(4) - 6\zeta(5) + 4\zeta(2)\zeta(3)\right) \tag{254}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k^2(k+2)^3} = \frac{1}{32}\left(-78 + 2\zeta(2) + 22\zeta(3) + 43\zeta(4) - 12\zeta(5) + 8\zeta(2)\zeta(3)\right) \tag{255}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k(k+1)(k+2)^3} = \frac{1}{16}\left(98 + 6\zeta(2) - 54\zeta(3) - 37\zeta(4) + 12\zeta(5) - 8\zeta(2)\zeta(3)\right) \tag{256}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{(k+1)^2(k+2)^3} = \frac{1}{4}\left(60 + 8\zeta(2) - 28\zeta(3) - 35\zeta(4) + 6\zeta(5) - 4\zeta(2)\zeta(3)\right) \tag{257}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k(k+2)^4} = \frac{1}{96}\big(666 - 114\zeta(2) - 270\zeta(3) - 117\zeta(4) - 156\zeta(5) + 72\zeta(2)\zeta(3) -74\zeta(6) + 48\zeta(3)^2\big) \tag{258}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{(k+1)(k+2)^4} = \frac{1}{24}\big(480 - 48\zeta(2) - 216\zeta(3) - 114\zeta(4) - 60\zeta(5) + 24\zeta(2)\zeta(3) -37\zeta(6) + 24\zeta(3)^2\big) \tag{259}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{(k+2)^5} = \frac{1}{2}\big(30 - 6\zeta(2) - 10\zeta(3) - 5\zeta(4) - 10\zeta(5) + 4\zeta(2)\zeta(3) - 3\zeta(6) +2\zeta(3)^2 + 2\zeta(7) - 2\zeta(2)\zeta(5) + \zeta(3)\zeta(4)\big) \tag{260}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^3}{k^4} = \frac{1}{16}\left(231\zeta(7) + 32\zeta(2)\zeta(5) - 204\zeta(3)\zeta(4)\right) \tag{261}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^3}{k^3(k+1)} = \frac{1}{16}\left(160\zeta(4) - 160\zeta(5) - 16\zeta(2)\zeta(3) + 93\zeta(6) - 40\zeta(3)^2\right) \tag{262}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^3}{k^2(k+1)^2} = \frac{1}{2}\left(-40\zeta(4) + 35\zeta(5) + 4\zeta(2)\zeta(3)\right) \tag{263}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^3}{k(k+1)^3} = \frac{1}{16}\left(160\zeta(4) - 120\zeta(5) - 16\zeta(2)\zeta(3) + 33\zeta(6) - 32\zeta(3)^2\right) \tag{264}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^3}{(k+1)^4} = \frac{1}{16}\left(119\zeta(7) + 32\zeta(2)\zeta(5) - 132\zeta(3)\zeta(4)\right) \tag{265}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^3}{k^3(k+2)} = \frac{1}{32}\big(4 + 8\zeta(2) + 16\zeta(3) + 40\zeta(4) - 80\zeta(5) - 8\zeta(2)\zeta(3) + 93\zeta(6) -40\zeta(3)^2\big) \tag{266}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^3}{k^2(k+1)(k+2)} = \frac{1}{4}\left(1 + 2\zeta(2) + 4\zeta(3) - 30\zeta(4) + 20\zeta(5) + 2\zeta(2)\zeta(3)\right) \tag{267}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^3}{k(k+1)^2(k+2)} = \frac{1}{2}\left(1 + 2\zeta(2) + 4\zeta(3) + 10\zeta(4) - 15\zeta(5) - 2\zeta(2)\zeta(3)\right) \tag{268}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^3}{(k+1)^3(k+2)} = \frac{1}{16}\big(16 + 32\zeta(2) + 64\zeta(3) - 120\zeta(5) - 16\zeta(2)\zeta(3) - 33\zeta(6) +32\zeta(3)^2\big) \tag{269}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^3}{k^2(k+2)^2} = \frac{1}{16}\left(-20 - 20\zeta(2) - 20\zeta(3) - 7\zeta(4) + 70\zeta(5) + 8\zeta(2)\zeta(3)\right) \tag{270}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^3}{k(k+1)(k+2)^2} = \frac{1}{8}\left(-22 - 24\zeta(2) - 28\zeta(3) + 53\zeta(4) + 30\zeta(5) + 4\zeta(2)\zeta(3)\right) \tag{271}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^3}{(k+1)^2(k+2)^2} = \frac{1}{4}\left(24 + 28\zeta(2) + 36\zeta(3) - 33\zeta(4) - 60\zeta(5) - 8\zeta(2)\zeta(3)\right) \tag{272}$$

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{H(k)^3}{k(k+2)^3} = \frac{1}{32}\big(&196 + 80\zeta(2) - 24\zeta(3) - 170\zeta(4) + 12\zeta(5) - 56\zeta(2)\zeta(3) \\ &+33\zeta(6) - 32\zeta(3)^2\big)\end{aligned} \tag{273}$$

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{H(k)^3}{(k+1)(k+2)^3} = \frac{1}{16}\big(&240 + 128\zeta(2) + 32\zeta(3) - 276\zeta(4) - 48\zeta(5) - 64\zeta(2)\zeta(3) \\ &+33\zeta(6) - 32\zeta(3)^2\big)\end{aligned} \tag{274}$$

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{H(k)^3}{(k+2)^4} = \frac{1}{16}\big(&320 + 32\zeta(2) - 128\zeta(3) - 172\zeta(4) - 24\zeta(5) - 74\zeta(6) + 48\zeta(3)^2 \\ &-119\zeta(7) - 32\zeta(2)\zeta(5) + 132\zeta(3)\zeta(4)\big)\end{aligned} \tag{275}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^4}{k^3} = \frac{1}{8}\left(185\zeta(7) + 40\zeta(2)\zeta(5) - 172\zeta(3)\zeta(4)\right) \tag{276}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^4}{k^2(k+1)} = \frac{1}{24}\left(720\zeta(5) + 144\zeta(2)\zeta(3) - 979\zeta(6) - 72\zeta(3)^2\right) \tag{277}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^4}{k(k+1)^2} = \frac{1}{24}\left(720\zeta(5) + 144\zeta(2)\zeta(3) - 859\zeta(6) - 72\zeta(3)^2\right) \tag{278}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^4}{(k+1)^3} = \frac{1}{8}\left(109\zeta(7) + 40\zeta(2)\zeta(5) - 148\zeta(3)\zeta(4)\right) \tag{279}$$

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{H(k)^4}{k^2(k+2)} = \frac{1}{48}\big(&-12 - 36\zeta(2) - 132\zeta(3) - 222\zeta(4) - 360\zeta(5) - 72\zeta(2)\zeta(3) \\ &+979\zeta(6) + 72\zeta(3)^2\big)\end{aligned} \tag{280}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^4}{k(k+1)(k+2)} = \frac{1}{4}\left(-2 - 6\zeta(2) - 22\zeta(3) - 37\zeta(4) + 60\zeta(5) + 12\zeta(2)\zeta(3)\right) \tag{281}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^4}{(k+1)^2(k+2)} = \frac{1}{24}\left(24 + 72\zeta(2) + 264\zeta(3) + 444\zeta(4) - 859\zeta(6) - 72\zeta(3)^2\right) \tag{282}$$

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{H(k)^4}{k(k+2)^2} = \frac{1}{48}\big(&132 + 228\zeta(2) + 564\zeta(3) + 270\zeta(4) - 360\zeta(5) - 24\zeta(2)\zeta(3) \\ &-859\zeta(6) - 72\zeta(3)^2\big)\end{aligned} \tag{283}$$

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{H(k)^4}{(k+1)(k+2)^2} = \frac{1}{24}\big(&144 + 264\zeta(2) + 696\zeta(3) + 492\zeta(4) - 720\zeta(5) - 96\zeta(2)\zeta(3) \\ &-859\zeta(6) - 72\zeta(3)^2\big)\end{aligned} \tag{284}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^4}{(k+2)^3} = \frac{1}{8}\left(-120 - 112\zeta(2) - 160\zeta(3) + 124\zeta(4) + 168\zeta(5) + 80\zeta(2)\zeta(3) \right.$$
$$\left. -66\zeta(6) + 64\zeta(3)^2 - 109\zeta(7) - 40\zeta(2)\zeta(5) + 148\zeta(3)\zeta(4)\right) \quad (285)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^5}{k^2} = \frac{1}{16}\left(2051\zeta(7) + 456\zeta(2)\zeta(5) + 528\zeta(3)\zeta(4)\right) \quad (286)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^5}{k(k+1)} = \frac{1}{2}\left(357\zeta(6) + 45\zeta(3)^2\right) \quad (287)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^5}{(k+1)^2} = \frac{1}{16}\left(1855\zeta(7) + 456\zeta(2)\zeta(5) + 528\zeta(3)\zeta(4)\right) \quad (288)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^5}{k(k+2)} = \frac{1}{8}\left(4 + 16\zeta(2) + 84\zeta(3) + 251\zeta(4) + 284\zeta(5) + 60\zeta(2)\zeta(3) + 714\zeta(6) \right.$$
$$\left. +90\zeta(3)^2\right) \quad (289)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^5}{(k+1)(k+2)} = \frac{1}{4}\left(4 + 16\zeta(2) + 84\zeta(3) + 251\zeta(4) + 284\zeta(5) + 60\zeta(2)\zeta(3)\right) \quad (290)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^5}{(k+2)^2} = \frac{1}{48}\left(-288 - 720\zeta(2) - 2784\zeta(3) - 4704\zeta(4) + 192\zeta(5) - 240\zeta(2)\zeta(3) \right.$$
$$\left. +8590\zeta(6) + 720\zeta(3)^2 + 5565\zeta(7) + 1368\zeta(2)\zeta(5) + 1584\zeta(3)\zeta(4)\right) \quad (291)$$

Formulas for order $r = m + n + p + q = 8$:

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^7} = \frac{1}{4}\left(9\zeta(8) - 4\zeta(3)\zeta(5)\right) \tag{292}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^6(k+1)} = \frac{1}{4}\big(4\zeta(2) - 8\zeta(3) + 5\zeta(4) - 12\zeta(5) + 4\zeta(2)\zeta(3) + 7\zeta(6) - 2\zeta(3)^2$$
$$-16\zeta(7) + 4\zeta(2)\zeta(5) + 4\zeta(3)\zeta(4)\big) \tag{293}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^5(k+1)^2} = \frac{1}{4}\big(20\zeta(2) - 36\zeta(3) + 15\zeta(4) - 24\zeta(5) + 8\zeta(2)\zeta(3) + 7\zeta(6)$$
$$-2\zeta(3)^2\big) \tag{294}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^4(k+1)^3} = \frac{1}{2}\left(20\zeta(2) - 32\zeta(3) + 7\zeta(4) - 6\zeta(5) + 2\zeta(2)\zeta(3)\right) \tag{295}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^3(k+1)^4} = \frac{1}{2}\left(20\zeta(2) - 28\zeta(3) + \zeta(4) - 4\zeta(5) + 2\zeta(2)\zeta(3)\right) \tag{296}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^2(k+1)^5} = \frac{1}{4}\big(20\zeta(2) - 24\zeta(3) - 3\zeta(4) - 16\zeta(5) + 8\zeta(2)\zeta(3) - 3\zeta(6)$$
$$+2\zeta(3)^2\big) \tag{297}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k(k+1)^6} = \frac{1}{4}\big(4\zeta(2) - 4\zeta(3) - \zeta(4) - 8\zeta(5) + 4\zeta(2)\zeta(3) - 3\zeta(6) + 2\zeta(3)^2$$
$$-12\zeta(7) + 4\zeta(2)\zeta(5) + 4\zeta(3)\zeta(4)\big) \tag{298}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{(k+1)^7} = \frac{1}{4}\left(5\zeta(8) - 4\zeta(3)\zeta(5)\right) \tag{299}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^6(k+2)} = \frac{1}{64}\big(-1 - \zeta(2) + 4\zeta(3) - 5\zeta(4) + 24\zeta(5) - 8\zeta(2)\zeta(3) - 28\zeta(6)$$
$$+8\zeta(3)^2 + 128\zeta(7) - 32\zeta(2)\zeta(5) - 32\zeta(3)\zeta(4)\big) \tag{300}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^5(k+1)(k+2)} = \frac{1}{32}\big(-1 + 31\zeta(2) - 60\zeta(3) + 35\zeta(4) - 72\zeta(5) + 24\zeta(2)\zeta(3) + 28\zeta(6)$$
$$-8\zeta(3)^2\big) \tag{301}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^4(k+1)^2(k+2)} = \frac{1}{16}\left(-1 - 49\zeta(2) + 84\zeta(3) - 25\zeta(4) + 24\zeta(5) - 8\zeta(2)\zeta(3)\right) \tag{302}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^3(k+1)^3(k+2)} = \frac{1}{8}\left(-1 + 31\zeta(2) - 44\zeta(3) + 3\zeta(4)\right) \tag{303}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^2(k+1)^4(k+2)} = \frac{1}{4}\left(-1 - 9\zeta(2) + 12\zeta(3) + \zeta(4) + 8\zeta(5) - 4\zeta(2)\zeta(3)\right) \tag{304}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k(k+1)^5(k+2)} = \frac{1}{4}\left(2 - 2\zeta(2) + \zeta(4) + 3\zeta(6) - 2\zeta(3)^2\right) \tag{305}$$

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{H(k)}{(k+1)^6(k+2)} &= \frac{1}{4}\big(4 - 4\zeta(3) + \zeta(4) - 8\zeta(5) + 4\zeta(2)\zeta(3) + 3\zeta(6) - 2\zeta(3)^2 \\ &\quad -12\zeta(7) + 4\zeta(2)\zeta(5) + 4\zeta(3)\zeta(4)\big)\end{aligned} \tag{306}$$

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{H(k)}{k^5(k+2)^2} &= \frac{1}{64}\big(9 + 3\zeta(2) - 18\zeta(3) + 15\zeta(4) - 48\zeta(5) + 16\zeta(2)\zeta(3) + 28\zeta(6) \\ &\quad -8\zeta(3)^2\big)\end{aligned} \tag{307}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^4(k+1)(k+2)^2} = \frac{1}{16}\left(5 - 14\zeta(2) + 21\zeta(3) - 10\zeta(4) + 12\zeta(5) - 4\zeta(2)\zeta(3)\right) \tag{308}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^3(k+1)^2(k+2)^2} = \frac{1}{16}\left(11 + 21\zeta(2) - 42\zeta(3) + 5\zeta(4)\right) \tag{309}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^2(k+1)^3(k+2)^2} = \frac{1}{4}\left(6 - 5\zeta(2) + \zeta(3) + \zeta(4)\right) \tag{310}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k(k+1)^4(k+2)^2} = \frac{1}{4}\left(13 - \zeta(2) - 10\zeta(3) + \zeta(4) - 8\zeta(5) + 4\zeta(2)\zeta(3)\right) \tag{311}$$

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{H(k)}{(k+1)^5(k+2)^2} &= \frac{1}{4}\big(28 - 4\zeta(2) - 20\zeta(3) + 3\zeta(4) - 16\zeta(5) + 8\zeta(2)\zeta(3) + 3\zeta(6) \\ &\quad -2\zeta(3)^2\big)\end{aligned} \tag{312}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^4(k+2)^3} = \frac{1}{32}\left(19 - \zeta(2) - 18\zeta(3) + 7\zeta(4) - 12\zeta(5) + 4\zeta(2)\zeta(3)\right) \tag{313}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^3(k+1)(k+2)^3} = \frac{1}{16}\left(-24 + 15\zeta(2) - 3\zeta(3) + 3\zeta(4)\right) \tag{314}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^2(k+1)^2(k+2)^3} = \frac{1}{16}\left(-59 + 9\zeta(2) + 36\zeta(3) + \zeta(4)\right) \tag{315}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k(k+1)^3(k+2)^3} = \frac{1}{8}\left(-71 + 19\zeta(2) + 34\zeta(3) - \zeta(4)\right) \tag{316}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{(k+1)^4(k+2)^3} = \frac{1}{2}\left(42 - 10\zeta(2) - 22\zeta(3) + \zeta(4) - 4\zeta(5) + 2\zeta(2)\zeta(3)\right) \tag{317}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^3(k+2)^4} = \frac{1}{32}\left(51 - 11\zeta(2) - 24\zeta(3) - 3\zeta(4) - 8\zeta(5) + 4\zeta(2)\zeta(3)\right) \tag{318}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^2(k+1)(k+2)^4} = \frac{1}{16}\left(75 - 26\zeta(2) - 21\zeta(3) - 6\zeta(4) - 8\zeta(5) + 4\zeta(2)\zeta(3)\right) \tag{319}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k(k+1)^2(k+2)^4} = \frac{1}{16}\left(209 - 61\zeta(2) - 78\zeta(3) - 13\zeta(4) - 16\zeta(5) + 8\zeta(2)\zeta(3)\right) \tag{320}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{(k+1)^3(k+2)^4} = \frac{1}{2}\left(70 - 20\zeta(2) - 28\zeta(3) - 3\zeta(4) - 4\zeta(5) + 2\zeta(2)\zeta(3)\right) \tag{321}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^2(k+2)^5} = \frac{1}{64}\left(-201 + 47\zeta(2) + 56\zeta(3) + 35\zeta(4) + 48\zeta(5) - 16\zeta(2)\zeta(3) + 12\zeta(6) -8\zeta(3)^2\right) \tag{322}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k(k+1)(k+2)^5} = \frac{1}{32}\left(351 - 99\zeta(2) - 98\zeta(3) - 47\zeta(4) - 64\zeta(5) + 24\zeta(2)\zeta(3) - 12\zeta(6) +8\zeta(3)^2\right) \tag{323}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{(k+1)^2(k+2)^5} = \frac{1}{4}\left(140 - 40\zeta(2) - 44\zeta(3) - 15\zeta(4) - 20\zeta(5) + 8\zeta(2)\zeta(3) - 3\zeta(6) +2\zeta(3)^2\right) \tag{324}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k(k+2)^6} = \frac{1}{64}\left(321 - 61\zeta(2) - 62\zeta(3) - 57\zeta(4) - 64\zeta(5) + 8\zeta(2)\zeta(3) - 44\zeta(6) +8\zeta(3)^2 - 96\zeta(7) + 32\zeta(2)\zeta(5) + 32\zeta(3)\zeta(4)\right) \tag{325}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{(k+1)(k+2)^6} = \frac{1}{4}\left(84 - 20\zeta(2) - 20\zeta(3) - 13\zeta(4) - 16\zeta(5) + 4\zeta(2)\zeta(3) - 7\zeta(6) +2\zeta(3)^2 - 12\zeta(7) + 4\zeta(2)\zeta(5) + 4\zeta(3)\zeta(4)\right) \tag{326}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{(k+2)^7} = \frac{1}{4}\left(28 - 4\zeta(2) - 4\zeta(3) - 4\zeta(4) - 4\zeta(5) - 4\zeta(6) - 4\zeta(7) - 5\zeta(8) +4\zeta(3)\zeta(5)\right) \tag{327}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k^6} = M(2,6) \tag{328}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k^5(k+1)} = \frac{1}{24}\left(72\zeta(3) - 102\zeta(4) + 84\zeta(5) - 24\zeta(2)\zeta(3) - 97\zeta(6) + 48\zeta(3)^2 +144\zeta(7) - 24\zeta(2)\zeta(5) - 60\zeta(3)\zeta(4)\right) \tag{329}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k^4(k+1)^2} = \frac{1}{24}\left(-288\zeta(3) + 372\zeta(4) - 168\zeta(5) + 48\zeta(2)\zeta(3) + 97\zeta(6) -48\zeta(3)^2\right) \tag{330}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k^3(k+1)^3} = 18\zeta(3) - 21\zeta(4) + 5\zeta(5) - 2\zeta(2)\zeta(3) \tag{331}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k^2(k+1)^4} = \frac{1}{24}\left(-288\zeta(3) + 300\zeta(4) - 72\zeta(5) + 48\zeta(2)\zeta(3) + 37\zeta(6) - 24\zeta(3)^2\right) \tag{332}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k(k+1)^5} = \frac{1}{24}\left(72\zeta(3) - 66\zeta(4) + 36\zeta(5) - 24\zeta(2)\zeta(3) - 37\zeta(6) + 24\zeta(3)^2 + 24\zeta(7) - 24\zeta(2)\zeta(5) + 12\zeta(3)\zeta(4)\right) \tag{333}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{(k+1)^6} = \frac{1}{2}\left(7\zeta(8) - 4\zeta(3)\zeta(5) - 2M(2,6)\right) \tag{334}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k^5(k+2)} = \frac{1}{192}\left(6 + 6\zeta(2) + 18\zeta(3) - 51\zeta(4) + 84\zeta(5) - 24\zeta(2)\zeta(3) - 194\zeta(6) + 96\zeta(3)^2 + 576\zeta(7) - 96\zeta(2)\zeta(5) - 240\zeta(3)\zeta(4)\right) \tag{335}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k^4(k+1)(k+2)} = \frac{1}{96}\left(6 + 6\zeta(2) - 270\zeta(3) + 357\zeta(4) - 252\zeta(5) + 72\zeta(2)\zeta(3) + 194\zeta(6) - 96\zeta(3)^2\right) \tag{336}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k^3(k+1)^2(k+2)} = \frac{1}{16}\left(2 + 2\zeta(2) + 102\zeta(3) - 129\zeta(4) + 28\zeta(5) - 8\zeta(2)\zeta(3)\right) \tag{337}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k^2(k+1)^3(k+2)} = \frac{1}{8}\left(2 + 2\zeta(2) - 42\zeta(3) + 39\zeta(4) - 12\zeta(5) + 8\zeta(2)\zeta(3)\right) \tag{338}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k(k+1)^4(k+2)} = \frac{1}{24}\left(12 + 12\zeta(2) + 36\zeta(3) - 66\zeta(4) - 37\zeta(6) + 24\zeta(3)^2\right) \tag{339}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{(k+1)^5(k+2)} = \frac{1}{24}\left(24 + 24\zeta(2) - 66\zeta(4) - 36\zeta(5) + 24\zeta(2)\zeta(3) - 37\zeta(6) + 24\zeta(3)^2 - 24\zeta(7) + 24\zeta(2)\zeta(5) - 12\zeta(3)\zeta(4)\right) \tag{340}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k^4(k+2)^2} = \frac{1}{96}\left(-30 - 12\zeta(2) - 24\zeta(3) + 93\zeta(4) - 84\zeta(5) + 24\zeta(2)\zeta(3) + 97\zeta(6) - 48\zeta(3)^2\right) \tag{341}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k^3(k+1)(k+2)^2} = \frac{1}{32}\left(-22 - 10\zeta(2) + 74\zeta(3) - 57\zeta(4) + 28\zeta(5) - 8\zeta(2)\zeta(3)\right) \tag{342}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k^2(k+1)^2(k+2)^2} = \frac{1}{4}\left(-6 - 3\zeta(2) - 7\zeta(3) + 18\zeta(4)\right) \tag{343}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^2}{k(k+1)^3(k+2)^2} = \frac{1}{8}\left(26+14\zeta(2)-14\zeta(3)-33\zeta(4)-12\zeta(5)\right. \\ \left.+8\zeta(2)\zeta(3)\right) \quad (344)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^2}{(k+1)^4(k+2)^2} = \frac{1}{24}\left(-168-96\zeta(2)+48\zeta(3)+264\zeta(4)+72\zeta(5)-48\zeta(2)\zeta(3)\right. \\ \left.+37\zeta(6)-24\zeta(3)^2\right) \quad (345)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^2}{k^3(k+2)^3} = \frac{1}{16}\left(24+\zeta(2)-3\zeta(3)-22\zeta(4)+10\zeta(5)-4\zeta(2)\zeta(3)\right) \quad (346)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^2}{k^2(k+1)(k+2)^3} = \frac{1}{32}\left(118+14\zeta(2)-86\zeta(3)-31\zeta(4)+12\zeta(5)\right. \\ \left.-8\zeta(2)\zeta(3)\right) \quad (347)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^2}{k(k+1)^2(k+2)^3} = \frac{1}{16}\left(142+26\zeta(2)-58\zeta(3)-103\zeta(4)+12\zeta(5)\right. \\ \left.-8\zeta(2)\zeta(3)\right) \quad (348)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^2}{(k+1)^3(k+2)^3} = 21+5\zeta(2)-9\zeta(3)-17\zeta(4) \quad (349)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^2}{k^2(k+2)^4} = \frac{1}{96}\left(450-60\zeta(2)-168\zeta(3)-123\zeta(4)-60\zeta(5)+24\zeta(2)\zeta(3)\right. \\ \left.-37\zeta(6)+24\zeta(3)^2\right) \quad (350)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^2}{k(k+1)(k+2)^4} = \frac{1}{96}\left(1254-78\zeta(2)-594\zeta(3)-339\zeta(4)-84\zeta(5)+24\zeta(2)\zeta(3)\right. \\ \left.-74\zeta(6)+48\zeta(3)^2\right) \quad (351)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^2}{(k+1)^2(k+2)^4} = \frac{1}{24}\left(-840+384\zeta(3)+324\zeta(4)+24\zeta(5)+37\zeta(6)-24\zeta(3)^2\right) \quad (352)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^2}{k(k+2)^5} = \frac{1}{192}\left(2106-402\zeta(2)-750\zeta(3)-357\zeta(4)-636\zeta(5)+264\zeta(2)\zeta(3)\right. \\ \left.-218\zeta(6)+144\zeta(3)^2+96\zeta(7)-96\zeta(2)\zeta(5)+48\zeta(3)\zeta(4)\right) \quad (353)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^2}{(k+1)(k+2)^5} = \frac{1}{24}\left(840-120\zeta(2)-336\zeta(3)-174\zeta(4)-180\zeta(5)+72\zeta(2)\zeta(3)\right. \\ \left.-73\zeta(6)+48\zeta(3)^2+24\zeta(7)-24\zeta(2)\zeta(5)+12\zeta(3)\zeta(4)\right) \quad (354)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^2}{(k+2)^6} = \frac{1}{2}\left(42-8\zeta(2)-12\zeta(3)-7\zeta(4)-12\zeta(5)+4\zeta(2)\zeta(3)-5\zeta(6)\right. \\ +2\zeta(3)^2-12\zeta(7)+4\zeta(2)\zeta(5)+4\zeta(3)\zeta(4)+7\zeta(8)-4\zeta(3)\zeta(5) \\ \left.-2M(2,6)\right) \quad (355)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^3}{k^5} = \frac{1}{96}\left(595\zeta(8) + 120\zeta(2)\zeta(3)^2 - 576\zeta(3)\zeta(5) - 264M(2,6)\right) \tag{356}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^3}{k^4(k+1)} = \frac{1}{16}\left(-160\zeta(4) + 160\zeta(5) + 16\zeta(2)\zeta(3) - 93\zeta(6) + 40\zeta(3)^2 + 231\zeta(7)\right.$$
$$\left. +32\zeta(2)\zeta(5) - 204\zeta(3)\zeta(4)\right) \tag{357}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^3}{k^3(k+1)^2} = \frac{1}{16}\left(480\zeta(4) - 440\zeta(5) - 48\zeta(2)\zeta(3) + 93\zeta(6) - 40\zeta(3)^2\right) \tag{358}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^3}{k^2(k+1)^3} = \frac{1}{16}\left(-480\zeta(4) + 400\zeta(5) + 48\zeta(2)\zeta(3) - 33\zeta(6) + 32\zeta(3)^2\right) \tag{359}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^3}{k(k+1)^4} = \frac{1}{16}\left(160\zeta(4) - 120\zeta(5) - 16\zeta(2)\zeta(3) + 33\zeta(6) - 32\zeta(3)^2 - 119\zeta(7)\right.$$
$$\left. -32\zeta(2)\zeta(5) + 132\zeta(3)\zeta(4)\right) \tag{360}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^3}{(k+1)^5} = \frac{1}{96}\left(43\zeta(8) + 120\zeta(2)\zeta(3)^2 - 288\zeta(3)\zeta(5) + 24M(2,6)\right) \tag{361}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^3}{k^4(k+2)} = \frac{1}{64}\left(-4 - 8\zeta(2) - 16\zeta(3) - 40\zeta(4) + 80\zeta(5) + 8\zeta(2)\zeta(3) - 93\zeta(6)\right.$$
$$\left. +40\zeta(3)^2 + 462\zeta(7) + 64\zeta(2)\zeta(5) - 408\zeta(3)\zeta(4)\right) \tag{362}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^3}{k^3(k+1)(k+2)} = \frac{1}{32}\left(-4 - 8\zeta(2) - 16\zeta(3) + 280\zeta(4) - 240\zeta(5) - 24\zeta(2)\zeta(3)\right.$$
$$\left. +93\zeta(6) - 40\zeta(3)^2\right) \tag{363}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^3}{k^2(k+1)^2(k+2)} = \frac{1}{4}\left(1 + 2\zeta(2) + 4\zeta(3) + 50\zeta(4) - 50\zeta(5) - 6\zeta(2)\zeta(3)\right) \tag{364}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^3}{k(k+1)^3(k+2)} = \frac{1}{16}\left(8 + 16\zeta(2) + 32\zeta(3) - 80\zeta(4) - 33\zeta(6) + 32\zeta(3)^2\right) \tag{365}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^3}{(k+1)^4(k+2)} = \frac{1}{16}\left(-16 - 32\zeta(2) - 64\zeta(3) + 120\zeta(5) + 16\zeta(2)\zeta(3) + 33\zeta(6)\right.$$
$$\left. -32\zeta(3)^2 + 119\zeta(7) + 32\zeta(2)\zeta(5) - 132\zeta(3)\zeta(4)\right) \tag{366}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^3}{k^3(k+2)^2} = \frac{1}{64}\left(44 + 48\zeta(2) + 56\zeta(3) + 54\zeta(4) - 220\zeta(5) - 24\zeta(2)\zeta(3)\right.$$
$$\left. +93\zeta(6) - 40\zeta(3)^2\right) \tag{367}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^3}{k^2(k+1)(k+2)^2} = \frac{1}{16}\left(24 + 28\zeta(2) + 36\zeta(3) - 113\zeta(4) + 10\zeta(5)\right) \tag{368}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^3}{k(k+1)^2(k+2)^2} = \frac{1}{8} (26 + 32\zeta(2) + 44\zeta(3) - 13\zeta(4) - 90\zeta(5) \\ -12\zeta(2)\zeta(3)) \tag{369}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^3}{(k+1)^3(k+2)^2} = \frac{1}{16} (112 + 144\zeta(2) + 208\zeta(3) - 132\zeta(4) - 360\zeta(5) - 48\zeta(2)\zeta(3) \\ -33\zeta(6) + 32\zeta(3)^2\big) \tag{370}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^3}{k^2(k+2)^3} = \frac{1}{64} (236 + 120\zeta(2) + 16\zeta(3) - 156\zeta(4) - 128\zeta(5) - 72\zeta(2)\zeta(3) \\ +33\zeta(6) - 32\zeta(3)^2\big) \tag{371}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^3}{k(k+1)(k+2)^3} = \frac{1}{32} (-284 - 176\zeta(2) - 88\zeta(3) + 382\zeta(4) + 108\zeta(5) + 72\zeta(2)\zeta(3) \\ -33\zeta(6) + 32\zeta(3)^2\big) \tag{372}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^3}{(k+1)^2(k+2)^3} = \frac{1}{16} (-336 - 240\zeta(2) - 176\zeta(3) + 408\zeta(4) + 288\zeta(5) + 96\zeta(2)\zeta(3) \\ -33\zeta(6) + 32\zeta(3)^2\big) \tag{373}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^3}{k(k+2)^4} = \frac{1}{64} (836 + 144\zeta(2) - 280\zeta(3) - 514\zeta(4) - 36\zeta(5) - 56\zeta(2)\zeta(3) \\ -115\zeta(6) + 64\zeta(3)^2 - 238\zeta(7) - 64\zeta(2)\zeta(5) + 264\zeta(3)\zeta(4)\big) \tag{374}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^3}{(k+1)(k+2)^4} = \frac{1}{16} (560 + 160\zeta(2) - 96\zeta(3) - 448\zeta(4) - 72\zeta(5) - 64\zeta(2)\zeta(3) \\ -41\zeta(6) + 16\zeta(3)^2 - 119\zeta(7) - 32\zeta(2)\zeta(5) + 132\zeta(3)\zeta(4)\big) \tag{375}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^3}{(k+2)^5} = \frac{1}{96} (-3360 + 1344\zeta(3) + 1296\zeta(4) + 816\zeta(5) - 288\zeta(2)\zeta(3) + 660\zeta(6) \\ -432\zeta(3)^2 - 288\zeta(7) + 288\zeta(2)\zeta(5) - 144\zeta(3)\zeta(4) - 43\zeta(8) \\ -120\zeta(2)\zeta(3)^2 + 288\zeta(3)\zeta(5) - 24M(2,6)\big) \tag{376}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^4}{k^4} = \frac{1}{144} \big(-14833\zeta(8) - 4032\zeta(2)\zeta(3)^2 + 16704\zeta(3)\zeta(5) + 3744M(2,6)\big) \tag{377}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^4}{k^3(k+1)} = \frac{1}{24} \big(720\zeta(5) + 144\zeta(2)\zeta(3) - 979\zeta(6) - 72\zeta(3)^2 + 555\zeta(7) \\ +120\zeta(2)\zeta(5) - 516\zeta(3)\zeta(4)) \tag{378}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^4}{k^2(k+1)^2} = \frac{1}{12} \big(720\zeta(5) + 144\zeta(2)\zeta(3) - 919\zeta(6) - 72\zeta(3)^2\big) \tag{379}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^4}{k(k+1)^3} = \frac{1}{24} \big(720\zeta(5) + 144\zeta(2)\zeta(3) - 859\zeta(6) - 72\zeta(3)^2 + 327\zeta(7) + 120\zeta(2)\zeta(5) - 444\zeta(3)\zeta(4)\big) \quad (380)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^4}{(k+1)^4} = \frac{1}{144} \big(12415\zeta(8) + 3312\zeta(2)\zeta(3)^2 - 13824\zeta(3)\zeta(5) - 3024M(2,6)\big) \quad (381)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^4}{k^3(k+2)} = \frac{1}{96} \big(12 + 36\zeta(2) + 132\zeta(3) + 222\zeta(4) + 360\zeta(5) + 72\zeta(2)\zeta(3) - 979\zeta(6) - 72\zeta(3)^2 + 1110\zeta(7) + 240\zeta(2)\zeta(5) - 1032\zeta(3)\zeta(4)\big) \quad (382)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^4}{k^2(k+1)(k+2)} = \frac{1}{48} \big(12 + 36\zeta(2) + 132\zeta(3) + 222\zeta(4) - 1080\zeta(5) - 216\zeta(2)\zeta(3) + 979\zeta(6) + 72\zeta(3)^2\big) \quad (383)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^4}{k(k+1)^2(k+2)} = \frac{1}{24} \big(12 + 36\zeta(2) + 132\zeta(3) + 222\zeta(4) + 360\zeta(5) + 72\zeta(2)\zeta(3) - 859\zeta(6) - 72\zeta(3)^2\big) \quad (384)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^4}{(k+1)^3(k+2)} = \frac{1}{24} \big(24 + 72\zeta(2) + 264\zeta(3) + 444\zeta(4) - 859\zeta(6) - 72\zeta(3)^2 - 327\zeta(7) - 120\zeta(2)\zeta(5) + 444\zeta(3)\zeta(4)\big) \quad (385)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^4}{k^2(k+2)^2} = \frac{1}{48} \big(72 + 132\zeta(2) + 348\zeta(3) + 246\zeta(4) + 24\zeta(2)\zeta(3) - 919\zeta(6) - 72\zeta(3)^2\big) \quad (386)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^4}{k(k+1)(k+2)^2} = \frac{1}{48} \big(-156 - 300\zeta(2) - 828\zeta(3) - 714\zeta(4) + 1080\zeta(5) + 168\zeta(2)\zeta(3) + 859\zeta(6) + 72\zeta(3)^2\big) \quad (387)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^4}{(k+1)^2(k+2)^2} = \frac{1}{12} \big(84 + 168\zeta(2) + 480\zeta(3) + 468\zeta(4) - 360\zeta(5) - 48\zeta(2)\zeta(3) - 859\zeta(6) - 72\zeta(3)^2\big) \quad (388)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^4}{k(k+2)^3} = \frac{1}{96} \big(852 + 900\zeta(2) + 1524\zeta(3) - 474\zeta(4) - 1368\zeta(5) - 504\zeta(2)\zeta(3) - 463\zeta(6) - 456\zeta(3)^2 + 654\zeta(7) + 240\zeta(2)\zeta(5) - 888\zeta(3)\zeta(4)\big) \quad (389)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^4}{(k+1)(k+2)^3} = \frac{1}{24} \big(504 + 600\zeta(2) + 1176\zeta(3) + 120\zeta(4) - 1224\zeta(5) - 336\zeta(2)\zeta(3) - 661\zeta(6) - 264\zeta(3)^2 + 327\zeta(7) + 120\zeta(2)\zeta(5) - 444\zeta(3)\zeta(4)\big) \quad (390)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^4}{(k+2)^4} = \frac{1}{144}\big(-5040-2880\zeta(2)-2304\zeta(3)+5040\zeta(4)+2880\zeta(5)+1728\zeta(2)\zeta(3)$$
$$+144\zeta(6)+288\zeta(3)^2+4284\zeta(7)+1152\zeta(2)\zeta(5)-4752\zeta(3)\zeta(4)-12415\zeta(8)$$
$$-3312\zeta(2)\zeta(3)^2+13824\zeta(3)\zeta(5)+3024M(2,6)\big) \quad (391)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^5}{k^3} = \frac{1}{288}\big(67811\zeta(8)+19080\zeta(2)\zeta(3)^2-78768\zeta(3)\zeta(5)-16920M(2,6)\big) \quad (392)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^5}{k^2(k+1)} = \frac{1}{16}\big(-2856\zeta(6)-360\zeta(3)^2+2051\zeta(7)+456\zeta(2)\zeta(5)$$
$$+528\zeta(3)\zeta(4)\big) \quad (393)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^5}{k(k+1)^2} = \frac{1}{16}\big(2856\zeta(6)+360\zeta(3)^2-1855\zeta(7)-456\zeta(2)\zeta(5)$$
$$-528\zeta(3)\zeta(4)\big) \quad (394)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^5}{(k+1)^3} = \frac{1}{288}\big(65621\zeta(8)+17640\zeta(2)\zeta(3)^2-72432\zeta(3)\zeta(5)-15480M(2,6)\big) \quad (395)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^5}{k^2(k+2)} = \frac{1}{32}\big(-8-32\zeta(2)-168\zeta(3)-502\zeta(4)-568\zeta(5)-120\zeta(2)\zeta(3)$$
$$-1428\zeta(6)-180\zeta(3)^2+2051\zeta(7)+456\zeta(2)\zeta(5)+528\zeta(3)\zeta(4)\big) \quad (396)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^5}{k(k+1)(k+2)} = \frac{1}{8}\big(4+16\zeta(2)+84\zeta(3)+251\zeta(4)+284\zeta(5)+60\zeta(2)\zeta(3)-714\zeta(6)$$
$$-90\zeta(3)^2\big) \quad (397)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^5}{(k+1)^2(k+2)} = \frac{1}{16}\big(16+64\zeta(2)+336\zeta(3)+1004\zeta(4)+1136\zeta(5)+240\zeta(2)\zeta(3)$$
$$-1855\zeta(7)-456\zeta(2)\zeta(5)-528\zeta(3)\zeta(4)\big) \quad (398)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^5}{k(k+2)^2} = \frac{1}{96}\big(312+816\zeta(2)+3288\zeta(3)+6210\zeta(4)+1512\zeta(5)+600\zeta(2)\zeta(3)$$
$$-4306\zeta(6)-180\zeta(3)^2-5565\zeta(7)-1368\zeta(2)\zeta(5)-1584\zeta(3)\zeta(4)\big) \quad (399)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^5}{(k+1)(k+2)^2} = \frac{1}{48}\big(336+912\zeta(2)+3792\zeta(3)+7716\zeta(4)+3216\zeta(5)+960\zeta(2)\zeta(3)$$
$$-8590\zeta(6)-720\zeta(3)^2-5565\zeta(7)-1368\zeta(2)\zeta(5)-1584\zeta(3)\zeta(4)\big) \quad (400)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^5}{(k+2)^3} = \frac{1}{288}\big(-6048-10080\zeta(2)-30240\zeta(3)-30096\zeta(4)+18432\zeta(5)$$
$$+4320\zeta(2)\zeta(3)+45600\zeta(6)+10080\zeta(3)^2-19620\zeta(7)-7200\zeta(2)\zeta(5)$$
$$+26640\zeta(3)\zeta(4)+65621\zeta(8)+17640\zeta(2)\zeta(3)^2-72432\zeta(3)\zeta(5)$$
$$-15480M(2,6)\big) \quad (401)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^6}{k^2} = \frac{1}{8}\left(5843\zeta(8) - 328\zeta(2)\zeta(3)^2 + 3896\zeta(3)\zeta(5) + 456M(2,6)\right) \tag{402}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^6}{k(k+1)} = 644\zeta(7) + 145\zeta(2)\zeta(5) + 297\zeta(3)\zeta(4) \tag{403}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^6}{(k+1)^2} = \frac{1}{24}\left(17027\zeta(8) - 924\zeta(2)\zeta(3)^2 + 11328\zeta(3)\zeta(5) + 1308M(2,6)\right) \tag{404}$$

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{H(k)^6}{k(k+2)} &= \frac{1}{8}\left(4 + 20\zeta(2) + 136\zeta(3) + 571\zeta(4) + 1142\zeta(5) + 244\zeta(2)\zeta(3) + 2097\zeta(6)\right. \\ &\quad \left. +268\zeta(3)^2 + 2576\zeta(7) + 580\zeta(2)\zeta(5) + 1188\zeta(3)\zeta(4)\right)\end{aligned} \tag{405}$$

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{H(k)^6}{(k+1)(k+2)} &= \frac{1}{4}\left(4 + 20\zeta(2) + 136\zeta(3) + 571\zeta(4) + 1142\zeta(5) + 244\zeta(2)\zeta(3)\right. \\ &\quad \left. +2097\zeta(6) + 268\zeta(3)^2\right)\end{aligned} \tag{406}$$

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{H(k)^6}{(k+2)^2} &= \frac{1}{24}\left(-168 - 576\zeta(2) - 3120\zeta(3) - 9288\zeta(4) - 10104\zeta(5) - 2448\zeta(2)\zeta(3)\right. \\ &\quad +303\zeta(6) - 528\zeta(3)^2 + 16695\zeta(7) + 4104\zeta(2)\zeta(5) + 4752\zeta(3)\zeta(4) + 17027\zeta(8) \\ &\quad \left. -924\zeta(2)\zeta(3)^2 + 11328\zeta(3)\zeta(5) + 1308M(2,6)\right)\end{aligned} \tag{407}$$

Formulas for order $r = m + n + p + q = 9$:

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^8} = 5\zeta(9) - \zeta(3)\zeta(6) - \zeta(4)\zeta(5) - \zeta(2)\zeta(7) \tag{408}$$

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{H(k)}{k^7(k+1)} &= \frac{1}{4}\big(4\zeta(2) - 8\zeta(3) + 5\zeta(4) - 12\zeta(5) + 4\zeta(2)\zeta(3) + 7\zeta(6) - 2\zeta(3)^2 \\ &\quad -16\zeta(7) + 4\zeta(2)\zeta(5) + 4\zeta(3)\zeta(4) + 9\zeta(8) - 4\zeta(3)\zeta(5)\big)\end{aligned} \tag{409}$$

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{H(k)}{k^6(k+1)^2} &= \frac{1}{2}\big(-12\zeta(2) + 22\zeta(3) - 10\zeta(4) + 18\zeta(5) - 6\zeta(2)\zeta(3) - 7\zeta(6) \\ &\quad +2\zeta(3)^2 + 8\zeta(7) - 2\zeta(2)\zeta(5) - 2\zeta(3)\zeta(4)\big)\end{aligned} \tag{410}$$

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{H(k)}{k^5(k+1)^3} &= \frac{1}{4}\big(60\zeta(2) - 100\zeta(3) + 29\zeta(4) - 36\zeta(5) + 12\zeta(2)\zeta(3) + 7\zeta(6) \\ &\quad -2\zeta(3)^2\big)\end{aligned} \tag{411}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^4(k+1)^4} = 20\zeta(2) - 30\zeta(3) + 4\zeta(4) - 5\zeta(5) + 2\zeta(2)\zeta(3) \tag{412}$$

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{H(k)}{k^3(k+1)^5} &= \frac{1}{4}\big(60\zeta(2) - 80\zeta(3) - \zeta(4) - 24\zeta(5) + 12\zeta(2)\zeta(3) - 3\zeta(6) \\ &\quad +2\zeta(3)^2\big)\end{aligned} \tag{413}$$

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{H(k)}{k^2(k+1)^6} &= \frac{1}{2}\big(-12\zeta(2) + 14\zeta(3) + 2\zeta(4) + 12\zeta(5) - 6\zeta(2)\zeta(3) + 3\zeta(6) \\ &\quad -2\zeta(3)^2 + 6\zeta(7) - 2\zeta(2)\zeta(5) - 2\zeta(3)\zeta(4)\big)\end{aligned} \tag{414}$$

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{H(k)}{k(k+1)^7} &= \frac{1}{4}\big(4\zeta(2) - 4\zeta(3) - \zeta(4) - 8\zeta(5) + 4\zeta(2)\zeta(3) - 3\zeta(6) + 2\zeta(3)^2 \\ &\quad -12\zeta(7) + 4\zeta(2)\zeta(5) + 4\zeta(3)\zeta(4) - 5\zeta(8) + 4\zeta(3)\zeta(5)\big)\end{aligned} \tag{415}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{(k+1)^8} = 4\zeta(9) - \zeta(3)\zeta(6) - \zeta(4)\zeta(5) - \zeta(2)\zeta(7) \tag{416}$$

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{H(k)}{k^7(k+2)} &= \frac{1}{128}\big(1 + \zeta(2) - 4\zeta(3) + 5\zeta(4) - 24\zeta(5) + 8\zeta(2)\zeta(3) + 28\zeta(6) \\ &\quad -8\zeta(3)^2 - 128\zeta(7) + 32\zeta(2)\zeta(5) + 32\zeta(3)\zeta(4) + 144\zeta(8) \\ &\quad -64\zeta(3)\zeta(5)\big)\end{aligned} \tag{417}$$

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{H(k)}{k^6(k+1)(k+2)} &= \frac{1}{64}\big(1 - 63\zeta(2) + 124\zeta(3) - 75\zeta(4) + 168\zeta(5) - 56\zeta(2)\zeta(3) \\ &\quad -84\zeta(6) + 24\zeta(3)^2 + 128\zeta(7) - 32\zeta(2)\zeta(5) - 32\zeta(3)\zeta(4)\big)\end{aligned} \tag{418}$$

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{H(k)}{k^5(k+1)^2(k+2)} &= \frac{1}{32}\big(1 + 129\zeta(2) - 228\zeta(3) + 85\zeta(4) - 120\zeta(5) + 40\zeta(2)\zeta(3) \\ &\quad +28\zeta(6) - 8\zeta(3)^2\big)\end{aligned} \tag{419}$$

$$\sum_{k=1}^{\infty}\frac{H(k)}{k^4(k+1)^3(k+2)} = \frac{1}{16}\left(1-111\zeta(2)+172\zeta(3)-31\zeta(4)+24\zeta(5)-8\zeta(2)\zeta(3)\right) \tag{420}$$

$$\sum_{k=1}^{\infty}\frac{H(k)}{k^3(k+1)^4(k+2)} = \frac{1}{8}\left(1+49\zeta(2)-68\zeta(3)+\zeta(4)-16\zeta(5)+8\zeta(2)\zeta(3)\right) \tag{421}$$

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{H(k)}{k^2(k+1)^5(k+2)} &= \frac{1}{4}\big(1-11\zeta(2)+12\zeta(3)+2\zeta(4)+8\zeta(5)-4\zeta(2)\zeta(3)+3\zeta(6)\\ &\quad -2\zeta(3)^2\big)\end{aligned} \tag{422}$$

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{H(k)}{k(k+1)^6(k+2)} &= \frac{1}{2}\big(1+\zeta(2)-2\zeta(3)-4\zeta(5)+2\zeta(2)\zeta(3)-6\zeta(7)+2\zeta(2)\zeta(5)\\ &\quad +2\zeta(3)\zeta(4)\big)\end{aligned} \tag{423}$$

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{H(k)}{(k+1)^7(k+2)} &= \frac{1}{4}\big(4-4\zeta(3)+\zeta(4)-8\zeta(5)+4\zeta(2)\zeta(3)+3\zeta(6)-2\zeta(3)^2\\ &\quad -12\zeta(7)+4\zeta(2)\zeta(5)+4\zeta(3)\zeta(4)+5\zeta(8)-4\zeta(3)\zeta(5)\big)\end{aligned} \tag{424}$$

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{H(k)}{k^6(k+2)^2} &= \frac{1}{64}\big(5+2\zeta(2)-11\zeta(3)+10\zeta(4)-36\zeta(5)+12\zeta(2)\zeta(3)+28\zeta(6)\\ &\quad -8\zeta(3)^2-64\zeta(7)+16\zeta(2)\zeta(5)+16\zeta(3)\zeta(4)\big)\end{aligned} \tag{425}$$

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{H(k)}{k^5(k+1)(k+2)^2} &= \frac{1}{64}\big(11-59\zeta(2)+102\zeta(3)-55\zeta(4)+96\zeta(5)-32\zeta(2)\zeta(3)\\ &\quad -28\zeta(6)+8\zeta(3)^2\big)\end{aligned} \tag{426}$$

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{H(k)}{k^4(k+1)^2(k+2)^2} &= \frac{1}{16}\big(-6-35\zeta(2)+63\zeta(3)-15\zeta(4)+12\zeta(5)\\ &\quad -4\zeta(2)\zeta(3)\big)\end{aligned} \tag{427}$$

$$\sum_{k=1}^{\infty}\frac{H(k)}{k^3(k+1)^3(k+2)^2} = \frac{1}{16}\left(13-41\zeta(2)+46\zeta(3)-\zeta(4)\right) \tag{428}$$

$$\sum_{k=1}^{\infty}\frac{H(k)}{k^2(k+1)^4(k+2)^2} = \frac{1}{4}\left(-7-4\zeta(2)+11\zeta(3)+8\zeta(5)-4\zeta(2)\zeta(3)\right) \tag{429}$$

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{H(k)}{k(k+1)^5(k+2)^2} &= \frac{1}{4}\big(15-3\zeta(2)-10\zeta(3)+2\zeta(4)-8\zeta(5)+4\zeta(2)\zeta(3)+3\zeta(6)\\ &\quad -2\zeta(3)^2\big)\end{aligned} \tag{430}$$

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{H(k)}{(k+1)^6(k+2)^2} &= \frac{1}{2}\big(16-2\zeta(2)-12\zeta(3)+2\zeta(4)-12\zeta(5)+6\zeta(2)\zeta(3)+3\zeta(6)\\ &\quad -2\zeta(3)^2-6\zeta(7)+2\zeta(2)\zeta(5)+2\zeta(3)\zeta(4)\big)\end{aligned} \tag{431}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^5(k+2)^3} = \frac{1}{128}\left(47+\zeta(2)-54\zeta(3)+29\zeta(4)-72\zeta(5)+24\zeta(2)\zeta(3)+28\zeta(6)\right.$$
$$\left.-8\zeta(3)^2\right) \quad (432)$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^4(k+1)(k+2)^3} = \frac{1}{32}\left(29-29\zeta(2)+24\zeta(3)-13\zeta(4)+12\zeta(5)-4\zeta(2)\zeta(3)\right) \quad (433)$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^3(k+1)^2(k+2)^3} = \frac{1}{16}\left(35+6\zeta(2)-39\zeta(3)+2\zeta(4)\right) \quad (434)$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^2(k+1)^3(k+2)^3} = \frac{1}{16}\left(83-29\zeta(2)-32\zeta(3)+3\zeta(4)\right) \quad (435)$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k(k+1)^4(k+2)^3} = \frac{1}{8}\left(97-21\zeta(2)-54\zeta(3)+3\zeta(4)-16\zeta(5)+8\zeta(2)\zeta(3)\right) \quad (436)$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{(k+1)^5(k+2)^3} = \frac{1}{4}\left(112-24\zeta(2)-64\zeta(3)+5\zeta(4)-24\zeta(5)+12\zeta(2)\zeta(3)\right.$$
$$\left.+3\zeta(6)-2\zeta(3)^2\right) \quad (437)$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^4(k+2)^4} = \frac{1}{32}\left(-35+6\zeta(2)+21\zeta(3)-2\zeta(4)+10\zeta(5)-4\zeta(2)\zeta(3)\right) \quad (438)$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^3(k+1)(k+2)^4} = \frac{1}{32}\left(-99+41\zeta(2)+18\zeta(3)+9\zeta(4)+8\zeta(5)-4\zeta(2)\zeta(3)\right) \quad (439)$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^2(k+1)^2(k+2)^4} = \frac{1}{16}\left(-134+35\zeta(2)+57\zeta(3)+7\zeta(4)+8\zeta(5)\right.$$
$$\left.-4\zeta(2)\zeta(3)\right) \quad (440)$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k(k+1)^3(k+2)^4} = \frac{1}{16}\left(-351+99\zeta(2)+146\zeta(3)+11\zeta(4)+16\zeta(5)\right.$$
$$\left.-8\zeta(2)\zeta(3)\right) \quad (441)$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{(k+1)^4(k+2)^4} = -56+15\zeta(2)+25\zeta(3)+\zeta(4)+4\zeta(5)-2\zeta(2)\zeta(3) \quad (442)$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^3(k+2)^5} = \frac{1}{128}\left(303-69\zeta(2)-104\zeta(3)-41\zeta(4)-64\zeta(5)+24\zeta(2)\zeta(3)\right.$$
$$\left.-12\zeta(6)+8\zeta(3)^2\right) \quad (443)$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^2(k+1)(k+2)^5} = \frac{1}{64}\left(501-151\zeta(2)-140\zeta(3)-59\zeta(4)-80\zeta(5)+32\zeta(2)\zeta(3)\right.$$
$$\left.-12\zeta(6)+8\zeta(3)^2\right) \quad (444)$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k(k+1)^2(k+2)^5} = \frac{1}{32} (769 - 221\zeta(2) - 254\zeta(3) - 73\zeta(4) - 96\zeta(5) + 40\zeta(2)\zeta(3) - 12\zeta(6) + 8\zeta(3)^2) \tag{445}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{(k+1)^3(k+2)^5} = \frac{1}{4} (280 - 80\zeta(2) - 100\zeta(3) - 21\zeta(4) - 28\zeta(5) + 12\zeta(2)\zeta(3) - 3\zeta(6) + 2\zeta(3)^2) \tag{446}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^2(k+2)^6} = \frac{1}{64} (261 - 54\zeta(2) - 59\zeta(3) - 46\zeta(4) - 56\zeta(5) + 12\zeta(2)\zeta(3) - 28\zeta(6) + 8\zeta(3)^2 - 48\zeta(7) + 16\zeta(2)\zeta(5) + 16\zeta(3)\zeta(4)) \tag{447}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k(k+1)(k+2)^6} = \frac{1}{64} (-1023 + 259\zeta(2) + 258\zeta(3) + 151\zeta(4) + 192\zeta(5) - 56\zeta(2)\zeta(3) + 68\zeta(6) - 24\zeta(3)^2 + 96\zeta(7) - 32\zeta(2)\zeta(5) - 32\zeta(3)\zeta(4)) \tag{448}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{(k+1)^2(k+2)^6} = \frac{1}{2} (112 - 30\zeta(2) - 32\zeta(3) - 14\zeta(4) - 18\zeta(5) + 6\zeta(2)\zeta(3) - 5\zeta(6) + 2\zeta(3)^2 - 6\zeta(7) + 2\zeta(2)\zeta(5) + 2\zeta(3)\zeta(4)) \tag{449}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k(k+2)^7} = \frac{1}{128} (769 - 125\zeta(2) - 126\zeta(3) - 121\zeta(4) - 128\zeta(5) + 8\zeta(2)\zeta(3) - 108\zeta(6) + 8\zeta(3)^2 - 160\zeta(7) + 32\zeta(2)\zeta(5) + 32\zeta(3)\zeta(4) - 80\zeta(8) + 64\zeta(3)\zeta(5)) \tag{450}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{(k+1)(k+2)^7} = \frac{1}{4} (112 - 24\zeta(2) - 24\zeta(3) - 17\zeta(4) - 20\zeta(5) + 4\zeta(2)\zeta(3) - 11\zeta(6) + 2\zeta(3)^2 - 16\zeta(7) + 4\zeta(2)\zeta(5) + 4\zeta(3)\zeta(4) - 5\zeta(8) + 4\zeta(3)\zeta(5)) \tag{451}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{(k+2)^8} = -8 + \zeta(2) + \zeta(3) + \zeta(4) + \zeta(5) + \zeta(6) + \zeta(7) + \zeta(8) + 4\zeta(9) - \zeta(3)\zeta(6) - \zeta(4)\zeta(5) - \zeta(2)\zeta(7) \tag{452}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k^7} = \frac{1}{6} (55\zeta(9) - 21\zeta(3)\zeta(6) - 15\zeta(4)\zeta(5) - 6\zeta(2)\zeta(7) + 2\zeta(3)^3) \tag{453}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k^6(k+1)} = \frac{1}{24} (72\zeta(3) - 102\zeta(4) + 84\zeta(5) - 24\zeta(2)\zeta(3) - 97\zeta(6) + 48\zeta(3)^2 + 144\zeta(7) - 24\zeta(2)\zeta(5) - 60\zeta(3)\zeta(4) - 24M(2,6)) \tag{454}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k^5(k+1)^2} = \frac{1}{12} (180\zeta(3) - 237\zeta(4) + 126\zeta(5) - 36\zeta(2)\zeta(3) - 97\zeta(6) + 48\zeta(3)^2 + 72\zeta(7) - 12\zeta(2)\zeta(5) - 30\zeta(3)\zeta(4)) \tag{455}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^2}{k^4(k+1)^3} = \frac{1}{24}\left(720\zeta(3) - 876\zeta(4) + 288\zeta(5) - 96\zeta(2)\zeta(3) - 97\zeta(6) + 48\zeta(3)^2\right) \quad (456)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^2}{k^3(k+1)^4} = \frac{1}{24}\left(720\zeta(3) - 804\zeta(4) + 192\zeta(5) - 96\zeta(2)\zeta(3) - 37\zeta(6) + 24\zeta(3)^2\right) \quad (457)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^2}{k^2(k+1)^5} = \frac{1}{12}\left(180\zeta(3) - 183\zeta(4) + 54\zeta(5) - 36\zeta(2)\zeta(3) - 37\zeta(6) + 24\zeta(3)^2 + 12\zeta(7) - 12\zeta(2)\zeta(5) + 6\zeta(3)\zeta(4)\right) \quad (458)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^2}{k(k+1)^6} = \frac{1}{24}\left(72\zeta(3) - 66\zeta(4) + 36\zeta(5) - 24\zeta(2)\zeta(3) - 37\zeta(6) + 24\zeta(3)^2 + 24\zeta(7) - 24\zeta(2)\zeta(5) + 12\zeta(3)\zeta(4) + 84\zeta(8) - 48\zeta(3)\zeta(5) - 24M(2,6)\right) \quad (459)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^2}{(k+1)^7} = \frac{1}{6}\left(\zeta(9) - 9\zeta(3)\zeta(6) - 3\zeta(4)\zeta(5) + 6\zeta(2)\zeta(7) + 2\zeta(3)^3\right) \quad (460)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^2}{k^6(k+2)} = \frac{1}{384}\left(6 + 6\zeta(2) + 18\zeta(3) - 51\zeta(4) + 84\zeta(5) - 24\zeta(2)\zeta(3) - 194\zeta(6) + 96\zeta(3)^2 + 576\zeta(7) - 96\zeta(2)\zeta(5) - 240\zeta(3)\zeta(4) - 192M(2,6)\right) \quad (461)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^2}{k^5(k+1)(k+2)} = \frac{1}{64}\left(-2 - 2\zeta(2) + 186\zeta(3) - 255\zeta(4) + 196\zeta(5) - 56\zeta(2)\zeta(3) - 194\zeta(6) + 96\zeta(3)^2 + 192\zeta(7) - 32\zeta(2)\zeta(5) - 80\zeta(3)\zeta(4)\right) \quad (462)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^2}{k^4(k+1)^2(k+2)} = \frac{1}{96}\left(6 + 6\zeta(2) + 882\zeta(3) - 1131\zeta(4) + 420\zeta(5) - 120\zeta(2)\zeta(3) - 194\zeta(6) + 96\zeta(3)^2\right) \quad (463)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^2}{k^3(k+1)^3(k+2)} = \frac{1}{16}\left(-2 - 2\zeta(2) + 186\zeta(3) - 207\zeta(4) + 52\zeta(5) - 24\zeta(2)\zeta(3)\right) \quad (464)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^2}{k^2(k+1)^4(k+2)} = \frac{1}{24}\left(-6 - 6\zeta(2) - 162\zeta(3) + 183\zeta(4) - 36\zeta(5) + 24\zeta(2)\zeta(3) + 37\zeta(6) - 24\zeta(3)^2\right) \quad (465)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^2}{k(k+1)^5(k+2)} = \frac{1}{2}\left(-1 - \zeta(2) + 3\zeta(3) + 3\zeta(5) - 2\zeta(2)\zeta(3) + 2\zeta(7) - 2\zeta(2)\zeta(5) + \zeta(3)\zeta(4)\right) \quad (466)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{(k+1)^6(k+2)} = \frac{1}{24}\,(24 + 24\zeta(2) - 66\zeta(4) - 36\zeta(5) + 24\zeta(2)\zeta(3) - 37\zeta(6) + 24\zeta(3)^2 - 24\zeta(7) + 24\zeta(2)\zeta(5) - 12\zeta(3)\zeta(4) + 84\zeta(8) - 48\zeta(3)\zeta(5) - 24M(2,6)) \tag{467}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k^5(k+2)^2} = \frac{1}{384}\,(66 + 30\zeta(2) + 66\zeta(3) - 237\zeta(4) + 252\zeta(5) - 72\zeta(2)\zeta(3) - 388\zeta(6) + 192\zeta(3)^2 + 576\zeta(7) - 96\zeta(2)\zeta(5) - 240\zeta(3)\zeta(4)\big) \tag{468}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k^4(k+1)(k+2)^2} = \frac{1}{96}\,(36 + 18\zeta(2) - 246\zeta(3) + 264\zeta(4) - 168\zeta(5) + 48\zeta(2)\zeta(3) + 97\zeta(6) - 48\zeta(3)^2\big) \tag{469}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k^3(k+1)^2(k+2)^2} = \frac{1}{32}\,(26 + 14\zeta(2) + 130\zeta(3) - 201\zeta(4) + 28\zeta(5) - 8\zeta(2)\zeta(3)) \tag{470}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k^2(k+1)^3(k+2)^2} = \frac{1}{8}\,(14 + 8\zeta(2) - 28\zeta(3) + 3\zeta(4) - 12\zeta(5) + 8\zeta(2)\zeta(3)) \tag{471}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k(k+1)^4(k+2)^2} = \frac{1}{24}\,(90 + 54\zeta(2) - 6\zeta(3) - 165\zeta(4) - 36\zeta(5) + 24\zeta(2)\zeta(3) - 37\zeta(6) + 24\zeta(3)^2\big) \tag{472}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{(k+1)^5(k+2)^2} = \frac{1}{12}\,(96 + 60\zeta(2) - 24\zeta(3) - 165\zeta(4) - 54\zeta(5) + 36\zeta(2)\zeta(3) - 37\zeta(6) + 24\zeta(3)^2 - 12\zeta(7) + 12\zeta(2)\zeta(5) - 6\zeta(3)\zeta(4)\big) \tag{473}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k^4(k+2)^3} = \frac{1}{192}\,(-174 - 18\zeta(2) - 6\zeta(3) + 225\zeta(4) - 144\zeta(5) + 48\zeta(2)\zeta(3) + 97\zeta(6) - 48\zeta(3)^2\big) \tag{474}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k^3(k+1)(k+2)^3} = \frac{1}{32}\,(-70 - 12\zeta(2) + 80\zeta(3) - 13\zeta(4) + 8\zeta(5)) \tag{475}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k^2(k+1)^2(k+2)^3} = \frac{1}{32}\,(-166 - 38\zeta(2) + 30\zeta(3) + 175\zeta(4) - 12\zeta(5) + 8\zeta(2)\zeta(3)) \tag{476}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k(k+1)^3(k+2)^3} = \frac{1}{16}\,(194 + 54\zeta(2) - 86\zeta(3) - 169\zeta(4) - 12\zeta(5) + 8\zeta(2)\zeta(3)) \tag{477}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{(k+1)^4(k+2)^3} = \frac{1}{24}\left(-672 - 216\zeta(2) + 264\zeta(3) + 672\zeta(4) + 72\zeta(5) - 48\zeta(2)\zeta(3) + 37\zeta(6) - 24\zeta(3)^2\right) \quad (478)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k^3(k+2)^4} = \frac{1}{192}\left(594 - 54\zeta(2) - 186\zeta(3) - 255\zeta(4) - 37\zeta(6) + 24\zeta(3)^2\right) \quad (479)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k^2(k+1)(k+2)^4} = \frac{1}{96}\left(804 - 18\zeta(2) - 426\zeta(3) - 216\zeta(4) - 24\zeta(5) - 37\zeta(6) + 24\zeta(3)^2\right) \quad (480)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k(k+1)^2(k+2)^4} = \frac{1}{96}\left(2106 + 78\zeta(2) - 942\zeta(3) - 957\zeta(4) - 12\zeta(5) - 24\zeta(2)\zeta(3) - 74\zeta(6) + 48\zeta(3)^2\right) \quad (481)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{(k+1)^3(k+2)^4} = \frac{1}{24}\left(1344 + 120\zeta(2) - 600\zeta(3) - 732\zeta(4) - 24\zeta(5) - 37\zeta(6) + 24\zeta(3)^2\right) \quad (482)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k^2(k+2)^5} = \frac{1}{384}\left(-3006 + 522\zeta(2) + 1086\zeta(3) + 603\zeta(4) + 756\zeta(5) - 312\zeta(2)\zeta(3) + 292\zeta(6) - 192\zeta(3)^2 - 96\zeta(7) + 96\zeta(2)\zeta(5) - 48\zeta(3)\zeta(4)\right) \quad (483)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k(k+1)(k+2)^5} = \frac{1}{64}\left(1538 - 186\zeta(2) - 646\zeta(3) - 345\zeta(4) - 268\zeta(5) + 104\zeta(2)\zeta(3) - 122\zeta(6) + 80\zeta(3)^2 + 32\zeta(7) - 32\zeta(2)\zeta(5) + 16\zeta(3)\zeta(4)\right) \quad (484)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{(k+1)^2(k+2)^5} = \frac{1}{12}\left(840 - 60\zeta(2) - 360\zeta(3) - 249\zeta(4) - 102\zeta(5) + 36\zeta(2)\zeta(3) - 55\zeta(6) + 36\zeta(3)^2 + 12\zeta(7) - 12\zeta(2)\zeta(5) + 6\zeta(3)\zeta(4)\right) \quad (485)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k(k+2)^6} = \frac{1}{384}\left(6138 - 1170\zeta(2) - 1902\zeta(3) - 1029\zeta(4) - 1788\zeta(5) + 648\zeta(2)\zeta(3) - 698\zeta(6) + 336\zeta(3)^2 - 1056\zeta(7) + 288\zeta(2)\zeta(5) + 432\zeta(3)\zeta(4) + 672\zeta(8) - 384\zeta(3)\zeta(5) - 192M(2,6)\right) \quad (486)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{(k+1)(k+2)^6} = \frac{1}{24}\left(1344 - 216\zeta(2) - 480\zeta(3) - 258\zeta(4) - 324\zeta(5) + 120\zeta(2)\zeta(3) - 133\zeta(6) + 72\zeta(3)^2 - 120\zeta(7) + 24\zeta(2)\zeta(5) + 60\zeta(3)\zeta(4) + 84\zeta(8) - 48\zeta(3)\zeta(5) - 24M(2,6)\right) \quad (487)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{(k+2)^7} = \frac{1}{6}\left(168 - 30\zeta(2) - 42\zeta(3) - 27\zeta(4) - 42\zeta(5) + 12\zeta(2)\zeta(3) - 21\zeta(6) + 6\zeta(3)^2 - 42\zeta(7) + 12\zeta(2)\zeta(5) + 12\zeta(3)\zeta(4) - 15\zeta(8) + 12\zeta(3)\zeta(5) - \zeta(9) + 9\zeta(3)\zeta(6) + 3\zeta(4)\zeta(5) - 6\zeta(2)\zeta(7) - 2\zeta(3)^3\right) \quad (488)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^3}{k^6} = \frac{1}{24}\left(521\zeta(9) - 291\zeta(3)\zeta(6) - 306\zeta(4)\zeta(5) + 72\zeta(2)\zeta(7) + 48\zeta(3)^3\right) \tag{489}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^3}{k^5(k+1)} = \frac{1}{96}\big(960\zeta(4) - 960\zeta(5) - 96\zeta(2)\zeta(3) + 558\zeta(6) - 240\zeta(3)^2 - 1386\zeta(7) - 192\zeta(2)\zeta(5) + 1224\zeta(3)\zeta(4) - 595\zeta(8) - 120\zeta(2)\zeta(3)^2 + 576\zeta(3)\zeta(5) + 264M(2,6)\big) \tag{490}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^3}{k^4(k+1)^2} = \frac{1}{16}\big(640\zeta(4) - 600\zeta(5) - 64\zeta(2)\zeta(3) + 186\zeta(6) - 80\zeta(3)^2 - 231\zeta(7) - 32\zeta(2)\zeta(5) + 204\zeta(3)\zeta(4)\big) \tag{491}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^3}{k^3(k+1)^3} = \frac{1}{8}\left(480\zeta(4) - 420\zeta(5) - 48\zeta(2)\zeta(3) + 63\zeta(6) - 36\zeta(3)^2\right) \tag{492}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^3}{k^2(k+1)^4} = \frac{1}{16}\big(640\zeta(4) - 520\zeta(5) - 64\zeta(2)\zeta(3) + 66\zeta(6) - 64\zeta(3)^2 - 119\zeta(7) - 32\zeta(2)\zeta(5) + 132\zeta(3)\zeta(4)\big) \tag{493}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^3}{k(k+1)^5} = \frac{1}{96}\big(960\zeta(4) - 720\zeta(5) - 96\zeta(2)\zeta(3) + 198\zeta(6) - 192\zeta(3)^2 - 714\zeta(7) - 192\zeta(2)\zeta(5) + 792\zeta(3)\zeta(4) + 43\zeta(8) + 120\zeta(2)\zeta(3)^2 - 288\zeta(3)\zeta(5) + 24M(2,6)\big) \tag{494}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^3}{(k+1)^6} = \frac{1}{24}\left(197\zeta(9) - 111\zeta(3)\zeta(6) - 198\zeta(4)\zeta(5) + 72\zeta(2)\zeta(7) + 24\zeta(3)^3\right) \tag{495}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^3}{k^5(k+2)} = \frac{1}{384}\big(12 + 24\zeta(2) + 48\zeta(3) + 120\zeta(4) - 240\zeta(5) - 24\zeta(2)\zeta(3) + 279\zeta(6) - 120\zeta(3)^2 - 1386\zeta(7) - 192\zeta(2)\zeta(5) + 1224\zeta(3)\zeta(4) - 1190\zeta(8) - 240\zeta(2)\zeta(3)^2 + 1152\zeta(3)\zeta(5) + 528M(2,6)\big) \tag{496}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^3}{k^4(k+1)(k+2)} = \frac{1}{64}\big(4 + 8\zeta(2) + 16\zeta(3) - 600\zeta(4) + 560\zeta(5) + 56\zeta(2)\zeta(3) - 279\zeta(6) + 120\zeta(3)^2 + 462\zeta(7) + 64\zeta(2)\zeta(5) - 408\zeta(3)\zeta(4)\big) \tag{497}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^3}{k^3(k+1)^2(k+2)} = \frac{1}{32}\big(4 + 8\zeta(2) + 16\zeta(3) + 680\zeta(4) - 640\zeta(5) - 72\zeta(2)\zeta(3) + 93\zeta(6) - 40\zeta(3)^2\big) \tag{498}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^3}{k^2(k+1)^3(k+2)} = \frac{1}{16}\left(4+8\zeta(2)+16\zeta(3)-280\zeta(4)+200\zeta(5)+24\zeta(2)\zeta(3)\right.$$
$$\left.-33\zeta(6)+32\zeta(3)^2\right) \tag{499}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^3}{k(k+1)^4(k+2)} = \frac{1}{16}\left(8+16\zeta(2)+32\zeta(3)+80\zeta(4)-120\zeta(5)-16\zeta(2)\zeta(3)\right.$$
$$\left.-119\zeta(7)-32\zeta(2)\zeta(5)+132\zeta(3)\zeta(4)\right) \tag{500}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^3}{(k+1)^5(k+2)} = \frac{1}{96}\left(96+192\zeta(2)+384\zeta(3)-720\zeta(5)-96\zeta(2)\zeta(3)-198\zeta(6)\right.$$
$$+192\zeta(3)^2-714\zeta(7)-192\zeta(2)\zeta(5)+792\zeta(3)\zeta(4)-43\zeta(8)$$
$$\left.-120\zeta(2)\zeta(3)^2+288\zeta(3)\zeta(5)-24M(2,6)\right) \tag{501}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^3}{k^4(k+2)^2} = \frac{1}{64}\left(-24-28\zeta(2)-36\zeta(3)-47\zeta(4)+150\zeta(5)+16\zeta(2)\zeta(3)\right.$$
$$\left.-93\zeta(6)+40\zeta(3)^2+231\zeta(7)+32\zeta(2)\zeta(5)-204\zeta(3)\zeta(4)\right) \tag{502}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^3}{k^3(k+1)(k+2)^2} = \frac{1}{64}\left(-52-64\zeta(2)-88\zeta(3)+506\zeta(4)-260\zeta(5)-24\zeta(2)\zeta(3)\right.$$
$$\left.+93\zeta(6)-40\zeta(3)^2\right) \tag{503}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^3}{k^2(k+1)^2(k+2)^2} = \frac{1}{16}\left(28+36\zeta(2)+52\zeta(3)+87\zeta(4)-190\zeta(5)\right.$$
$$\left.-24\zeta(2)\zeta(3)\right) \tag{504}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^3}{k(k+1)^3(k+2)^2} = \frac{1}{16}\left(60+80\zeta(2)+120\zeta(3)-106\zeta(4)-180\zeta(5)-24\zeta(2)\zeta(3)\right.$$
$$\left.-33\zeta(6)+32\zeta(3)^2\right) \tag{505}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^3}{(k+1)^4(k+2)^2} = \frac{1}{16}\left(-128-176\zeta(2)-272\zeta(3)+132\zeta(4)+480\zeta(5)+64\zeta(2)\zeta(3)\right.$$
$$\left.+66\zeta(6)-64\zeta(3)^2+119\zeta(7)+32\zeta(2)\zeta(5)-132\zeta(3)\zeta(4)\right) \tag{506}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^3}{k^3(k+2)^3} = \frac{1}{64}\left(140+84\zeta(2)+36\zeta(3)-51\zeta(4)-174\zeta(5)-48\zeta(2)\zeta(3)\right.$$
$$\left.+63\zeta(6)-36\zeta(3)^2\right) \tag{507}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^3}{k^2(k+1)(k+2)^3} = \frac{1}{64}\left(332+232\zeta(2)+160\zeta(3)-608\zeta(4)-88\zeta(5)-72\zeta(2)\zeta(3)\right.$$
$$\left.+33\zeta(6)-32\zeta(3)^2\right) \tag{508}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^3}{k(k+1)^2(k+2)^3} = \frac{1}{32}\left(388+304\zeta(2)+264\zeta(3)-434\zeta(4)-468\zeta(5)\right.$$
$$\left.-120\zeta(2)\zeta(3)+33\zeta(6)-32\zeta(3)^2\right) \tag{509}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^3}{(k+1)^3(k+2)^3} = \frac{1}{4}\left(112+96\zeta(2)+96\zeta(3)-135\zeta(4)-162\zeta(5)\right.$$
$$\left.-36\zeta(2)\zeta(3)\right) \tag{510}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^3}{k^2(k+2)^4} = \frac{1}{64}\left(-536-132\zeta(2)+132\zeta(3)+335\zeta(4)+82\zeta(5)+64\zeta(2)\zeta(3)\right.$$
$$\left.+41\zeta(6)-16\zeta(3)^2+119\zeta(7)+32\zeta(2)\zeta(5)-132\zeta(3)\zeta(4)\right) \tag{511}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^3}{k(k+1)(k+2)^4} = \frac{1}{64}\left(-1404-496\zeta(2)+104\zeta(3)+1278\zeta(4)+252\zeta(5)+200\zeta(2)\zeta(3)\right.$$
$$\left.+49\zeta(6)+238\zeta(7)+64\zeta(2)\zeta(5)-264\zeta(3)\zeta(4)\right) \tag{512}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^3}{(k+1)^2(k+2)^4} = \frac{1}{16}\left(896+400\zeta(2)+80\zeta(3)-856\zeta(4)-360\zeta(5)-160\zeta(2)\zeta(3)\right.$$
$$\left.-8\zeta(6)-16\zeta(3)^2-119\zeta(7)-32\zeta(2)\zeta(5)+132\zeta(3)\zeta(4)\right) \tag{513}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^3}{k(k+2)^5} = \frac{1}{384}\left(9228+432\zeta(2)-3528\zeta(3)-4134\zeta(4)-1740\zeta(5)+408\zeta(2)\zeta(3)\right.$$
$$-1665\zeta(6)+1056\zeta(3)^2-138\zeta(7)-768\zeta(2)\zeta(5)+1080\zeta(3)\zeta(4)+86\zeta(8)$$
$$\left.+240\zeta(2)\zeta(3)^2-576\zeta(3)\zeta(5)+48M(2,6)\right) \tag{514}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^3}{(k+1)(k+2)^5} = \frac{1}{96}\left(6720+960\zeta(2)-1920\zeta(3)-3984\zeta(4)-1248\zeta(5)-96\zeta(2)\zeta(3)\right.$$
$$-906\zeta(6)+528\zeta(3)^2-426\zeta(7)-480\zeta(2)\zeta(5)+936\zeta(3)\zeta(4)+43\zeta(8)$$
$$\left.+120\zeta(2)\zeta(3)^2-288\zeta(3)\zeta(5)+24M(2,6)\right) \tag{515}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^3}{(k+2)^6} = \frac{1}{24}\left(1344-72\zeta(2)-504\zeta(3)-414\zeta(4)-396\zeta(5)+144\zeta(2)\zeta(3)\right.$$
$$-243\zeta(6)+144\zeta(3)^2-144\zeta(7)+108\zeta(3)\zeta(4)+252\zeta(8)-144\zeta(3)\zeta(5)$$
$$-72M(2,6)-197\zeta(9)+111\zeta(3)\zeta(6)+198\zeta(4)\zeta(5)-72\zeta(2)\zeta(7)$$
$$\left.-24\zeta(3)^3\right) \tag{516}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^4}{k^5} = \frac{1}{12}\left(436\zeta(9)-279\zeta(3)\zeta(6)-258\zeta(4)\zeta(5)+84\zeta(2)\zeta(7)\right.$$
$$\left.+40\zeta(3)^3\right) \tag{517}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^4}{k^4(k+1)} = \frac{1}{144}\left(-4320\zeta(5)-864\zeta(2)\zeta(3)+5874\zeta(6)+432\zeta(3)^2-3330\zeta(7)\right.$$
$$-720\zeta(2)\zeta(5)+3096\zeta(3)\zeta(4)-14833\zeta(8)-4032\zeta(2)\zeta(3)^2$$
$$\left.+16704\zeta(3)\zeta(5)+3744M(2,6)\right) \tag{518}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^4}{k^3(k+1)^2} = \frac{1}{8}\big(720\zeta(5) + 144\zeta(2)\zeta(3) - 939\zeta(6) - 72\zeta(3)^2 + 185\zeta(7)$$
$$+40\zeta(2)\zeta(5) - 172\zeta(3)\zeta(4)\big) \tag{519}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^4}{k^2(k+1)^3} = \frac{1}{8}\big(-720\zeta(5) - 144\zeta(2)\zeta(3) + 899\zeta(6) + 72\zeta(3)^2 - 109\zeta(7)$$
$$-40\zeta(2)\zeta(5) + 148\zeta(3)\zeta(4)\big) \tag{520}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^4}{k(k+1)^4} = \frac{1}{144}\big(4320\zeta(5) + 864\zeta(2)\zeta(3) - 5154\zeta(6) - 432\zeta(3)^2 + 1962\zeta(7)$$
$$+720\zeta(2)\zeta(5) - 2664\zeta(3)\zeta(4) + 12415\zeta(8) + 3312\zeta(2)\zeta(3)^2$$
$$-13824\zeta(3)\zeta(5) - 3024M(2,6)\big) \tag{521}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^4}{(k+1)^5} = \frac{1}{12}\big(-174\zeta(9) + 99\zeta(3)\zeta(6) + 222\zeta(4)\zeta(5) - 84\zeta(2)\zeta(7)$$
$$-32\zeta(3)^3\big) \tag{522}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^4}{k^4(k+2)} = \frac{1}{576}\big(36 + 108\zeta(2) + 396\zeta(3) + 666\zeta(4) + 1080\zeta(5) + 216\zeta(2)\zeta(3)$$
$$-2937\zeta(6) - 216\zeta(3)^2 + 3330\zeta(7) + 720\zeta(2)\zeta(5) - 3096\zeta(3)\zeta(4) + 29666\zeta(8)$$
$$+8064\zeta(2)\zeta(3)^2 - 33408\zeta(3)\zeta(5) - 7488M(2,6)\big) \tag{523}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^4}{k^3(k+1)(k+2)} = \frac{1}{32}\big(-4 - 12\zeta(2) - 44\zeta(3) - 74\zeta(4) + 840\zeta(5) + 168\zeta(2)\zeta(3)$$
$$-979\zeta(6) - 72\zeta(3)^2 + 370\zeta(7) + 80\zeta(2)\zeta(5) - 344\zeta(3)\zeta(4)\big) \tag{524}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^4}{k^2(k+1)^2(k+2)} = \frac{1}{16}\big(-4 - 12\zeta(2) - 44\zeta(3) - 74\zeta(4) - 600\zeta(5) - 120\zeta(2)\zeta(3)$$
$$+899\zeta(6) + 72\zeta(3)^2\big) \tag{525}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^4}{k(k+1)^3(k+2)} = \frac{1}{8}\big(4 + 12\zeta(2) + 44\zeta(3) + 74\zeta(4) - 120\zeta(5) - 24\zeta(2)\zeta(3)$$
$$-109\zeta(7) - 40\zeta(2)\zeta(5) + 148\zeta(3)\zeta(4)\big) \tag{526}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^4}{(k+1)^4(k+2)} = \frac{1}{144}\big(144 + 432\zeta(2) + 1584\zeta(3) + 2664\zeta(4) - 5154\zeta(6) - 432\zeta(3)^2$$
$$-1962\zeta(7) - 720\zeta(2)\zeta(5) + 2664\zeta(3)\zeta(4) + 12415\zeta(8) + 3312\zeta(2)\zeta(3)^2$$
$$-13824\zeta(3)\zeta(5) - 3024M(2,6)\big) \tag{527}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^4}{k^3(k+2)^2} = \frac{1}{64}\big(52 + 100\zeta(2) + 276\zeta(3) + 238\zeta(4) + 120\zeta(5) + 40\zeta(2)\zeta(3)$$
$$-939\zeta(6) - 72\zeta(3)^2 + 370\zeta(7) + 80\zeta(2)\zeta(5) - 344\zeta(3)\zeta(4)\big) \tag{528}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^4}{k^2(k+1)(k+2)^2} = \frac{1}{4}\left(7 + 14\zeta(2) + 40\zeta(3) + 39\zeta(4) - 90\zeta(5) - 16\zeta(2)\zeta(3) \right.$$
$$\left. +5\zeta(6)\right) \quad (529)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^4}{k(k+1)^2(k+2)^2} = \frac{1}{16}\left(60 + 124\zeta(2) + 364\zeta(3) + 386\zeta(4) - 120\zeta(5) - 8\zeta(2)\zeta(3) \right.$$
$$\left. -859\zeta(6) - 72\zeta(3)^2\right) \quad (530)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^4}{(k+1)^3(k+2)^2} = \frac{1}{8}\left(64 + 136\zeta(2) + 408\zeta(3) + 460\zeta(4) - 240\zeta(5) - 32\zeta(2)\zeta(3) \right.$$
$$\left. -859\zeta(6) - 72\zeta(3)^2 - 109\zeta(7) - 40\zeta(2)\zeta(5) + 148\zeta(3)\zeta(4)\right) \quad (531)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^4}{k^2(k+2)^3} = \frac{1}{64}\left(-332 - 388\zeta(2) - 740\zeta(3) - 6\zeta(4) + 456\zeta(5) + 152\zeta(2)\zeta(3) \right.$$
$$\left. +767\zeta(6) + 200\zeta(3)^2 - 218\zeta(7) - 80\zeta(2)\zeta(5) + 296\zeta(3)\zeta(4)\right) \quad (532)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^4}{k(k+1)(k+2)^3} = \frac{1}{32}\left(-388 - 500\zeta(2) - 1060\zeta(3) - 318\zeta(4) + 1176\zeta(5) + 280\zeta(2)\zeta(3) \right.$$
$$\left. +727\zeta(6) + 200\zeta(3)^2 - 218\zeta(7) - 80\zeta(2)\zeta(5) + 296\zeta(3)\zeta(4)\right) \quad (533)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^4}{(k+1)^2(k+2)^3} = \frac{1}{8}\left(224 + 312\zeta(2) + 712\zeta(3) + 352\zeta(4) - 648\zeta(5) - 144\zeta(2)\zeta(3) \right.$$
$$\left. -793\zeta(6) - 136\zeta(3)^2 + 109\zeta(7) + 40\zeta(2)\zeta(5) - 148\zeta(3)\zeta(4)\right) \quad (534)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^4}{k(k+2)^4} = \frac{1}{576}\left(12636 + 8460\zeta(2) + 9180\zeta(3) - 11502\zeta(4) - 9864\zeta(5) \right.$$
$$-4968\zeta(2)\zeta(3) - 1677\zeta(6) - 1944\zeta(3)^2 - 6606\zeta(7) - 1584\zeta(2)\zeta(5)$$
$$+6840\zeta(3)\zeta(4) + 24830\zeta(8) + 6624\zeta(2)\zeta(3)^2 - 27648\zeta(3)\zeta(5)$$
$$\left. -6048M(2,6)\right) \quad (535)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^4}{(k+1)(k+2)^4} = \frac{1}{144}\left(8064 + 6480\zeta(2) + 9360\zeta(3) - 4320\zeta(4) - 10224\zeta(5) \right.$$
$$-3744\zeta(2)\zeta(3) - 4110\zeta(6) - 1872\zeta(3)^2 - 2322\zeta(7) - 432\zeta(2)\zeta(5)$$
$$+2088\zeta(3)\zeta(4) + 12415\zeta(8) + 3312\zeta(2)\zeta(3)^2 - 13824\zeta(3)\zeta(5)$$
$$\left. -3024M(2,6)\right) \quad (536)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^4}{(k+2)^5} = \frac{1}{24}\left(-1680 - 600\zeta(2) - 120\zeta(3) + 1380\zeta(4) + 672\zeta(5) + 240\zeta(2)\zeta(3) \right.$$
$$+318\zeta(6) - 144\zeta(3)^2 + 570\zeta(7) + 336\zeta(2)\zeta(5) - 864\zeta(3)\zeta(4) - 43\zeta(8)$$
$$-120\zeta(2)\zeta(3)^2 + 288\zeta(3)\zeta(5) - 24M(2,6) - 348\zeta(9) + 198\zeta(3)\zeta(6)$$
$$\left. +444\zeta(4)\zeta(5) - 168\zeta(2)\zeta(7) - 64\zeta(3)^3\right) \quad (537)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^5}{k^4} = \frac{1}{72}\left(9442\zeta(9) - 14685\zeta(3)\zeta(6) + 4752\zeta(4)\zeta(5) + 2385\zeta(2)\zeta(7)\right.$$
$$\left. -360\zeta(3)^3\right) \quad (538)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^5}{k^3(k+1)} = \frac{1}{288}\left(51408\zeta(6) + 6480\zeta(3)^2 - 36918\zeta(7) - 8208\zeta(2)\zeta(5)\right.$$
$$-9504\zeta(3)\zeta(4) - 67811\zeta(8) - 19080\zeta(2)\zeta(3)^2 + 78768\zeta(3)\zeta(5)$$
$$\left.+16920M(2,6)\right) \quad (539)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^5}{k^2(k+1)^2} = \frac{1}{8}\left(2856\zeta(6) + 360\zeta(3)^2 - 1953\zeta(7) - 456\zeta(2)\zeta(5)\right.$$
$$\left.-528\zeta(3)\zeta(4)\right) \quad (540)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^5}{k(k+1)^3} = \frac{1}{288}\left(51408\zeta(6) + 6480\zeta(3)^2 - 33390\zeta(7) - 8208\zeta(2)\zeta(5)\right.$$
$$-9504\zeta(3)\zeta(4) - 65621\zeta(8) - 17640\zeta(2)\zeta(3)^2 + 72432\zeta(3)\zeta(5)$$
$$\left.+15480M(2,6)\right) \quad (541)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^5}{(k+1)^4} = \frac{1}{72}\left(7120\zeta(9) - 12885\zeta(3)\zeta(6) + 4752\zeta(4)\zeta(5) + 2385\zeta(2)\zeta(7)\right.$$
$$\left.-360\zeta(3)^3\right) \quad (542)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^5}{k^3(k+2)} = \frac{1}{576}\left(72 + 288\zeta(2) + 1512\zeta(3) + 4518\zeta(4) + 5112\zeta(5) + 1080\zeta(2)\zeta(3)\right.$$
$$+12852\zeta(6) + 1620\zeta(3)^2 - 18459\zeta(7) - 4104\zeta(2)\zeta(5) - 4752\zeta(3)\zeta(4)$$
$$\left.-67811\zeta(8) - 19080\zeta(2)\zeta(3)^2 + 78768\zeta(3)\zeta(5) + 16920M(2,6)\right) \quad (543)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^5}{k^2(k+1)(k+2)} = \frac{1}{32}\left(8 + 32\zeta(2) + 168\zeta(3) + 502\zeta(4) + 568\zeta(5) + 120\zeta(2)\zeta(3)\right.$$
$$\left.-4284\zeta(6) - 540\zeta(3)^2 + 2051\zeta(7) + 456\zeta(2)\zeta(5) + 528\zeta(3)\zeta(4)\right) \quad (544)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^5}{k(k+1)^2(k+2)} = \frac{1}{16}\left(8 + 32\zeta(2) + 168\zeta(3) + 502\zeta(4) + 568\zeta(5) + 120\zeta(2)\zeta(3)\right.$$
$$\left.+1428\zeta(6) + 180\zeta(3)^2 - 1855\zeta(7) - 456\zeta(2)\zeta(5) - 528\zeta(3)\zeta(4)\right) \quad (545)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^5}{(k+1)^3(k+2)} = \frac{1}{288}\left(288 + 1152\zeta(2) + 6048\zeta(3) + 18072\zeta(4) + 20448\zeta(5)\right.$$
$$+4320\zeta(2)\zeta(3) - 33390\zeta(7) - 8208\zeta(2)\zeta(5) - 9504\zeta(3)\zeta(4) + 65621\zeta(8)$$
$$\left.+17640\zeta(2)\zeta(3)^2 - 72432\zeta(3)\zeta(5) - 15480M(2,6)\right) \quad (546)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^5}{k^2(k+2)^2} = \frac{1}{96}\left(-168 - 456\zeta(2) - 1896\zeta(3) - 3858\zeta(4) - 1608\zeta(5) - 480\zeta(2)\zeta(3)\right.$$
$$\left.+11\zeta(6) - 180\zeta(3)^2 + 5859\zeta(7) + 1368\zeta(2)\zeta(5) + 1584\zeta(3)\zeta(4)\right) \quad (547)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^5}{k(k+1)(k+2)^2} = \frac{1}{96} \left(-360 - 1008\zeta(2) - 4296\zeta(3) - 9222\zeta(4) - 4920\zeta(5) \right.$$
$$-1320\zeta(2)\zeta(3) + 12874\zeta(6) + 1260\zeta(3)^2 + 5565\zeta(7) + 1368\zeta(2)\zeta(5)$$
$$\left. +1584\zeta(3)\zeta(4)\right) \quad (548)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^5}{(k+1)^2(k+2)^2} = \frac{1}{24} \left(-192 - 552\zeta(2) - 2400\zeta(3) - 5364\zeta(4) - 3312\zeta(5) \right.$$
$$-840\zeta(2)\zeta(3) + 4295\zeta(6) + 360\zeta(3)^2 + 5565\zeta(7) + 1368\zeta(2)\zeta(5)$$
$$\left. +1584\zeta(3)\zeta(4)\right) \quad (549)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^5}{k(k+2)^3} = \frac{1}{576} \left(6984 + 12528\zeta(2) + 40104\zeta(3) + 48726\zeta(4) - 13896\zeta(5) \right.$$
$$-2520\zeta(2)\zeta(3) - 58518\zeta(6) - 10620\zeta(3)^2 + 2925\zeta(7) + 3096\zeta(2)\zeta(5)$$
$$-31392\zeta(3)\zeta(4) - 65621\zeta(8) - 17640\zeta(2)\zeta(3)^2 + 72432\zeta(3)\zeta(5)$$
$$\left. +15480M(2,6)\right) \quad (550)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^5}{(k+1)(k+2)^3} = \frac{1}{288} \left(8064 + 15552\zeta(2) + 52992\zeta(3) + 76392\zeta(4) + 864\zeta(5) \right.$$
$$+1440\zeta(2)\zeta(3) - 97140\zeta(6) - 14400\zeta(3)^2 - 13770\zeta(7) - 1008\zeta(2)\zeta(5)$$
$$-36144\zeta(3)\zeta(4) - 65621\zeta(8) - 17640\zeta(2)\zeta(3)^2 + 72432\zeta(3)\zeta(5)$$
$$\left. +15480M(2,6)\right) \quad (551)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^5}{(k+2)^4} = \frac{1}{144} \left(8064 + 9360\zeta(2) + 22320\zeta(3) + 11520\zeta(4) - 17136\zeta(5) \right.$$
$$-5760\zeta(2)\zeta(3) - 22050\zeta(6) - 6480\zeta(3)^2 - 900\zeta(7) + 720\zeta(2)\zeta(5)$$
$$-1440\zeta(3)\zeta(4) + 62075\zeta(8) + 16560\zeta(2)\zeta(3)^2 - 69120\zeta(3)\zeta(5) - 15120M(2,6)$$
$$-14240\zeta(9) + 25770\zeta(3)\zeta(6) - 9504\zeta(4)\zeta(5) - 4770\zeta(2)\zeta(7)$$
$$\left. +720\zeta(3)^3\right) \quad (552)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^6}{k^3} = \frac{1}{24} \left(7474\zeta(9) - 13122\zeta(3)\zeta(6) + 6048\zeta(4)\zeta(5) + 1953\zeta(2)\zeta(7) \right.$$
$$\left. -544\zeta(3)^3\right) \quad (553)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^6}{k^2(k+1)} = \frac{1}{8} \left(-5152\zeta(7) - 1160\zeta(2)\zeta(5) - 2376\zeta(3)\zeta(4) + 5843\zeta(8) \right.$$
$$\left. -328\zeta(2)\zeta(3)^2 + 3896\zeta(3)\zeta(5) + 456M(2,6)\right) \quad (554)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^6}{k(k+1)^2} = \frac{1}{24} \left(15456\zeta(7) + 3480\zeta(2)\zeta(5) + 7128\zeta(3)\zeta(4) - 17027\zeta(8) \right.$$
$$\left. +924\zeta(2)\zeta(3)^2 - 11328\zeta(3)\zeta(5) - 1308M(2,6)\right) \quad (555)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^6}{(k+1)^3} = \frac{1}{24}\big(-6146\zeta(9)+12582\zeta(3)\zeta(6)-5832\zeta(4)\zeta(5)-1953\zeta(2)\zeta(7)$$
$$+536\zeta(3)^3\big) \quad (556)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^6}{k^2(k+2)} = \frac{1}{16}\big(4+20\zeta(2)+136\zeta(3)+571\zeta(4)+1142\zeta(5)+244\zeta(2)\zeta(3)$$
$$+2097\zeta(6)+268\zeta(3)^2+2576\zeta(7)+580\zeta(2)\zeta(5)+1188\zeta(3)\zeta(4)-5843\zeta(8)$$
$$+328\zeta(2)\zeta(3)^2-3896\zeta(3)\zeta(5)-456M(2,6)\big) \quad (557)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^6}{k(k+1)(k+2)} = \frac{1}{8}\big(-4-20\zeta(2)-136\zeta(3)-571\zeta(4)-1142\zeta(5)-244\zeta(2)\zeta(3)$$
$$-2097\zeta(6)-268\zeta(3)^2+2576\zeta(7)+580\zeta(2)\zeta(5)+1188\zeta(3)\zeta(4)\big) \quad (558)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^6}{(k+1)^2(k+2)} = \frac{1}{24}\big(-24-120\zeta(2)-816\zeta(3)-3426\zeta(4)-6852\zeta(5)-1464\zeta(2)\zeta(3)$$
$$-12582\zeta(6)-1608\zeta(3)^2+17027\zeta(8)-924\zeta(2)\zeta(3)^2+11328\zeta(3)\zeta(5)$$
$$+1308M(2,6)\big) \quad (559)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^6}{k(k+2)^2} = \frac{1}{48}\big(180+636\zeta(2)+3528\zeta(3)+11001\zeta(4)+13530\zeta(5)+3180\zeta(2)\zeta(3)$$
$$+5988\zeta(6)+1332\zeta(3)^2-8967\zeta(7)-2364\zeta(2)\zeta(5)-1188\zeta(3)\zeta(4)-17027\zeta(8)$$
$$+924\zeta(2)\zeta(3)^2-11328\zeta(3)\zeta(5)-1308M(2,6)\big) \quad (560)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^6}{(k+1)(k+2)^2} = \frac{1}{24}\big(192+696\zeta(2)+3936\zeta(3)+12714\zeta(4)+16956\zeta(5)$$
$$+3912\zeta(2)\zeta(3)+12279\zeta(6)+2136\zeta(3)^2-16695\zeta(7)-4104\zeta(2)\zeta(5)$$
$$-4752\zeta(3)\zeta(4)-17027\zeta(8)+924\zeta(2)\zeta(3)^2-11328\zeta(3)\zeta(5)-1308M(2,6)\big) \quad (561)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^6}{(k+2)^3} = \frac{1}{48}\big(1344+3312\zeta(2)+14832\zeta(3)+33120\zeta(4)+20592\zeta(5)+5184\zeta(2)\zeta(3)$$
$$-24396\zeta(6)-3024\zeta(3)^2-23580\zeta(7)-4608\zeta(2)\zeta(5)-22824\zeta(3)\zeta(4)$$
$$-65621\zeta(8)-17640\zeta(2)\zeta(3)^2+72432\zeta(3)\zeta(5)+15480M(2,6)+12292\zeta(9)$$
$$-25164\zeta(3)\zeta(6)+11664\zeta(4)\zeta(5)+3906\zeta(2)\zeta(7)-1072\zeta(3)^3\big) \quad (562)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^7}{k^2} = \frac{1}{72}\big(276341\zeta(9)+88665\zeta(3)\zeta(6)+143163\zeta(4)\zeta(5)+59166\zeta(2)\zeta(7)$$
$$+4032\zeta(3)^3\big) \quad (563)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^7}{k(k+1)} = \frac{1}{18}\big(119774\zeta(8)+3024\zeta(2)\zeta(3)^2+27405\zeta(3)\zeta(5)\big) \quad (564)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^7}{(k+1)^2} = \frac{1}{72}\left(269402\zeta(9)+88665\zeta(3)\zeta(6)+141273\zeta(4)\zeta(5)+59166\zeta(2)\zeta(7)\right.$$
$$\left.+4032\zeta(3)^3\right) \quad (565)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^7}{k(k+2)} = \frac{1}{288}\left(144+864\zeta(2)+7200\zeta(3)+38664\zeta(4)+108504\zeta(5)+23184\zeta(2)\zeta(3)\right.$$
$$+352887\zeta(6)+45864\zeta(3)^2+319554\zeta(7)+73080\zeta(2)\zeta(5)+148932\zeta(3)\zeta(4)$$
$$\left.+958192\zeta(8)+24192\zeta(2)\zeta(3)^2+219240\zeta(3)\zeta(5)\right) \quad (566)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^7}{(k+1)(k+2)} = \frac{1}{48}\left(48+288\zeta(2)+2400\zeta(3)+12888\zeta(4)+36168\zeta(5)+7728\zeta(2)\zeta(3)\right.$$
$$\left.+117629\zeta(6)+15288\zeta(3)^2+106518\zeta(7)+24360\zeta(2)\zeta(5)+49644\zeta(3)\zeta(4)\right) \quad (567)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^7}{(k+2)^2} = \frac{1}{144}\left(-1152-5040\zeta(2)-34992\zeta(3)-146340\zeta(4)-287712\zeta(5)\right.$$
$$-64512\zeta(2)\zeta(3)-525384\zeta(6)-76608\zeta(3)^2+31041\zeta(7)+13104\zeta(2)\zeta(5)$$
$$-49140\zeta(3)\zeta(4)+715134\zeta(8)-38808\zeta(2)\zeta(3)^2+475776\zeta(3)\zeta(5)+54936M(2,6)$$
$$+538804\zeta(9)+177330\zeta(3)\zeta(6)+282546\zeta(4)\zeta(5)+118332\zeta(2)\zeta(7)$$
$$\left.+8064\zeta(3)^3\right) \quad (568)$$

Formulas for order $r = m + n + p + q = 10$:

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^9} = \frac{1}{4}\left(11\zeta(10) - 4\zeta(3)\zeta(7) - 2\zeta(5)^2\right) \tag{569}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^8(k+1)} = \frac{1}{4}\left(4\zeta(2) - 8\zeta(3) + 5\zeta(4) - 12\zeta(5) + 4\zeta(2)\zeta(3) + 7\zeta(6) - 2\zeta(3)^2\right.$$
$$-16\zeta(7) + 4\zeta(2)\zeta(5) + 4\zeta(3)\zeta(4) + 9\zeta(8) - 4\zeta(3)\zeta(5) - 20\zeta(9)$$
$$\left.+4\zeta(3)\zeta(6) + 4\zeta(4)\zeta(5) + 4\zeta(2)\zeta(7)\right) \tag{570}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^7(k+1)^2} = \frac{1}{4}\left(28\zeta(2) - 52\zeta(3) + 25\zeta(4) - 48\zeta(5) + 16\zeta(2)\zeta(3) + 21\zeta(6)\right.$$
$$\left.-6\zeta(3)^2 - 32\zeta(7) + 8\zeta(2)\zeta(5) + 8\zeta(3)\zeta(4) + 9\zeta(8) - 4\zeta(3)\zeta(5)\right) \tag{571}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^6(k+1)^3} = \frac{1}{4}\left(-84\zeta(2) + 144\zeta(3) - 49\zeta(4) + 72\zeta(5) - 24\zeta(2)\zeta(3) - 21\zeta(6)\right.$$
$$\left.+6\zeta(3)^2 + 16\zeta(7) - 4\zeta(2)\zeta(5) - 4\zeta(3)\zeta(4)\right) \tag{572}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^5(k+1)^4} = \frac{1}{4}\left(140\zeta(2) - 220\zeta(3) + 45\zeta(4) - 56\zeta(5) + 20\zeta(2)\zeta(3) + 7\zeta(6)\right.$$
$$\left.-2\zeta(3)^2\right) \tag{573}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^4(k+1)^5} = \frac{1}{4}\left(-140\zeta(2) + 200\zeta(3) - 15\zeta(4) + 44\zeta(5) - 20\zeta(2)\zeta(3) + 3\zeta(6)\right.$$
$$\left.-2\zeta(3)^2\right) \tag{574}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^3(k+1)^6} = \frac{1}{4}\left(84\zeta(2) - 108\zeta(3) - 5\zeta(4) - 48\zeta(5) + 24\zeta(2)\zeta(3) - 9\zeta(6)\right.$$
$$\left.+6\zeta(3)^2 - 12\zeta(7) + 4\zeta(2)\zeta(5) + 4\zeta(3)\zeta(4)\right) \tag{575}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^2(k+1)^7} = \frac{1}{4}\left(28\zeta(2) - 32\zeta(3) - 5\zeta(4) - 32\zeta(5) + 16\zeta(2)\zeta(3) - 9\zeta(6)\right.$$
$$\left.+6\zeta(3)^2 - 24\zeta(7) + 8\zeta(2)\zeta(5) + 8\zeta(3)\zeta(4) - 5\zeta(8) + 4\zeta(3)\zeta(5)\right) \tag{576}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k(k+1)^8} = \frac{1}{4}\left(4\zeta(2) - 4\zeta(3) - \zeta(4) - 8\zeta(5) + 4\zeta(2)\zeta(3) - 3\zeta(6) + 2\zeta(3)^2\right.$$
$$-12\zeta(7) + 4\zeta(2)\zeta(5) + 4\zeta(3)\zeta(4) - 5\zeta(8) + 4\zeta(3)\zeta(5) - 16\zeta(9)$$
$$\left.+4\zeta(3)\zeta(6) + 4\zeta(4)\zeta(5) + 4\zeta(2)\zeta(7)\right) \tag{577}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{(k+1)^9} = \frac{1}{4}\left(7\zeta(10) - 4\zeta(3)\zeta(7) - 2\zeta(5)^2\right) \tag{578}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^8(k+2)} = \frac{1}{256}\left(-1 - \zeta(2) + 4\zeta(3) - 5\zeta(4) + 24\zeta(5) - 8\zeta(2)\zeta(3) - 28\zeta(6)\right.$$
$$+8\zeta(3)^2 + 128\zeta(7) - 32\zeta(2)\zeta(5) - 32\zeta(3)\zeta(4) - 144\zeta(8) + 64\zeta(3)\zeta(5)$$
$$\left.+640\zeta(9) - 128\zeta(3)\zeta(6) - 128\zeta(4)\zeta(5) - 128\zeta(2)\zeta(7)\right) \tag{579}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^7(k+1)(k+2)} = \frac{1}{128} (1 - 127\zeta(2) + 252\zeta(3) - 155\zeta(4) + 360\zeta(5) - 120\zeta(2)\zeta(3) -196\zeta(6) + 56\zeta(3)^2 + 384\zeta(7) - 96\zeta(2)\zeta(5) - 96\zeta(3)\zeta(4) - 144\zeta(8) +64\zeta(3)\zeta(5)) \tag{580}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^6(k+1)^2(k+2)} = \frac{1}{64} \big(1 + 321\zeta(2) - 580\zeta(3) + 245\zeta(4) - 408\zeta(5) + 136\zeta(2)\zeta(3) +140\zeta(6) - 40\zeta(3)^2 - 128\zeta(7) + 32\zeta(2)\zeta(5) + 32\zeta(3)\zeta(4)\big) \tag{581}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^5(k+1)^3(k+2)} = \frac{1}{32} \big(-1 + 351\zeta(2) - 572\zeta(3) + 147\zeta(4) - 168\zeta(5) + 56\zeta(2)\zeta(3) +28\zeta(6) - 8\zeta(3)^2\big) \tag{582}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^4(k+1)^4(k+2)} = \frac{1}{16} (1 + 209\zeta(2) - 308\zeta(3) + 33\zeta(4) - 56\zeta(5) + 24\zeta(2)\zeta(3)) \tag{583}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^3(k+1)^5(k+2)} = \frac{1}{8} \big(-1 + 71\zeta(2) - 92\zeta(3) - 3\zeta(4) - 32\zeta(5) + 16\zeta(2)\zeta(3) - 6\zeta(6) +4\zeta(3)^2\big) \tag{584}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^2(k+1)^6(k+2)} = \frac{1}{4} \big(-1 - 13\zeta(2) + 16\zeta(3) + 2\zeta(4) + 16\zeta(5) - 8\zeta(2)\zeta(3) + 3\zeta(6) -2\zeta(3)^2 + 12\zeta(7) - 4\zeta(2)\zeta(5) - 4\zeta(3)\zeta(4)\big) \tag{585}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k(k+1)^7(k+2)} = \frac{1}{4} \big(-2 + 2\zeta(2) - \zeta(4) - 3\zeta(6) + 2\zeta(3)^2 - 5\zeta(8) +4\zeta(3)\zeta(5)\big) \tag{586}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{(k+1)^8(k+2)} = \frac{1}{4} \big(4 - 4\zeta(3) + \zeta(4) - 8\zeta(5) + 4\zeta(2)\zeta(3) + 3\zeta(6) - 2\zeta(3)^2 -12\zeta(7) + 4\zeta(2)\zeta(5) + 4\zeta(3)\zeta(4) + 5\zeta(8) - 4\zeta(3)\zeta(5) - 16\zeta(9) +4\zeta(3)\zeta(6) + 4\zeta(4)\zeta(5) + 4\zeta(2)\zeta(7)\big) \tag{587}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^7(k+2)^2} = \frac{1}{256} (11 + 5\zeta(2) - 26\zeta(3) + 25\zeta(4) - 96\zeta(5) + 32\zeta(2)\zeta(3) + 84\zeta(6) -24\zeta(3)^2 - 256\zeta(7) + 64\zeta(2)\zeta(5) + 64\zeta(3)\zeta(4) + 144\zeta(8) -64\zeta(3)\zeta(5)) \tag{588}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^6(k+1)(k+2)^2} = \frac{1}{64} \big(6 - 61\zeta(2) + 113\zeta(3) - 65\zeta(4) + 132\zeta(5) - 44\zeta(2)\zeta(3) -56\zeta(6) + 16\zeta(3)^2 + 64\zeta(7) - 16\zeta(2)\zeta(5) - 16\zeta(3)\zeta(4)\big) \tag{589}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^5(k+1)^2(k+2)^2} = \frac{1}{64}\left(13 + 199\zeta(2) - 354\zeta(3) + 115\zeta(4) - 144\zeta(5) + 48\zeta(2)\zeta(3) \right.$$
$$\left. +28\zeta(6) - 8\zeta(3)^2\right) \quad (590)$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^4(k+1)^3(k+2)^2} = \frac{1}{16}\left(7 - 76\zeta(2) + 109\zeta(3) - 16\zeta(4) + 12\zeta(5) \right.$$
$$\left. -4\zeta(2)\zeta(3)\right) \quad (591)$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^3(k+1)^4(k+2)^2} = \frac{1}{16}\left(15 + 57\zeta(2) - 90\zeta(3) + \zeta(4) - 32\zeta(5) \right.$$
$$\left. +16\zeta(2)\zeta(3)\right) \quad (592)$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^2(k+1)^5(k+2)^2} = \frac{1}{4}\left(8 - 7\zeta(2) + \zeta(3) + 2\zeta(4) + 3\zeta(6) - 2\zeta(3)^2\right) \quad (593)$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k(k+1)^6(k+2)^2} = \frac{1}{4}\left(17 - \zeta(2) - 14\zeta(3) + 2\zeta(4) - 16\zeta(5) + 8\zeta(2)\zeta(3) + 3\zeta(6) \right.$$
$$\left. -2\zeta(3)^2 - 12\zeta(7) + 4\zeta(2)\zeta(5) + 4\zeta(3)\zeta(4)\right) \quad (594)$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{(k+1)^7(k+2)^2} = \frac{1}{4}\left(36 - 4\zeta(2) - 28\zeta(3) + 5\zeta(4) - 32\zeta(5) + 16\zeta(2)\zeta(3) + 9\zeta(6) \right.$$
$$\left. -6\zeta(3)^2 - 24\zeta(7) + 8\zeta(2)\zeta(5) + 8\zeta(3)\zeta(4) + 5\zeta(8) - 4\zeta(3)\zeta(5)\right) \quad (595)$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^6(k+2)^3} = \frac{1}{256}\left(57 + 5\zeta(2) - 76\zeta(3) + 49\zeta(4) - 144\zeta(5) + 48\zeta(2)\zeta(3) + 84\zeta(6) \right.$$
$$\left. -24\zeta(3)^2 - 128\zeta(7) + 32\zeta(2)\zeta(5) + 32\zeta(3)\zeta(4)\right) \quad (596)$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^5(k+1)(k+2)^3} = \frac{1}{128}\left(69 - 117\zeta(2) + 150\zeta(3) - 81\zeta(4) + 120\zeta(5) - 40\zeta(2)\zeta(3) \right.$$
$$\left. -28\zeta(6) + 8\zeta(3)^2\right) \quad (597)$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^4(k+1)^2(k+2)^3} = \frac{1}{32}\left(41 + 41\zeta(2) - 102\zeta(3) + 17\zeta(4) - 12\zeta(5) \right.$$
$$\left. +4\zeta(2)\zeta(3)\right) \quad (598)$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^3(k+1)^3(k+2)^3} = \frac{1}{16}\left(-48 + 35\zeta(2) - 7\zeta(3) - \zeta(4)\right) \quad (599)$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^2(k+1)^4(k+2)^3} = \frac{1}{16}\left(111 - 13\zeta(2) - 76\zeta(3) + 3\zeta(4) - 32\zeta(5) \right.$$
$$\left. +16\zeta(2)\zeta(3)\right) \quad (600)$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k(k+1)^5(k+2)^3} = \frac{1}{8}\left(-127 + 27\zeta(2) + 74\zeta(3) - 7\zeta(4) + 32\zeta(5) - 16\zeta(2)\zeta(3) \right.$$
$$\left. -6\zeta(6) + 4\zeta(3)^2\right) \quad (601)$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{(k+1)^6(k+2)^3} = \frac{1}{4}\big(-144+28\zeta(2)+88\zeta(3)-9\zeta(4)+48\zeta(5)-24\zeta(2)\zeta(3)$$
$$-9\zeta(6)+6\zeta(3)^2+12\zeta(7)-4\zeta(2)\zeta(5)-4\zeta(3)\zeta(4)\big) \quad (602)$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^5(k+2)^4} = \frac{1}{256}\big(187-23\zeta(2)-138\zeta(3)+37\zeta(4)-112\zeta(5)+40\zeta(2)\zeta(3)$$
$$+28\zeta(6)-8\zeta(3)^2\big) \quad (603)$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^4(k+1)(k+2)^4} = \frac{1}{32}\left(64-35\zeta(2)+3\zeta(3)-11\zeta(4)+2\zeta(5)\right) \quad (604)$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^3(k+1)^2(k+2)^4} = \frac{1}{32}\big(169-29\zeta(2)-96\zeta(3)-5\zeta(4)-8\zeta(5)$$
$$+4\zeta(2)\zeta(3)\big) \quad (605)$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^2(k+1)^3(k+2)^4} = \frac{1}{16}\big(217-64\zeta(2)-89\zeta(3)-4\zeta(4)-8\zeta(5)$$
$$+4\zeta(2)\zeta(3)\big) \quad (606)$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k(k+1)^4(k+2)^4} = \frac{1}{16}\big(545-141\zeta(2)-254\zeta(3)-5\zeta(4)-48\zeta(5)$$
$$+24\zeta(2)\zeta(3)\big) \quad (607)$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{(k+1)^5(k+2)^4} = \frac{1}{4}\big(336-84\zeta(2)-164\zeta(3)+\zeta(4)-40\zeta(5)+20\zeta(2)\zeta(3)$$
$$+3\zeta(6)-2\zeta(3)^2\big) \quad (608)$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^4(k+2)^5} = \frac{1}{256}\big(443-93\zeta(2)-188\zeta(3)-33\zeta(4)-104\zeta(5)+40\zeta(2)\zeta(3)$$
$$-12\zeta(6)+8\zeta(3)^2\big) \quad (609)$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^3(k+1)(k+2)^5} = \frac{1}{128}\big(-699+233\zeta(2)+176\zeta(3)+77\zeta(4)+96\zeta(5)-40\zeta(2)\zeta(3)$$
$$+12\zeta(6)-8\zeta(3)^2\big) \quad (610)$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^2(k+1)^2(k+2)^5} = \frac{1}{64}\big(1037-291\zeta(2)-368\zeta(3)-87\zeta(4)-112\zeta(5)+48\zeta(2)\zeta(3)$$
$$-12\zeta(6)+8\zeta(3)^2\big) \quad (611)$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k(k+1)^3(k+2)^5} = \frac{1}{32}\big(1471-419\zeta(2)-546\zeta(3)-95\zeta(4)-128\zeta(5)+56\zeta(2)\zeta(3)$$
$$-12\zeta(6)+8\zeta(3)^2\big) \quad (612)$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{(k+1)^4(k+2)^5} = \frac{1}{4}\left(-504 + 140\zeta(2) + 200\zeta(3) + 25\zeta(4) + 44\zeta(5) - 20\zeta(2)\zeta(3)\right.$$
$$\left. +3\zeta(6) - 2\zeta(3)^2\right) \quad (613)$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^3(k+2)^6} = \frac{1}{256}\left(825 - 177\zeta(2) - 222\zeta(3) - 133\zeta(4) - 176\zeta(5) + 48\zeta(2)\zeta(3)\right.$$
$$\left. -68\zeta(6) + 24\zeta(3)^2 - 96\zeta(7) + 32\zeta(2)\zeta(5) + 32\zeta(3)\zeta(4)\right) \quad (614)$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^2(k+1)(k+2)^6} = \frac{1}{64}\left(762 - 205\zeta(2) - 199\zeta(3) - 105\zeta(4) - 136\zeta(5) + 44\zeta(2)\zeta(3)\right.$$
$$\left. -40\zeta(6) + 16\zeta(3)^2 - 48\zeta(7) + 16\zeta(2)\zeta(5) + 16\zeta(3)\zeta(4)\right) \quad (615)$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k(k+1)^2(k+2)^6} = \frac{1}{64}\left(2561 - 701\zeta(2) - 766\zeta(3) - 297\zeta(4) - 384\zeta(5) + 136\zeta(2)\zeta(3)\right.$$
$$\left. -92\zeta(6) + 40\zeta(3)^2 - 96\zeta(7) + 32\zeta(2)\zeta(5) + 32\zeta(3)\zeta(4)\right) \quad (616)$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{(k+1)^3(k+2)^6} = \frac{1}{4}\left(504 - 140\zeta(2) - 164\zeta(3) - 49\zeta(4) - 64\zeta(5) + 24\zeta(2)\zeta(3)\right.$$
$$\left. -13\zeta(6) + 6\zeta(3)^2 - 12\zeta(7) + 4\zeta(2)\zeta(5) + 4\zeta(3)\zeta(4)\right) \quad (617)$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^2(k+2)^7} = \frac{1}{256}\left(1291 - 233\zeta(2) - 244\zeta(3) - 213\zeta(4) - 240\zeta(5) + 32\zeta(2)\zeta(3)\right.$$
$$-164\zeta(6) + 24\zeta(3)^2 - 256\zeta(7) + 64\zeta(2)\zeta(5) + 64\zeta(3)\zeta(4) - 80\zeta(8)$$
$$\left. +64\zeta(3)\zeta(5)\right) \quad (618)$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k(k+1)(k+2)^7} = \frac{1}{128}\left(-2815 + 643\zeta(2) + 642\zeta(3) + 423\zeta(4) + 512\zeta(5) - 120\zeta(2)\zeta(3)\right.$$
$$+244\zeta(6) - 56\zeta(3)^2 + 352\zeta(7) - 96\zeta(2)\zeta(5) - 96\zeta(3)\zeta(4) + 80\zeta(8)$$
$$\left. -64\zeta(3)\zeta(5)\right) \quad (619)$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{(k+1)^2(k+2)^7} = \frac{1}{4}\left(336 - 84\zeta(2) - 88\zeta(3) - 45\zeta(4) - 56\zeta(5) + 16\zeta(2)\zeta(3)\right.$$
$$-21\zeta(6) + 6\zeta(3)^2 - 28\zeta(7) + 8\zeta(2)\zeta(5) + 8\zeta(3)\zeta(4) - 5\zeta(8)$$
$$\left. +4\zeta(3)\zeta(5)\right) \quad (620)$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k(k+2)^8} = \frac{1}{256}\left(1793 - 253\zeta(2) - 254\zeta(3) - 249\zeta(4) - 256\zeta(5) + 8\zeta(2)\zeta(3)\right.$$
$$-236\zeta(6) + 8\zeta(3)^2 - 288\zeta(7) + 32\zeta(2)\zeta(5) + 32\zeta(3)\zeta(4) - 208\zeta(8)$$
$$\left. +64\zeta(3)\zeta(5) - 512\zeta(9) + 128\zeta(3)\zeta(6) + 128\zeta(4)\zeta(5) + 128\zeta(2)\zeta(7)\right) \quad (621)$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{(k+1)(k+2)^8} = \frac{1}{4}\left(144 - 28\zeta(2) - 28\zeta(3) - 21\zeta(4) - 24\zeta(5) + 4\zeta(2)\zeta(3) - 15\zeta(6)\right.$$
$$+2\zeta(3)^2 - 20\zeta(7) + 4\zeta(2)\zeta(5) + 4\zeta(3)\zeta(4) - 9\zeta(8) + 4\zeta(3)\zeta(5)$$
$$\left. -16\zeta(9) + 4\zeta(3)\zeta(6) + 4\zeta(4)\zeta(5) + 4\zeta(2)\zeta(7)\right) \quad (622)$$

$$\sum_{k=1}^{\infty}\frac{H(k)}{(k+2)^9} = \frac{1}{4}\left(-36+4\zeta(2)+4\zeta(3)+4\zeta(4)+4\zeta(5)+4\zeta(6)+4\zeta(7)+4\zeta(8)\right.$$
$$\left.+4\zeta(9)+7\zeta(10)-4\zeta(3)\zeta(7)-2\zeta(5)^2\right) \tag{623}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^2}{k^8} = M(2,8) \tag{624}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^2}{k^7(k+1)} = \frac{1}{24}\left(72\zeta(3)-102\zeta(4)+84\zeta(5)-24\zeta(2)\zeta(3)-97\zeta(6)+48\zeta(3)^2\right.$$
$$+144\zeta(7)-24\zeta(2)\zeta(5)-60\zeta(3)\zeta(4)-24M(2,6)+220\zeta(9)-84\zeta(3)\zeta(6)$$
$$\left.-60\zeta(4)\zeta(5)-24\zeta(2)\zeta(7)+8\zeta(3)^3\right) \tag{625}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^2}{k^6(k+1)^2} = \frac{1}{8}\left(144\zeta(3)-192\zeta(4)+112\zeta(5)-32\zeta(2)\zeta(3)-97\zeta(6)\right.$$
$$\left.+48\zeta(3)^2+96\zeta(7)-16\zeta(2)\zeta(5)-40\zeta(3)\zeta(4)-8M(2,6)\right) \tag{626}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^2}{k^5(k+1)^3} = \frac{1}{8}\left(360\zeta(3)-450\zeta(4)+180\zeta(5)-56\zeta(2)\zeta(3)-97\zeta(6)\right.$$
$$\left.+48\zeta(3)^2+48\zeta(7)-8\zeta(2)\zeta(5)-20\zeta(3)\zeta(4)\right) \tag{627}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^2}{k^4(k+1)^4} = \frac{1}{12}\left(-720\zeta(3)+840\zeta(4)-240\zeta(5)+96\zeta(2)\zeta(3)+67\zeta(6)\right.$$
$$\left.-36\zeta(3)^2\right) \tag{628}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^2}{k^3(k+1)^5} = \frac{1}{8}\left(360\zeta(3)-390\zeta(4)+100\zeta(5)-56\zeta(2)\zeta(3)-37\zeta(6)\right.$$
$$\left.+24\zeta(3)^2+8\zeta(7)-8\zeta(2)\zeta(5)+4\zeta(3)\zeta(4)\right) \tag{629}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^2}{k^2(k+1)^6} = \frac{1}{8}\left(-144\zeta(3)+144\zeta(4)-48\zeta(5)+32\zeta(2)\zeta(3)+37\zeta(6)\right.$$
$$-24\zeta(3)^2-16\zeta(7)+16\zeta(2)\zeta(5)-8\zeta(3)\zeta(4)-28\zeta(8)+16\zeta(3)\zeta(5)$$
$$\left.+8M(2,6)\right) \tag{630}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^2}{k(k+1)^7} = \frac{1}{24}\left(72\zeta(3)-66\zeta(4)+36\zeta(5)-24\zeta(2)\zeta(3)-37\zeta(6)+24\zeta(3)^2\right.$$
$$+24\zeta(7)-24\zeta(2)\zeta(5)+12\zeta(3)\zeta(4)+84\zeta(8)-48\zeta(3)\zeta(5)-24M(2,6)$$
$$\left.-4\zeta(9)+36\zeta(3)\zeta(6)+12\zeta(4)\zeta(5)-24\zeta(2)\zeta(7)-8\zeta(3)^3\right) \tag{631}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^2}{(k+1)^8} = \frac{1}{2}\left(9\zeta(10)-4\zeta(3)\zeta(7)-2\zeta(5)^2-2M(2,8)\right) \tag{632}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^2}{k^7(k+2)} = \frac{1}{768}\left(6+6\zeta(2)+18\zeta(3)-51\zeta(4)+84\zeta(5)-24\zeta(2)\zeta(3)-194\zeta(6)\right.$$
$$+96\zeta(3)^2+576\zeta(7)-96\zeta(2)\zeta(5)-240\zeta(3)\zeta(4)-192M(2,6)+3520\zeta(9)$$
$$\left.-1344\zeta(3)\zeta(6)-960\zeta(4)\zeta(5)-384\zeta(2)\zeta(7)+128\zeta(3)^3\right) \tag{633}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k^6(k+1)(k+2)} = \frac{1}{384}\left(6+6\zeta(2)-1134\zeta(3)+1581\zeta(4)-1260\zeta(5)+360\zeta(2)\zeta(3)\right.$$
$$+1358\zeta(6)-672\zeta(3)^2-1728\zeta(7)+288\zeta(2)\zeta(5)+720\zeta(3)\zeta(4)$$
$$\left.+192M(2,6)\right) \tag{634}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k^5(k+1)^2(k+2)} = \frac{1}{192}\left(6+6\zeta(2)+2322\zeta(3)-3027\zeta(4)+1428\zeta(5)\right.$$
$$-408\zeta(2)\zeta(3)-970\zeta(6)+480\zeta(3)^2+576\zeta(7)-96\zeta(2)\zeta(5)$$
$$\left.-240\zeta(3)\zeta(4)\right) \tag{635}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k^4(k+1)^3(k+2)} = \frac{1}{96}\left(6+6\zeta(2)-1998\zeta(3)+2373\zeta(4)-732\zeta(5)+264\zeta(2)\zeta(3)\right.$$
$$\left.+194\zeta(6)-96\zeta(3)^2\right) \tag{636}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k^3(k+1)^4(k+2)} = \frac{1}{48}\left(6+6\zeta(2)+882\zeta(3)-987\zeta(4)+228\zeta(5)-120\zeta(2)\zeta(3)\right.$$
$$\left.-74\zeta(6)+48\zeta(3)^2\right) \tag{637}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k^2(k+1)^5(k+2)} = \frac{1}{24}\left(6+6\zeta(2)-198\zeta(3)+183\zeta(4)-72\zeta(5)+48\zeta(2)\zeta(3)\right.$$
$$\left.+37\zeta(6)-24\zeta(3)^2-24\zeta(7)+24\zeta(2)\zeta(5)-12\zeta(3)\zeta(4)\right) \tag{638}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k(k+1)^6(k+2)} = \frac{1}{24}\left(12+12\zeta(2)+36\zeta(3)-66\zeta(4)-37\zeta(6)+24\zeta(3)^2+84\zeta(8)\right.$$
$$\left.-48\zeta(3)\zeta(5)-24M(2,6)\right) \tag{639}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{(k+1)^7(k+2)} = \frac{1}{24}\left(24+24\zeta(2)-66\zeta(4)-36\zeta(5)+24\zeta(2)\zeta(3)-37\zeta(6)\right.$$
$$+24\zeta(3)^2-24\zeta(7)+24\zeta(2)\zeta(5)-12\zeta(3)\zeta(4)+84\zeta(8)-48\zeta(3)\zeta(5)$$
$$\left.-24M(2,6)+4\zeta(9)-36\zeta(3)\zeta(6)-12\zeta(4)\zeta(5)+24\zeta(2)\zeta(7)+8\zeta(3)^3\right) \tag{640}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k^6(k+2)^2} = \frac{1}{128}\left(12+6\zeta(2)+14\zeta(3)-48\zeta(4)+56\zeta(5)-16\zeta(2)\zeta(3)\right.$$
$$\left.-97\zeta(6)+48\zeta(3)^2+192\zeta(7)-32\zeta(2)\zeta(5)-80\zeta(3)\zeta(4)-32M(2,6)\right) \tag{641}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k^5(k+1)(k+2)^2} = \frac{1}{384}\left(78+42\zeta(2)-1050\zeta(3)+1293\zeta(4)-924\zeta(5)\right.$$
$$+264\zeta(2)\zeta(3)+776\zeta(6)-384\zeta(3)^2-576\zeta(7)+96\zeta(2)\zeta(5)$$
$$\left.+240\zeta(3)\zeta(4)\right) \tag{642}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k^4(k+1)^2(k+2)^2} = \frac{1}{96}\left(42+24\zeta(2)+636\zeta(3)-867\zeta(4)+252\zeta(5)\right.$$
$$\left.-72\zeta(2)\zeta(3)-97\zeta(6)+48\zeta(3)^2\right) \tag{643}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k^3(k+1)^3(k+2)^2} = \frac{1}{32}\left(30+18\zeta(2)-242\zeta(3)+213\zeta(4)-76\zeta(5)\right.$$
$$\left.+40\zeta(2)\zeta(3)\right) \quad (644)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k^2(k+1)^4(k+2)^2} = \frac{1}{24}\left(-48-30\zeta(2)-78\zeta(3)+174\zeta(4)+37\zeta(6)\right.$$
$$\left.-24\zeta(3)^2\right) \quad (645)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k(k+1)^5(k+2)^2} = \frac{1}{24}\left(-102-66\zeta(2)+42\zeta(3)+165\zeta(4)+72\zeta(5)-48\zeta(2)\zeta(3)\right.$$
$$\left.+37\zeta(6)-24\zeta(3)^2+24\zeta(7)-24\zeta(2)\zeta(5)+12\zeta(3)\zeta(4)\right) \quad (646)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{(k+1)^6(k+2)^2} = \frac{1}{8}\left(-72-48\zeta(2)+16\zeta(3)+132\zeta(4)+48\zeta(5)-32\zeta(2)\zeta(3)\right.$$
$$+37\zeta(6)-24\zeta(3)^2+16\zeta(7)-16\zeta(2)\zeta(5)+8\zeta(3)\zeta(4)-28\zeta(8)$$
$$\left.+16\zeta(3)\zeta(5)+8M(2,6)\right) \quad (647)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k^5(k+2)^3} = \frac{1}{256}\left(138+22\zeta(2)+26\zeta(3)-229\zeta(4)+180\zeta(5)-56\zeta(2)\zeta(3)\right.$$
$$\left.-194\zeta(6)+96\zeta(3)^2+192\zeta(7)-32\zeta(2)\zeta(5)-80\zeta(3)\zeta(4)\right) \quad (648)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k^4(k+1)(k+2)^3} = \frac{1}{192}\left(246+54\zeta(2)-486\zeta(3)+303\zeta(4)-192\zeta(5)+48\zeta(2)\zeta(3)\right.$$
$$\left.+97\zeta(6)-48\zeta(3)^2\right) \quad (649)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k^3(k+1)^2(k+2)^3} = \frac{1}{16}\left(48+13\zeta(2)+25\zeta(3)-94\zeta(4)+10\zeta(5)\right.$$
$$\left.-4\zeta(2)\zeta(3)\right) \quad (650)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k^2(k+1)^3(k+2)^3} = \frac{1}{32}\left(222+70\zeta(2)-142\zeta(3)-163\zeta(4)-36\zeta(5)\right.$$
$$\left.+24\zeta(2)\zeta(3)\right) \quad (651)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k(k+1)^4(k+2)^3} = \frac{1}{48}\left(762+270\zeta(2)-270\zeta(3)-837\zeta(4)-108\zeta(5)+72\zeta(2)\zeta(3)\right.$$
$$\left.-74\zeta(6)+48\zeta(3)^2\right) \quad (652)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{(k+1)^5(k+2)^3} = \frac{1}{8}\left(288+112\zeta(2)-104\zeta(3)-334\zeta(4)-60\zeta(5)+40\zeta(2)\zeta(3)\right.$$
$$\left.-37\zeta(6)+24\zeta(3)^2-8\zeta(7)+8\zeta(2)\zeta(5)-4\zeta(3)\zeta(4)\right) \quad (653)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k^4(k+2)^4} = \frac{1}{192}\left(-384+18\zeta(2)+90\zeta(3)+240\zeta(4)-72\zeta(5)+24\zeta(2)\zeta(3)\right.$$
$$\left.+67\zeta(6)-36\zeta(3)^2\right) \quad (654)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k^3(k+1)(k+2)^4} = \frac{1}{192}\left(-1014 - 18\zeta(2) + 666\zeta(3) + 177\zeta(4) + 48\zeta(5) + 37\zeta(6) -24\zeta(3)^2\right) \quad (655)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k^2(k+1)^2(k+2)^4} = \frac{1}{96}\left(-1302 - 96\zeta(2) + 516\zeta(3) + 741\zeta(4) - 12\zeta(5) +24\zeta(2)\zeta(3) + 37\zeta(6) - 24\zeta(3)^2\right) \quad (656)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k(k+1)^3(k+2)^4} = \frac{1}{96}\left(-3270 - 402\zeta(2) + 1458\zeta(3) + 1971\zeta(4) + 84\zeta(5) -24\zeta(2)\zeta(3) + 74\zeta(6) - 48\zeta(3)^2\right) \quad (657)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{(k+1)^4(k+2)^4} = \frac{1}{12}\left(1008 + 168\zeta(2) - 432\zeta(3) - 702\zeta(4) - 48\zeta(5) + 24\zeta(2)\zeta(3) -37\zeta(6) + 24\zeta(3)^2\right) \quad (658)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k^3(k+2)^5} = \frac{1}{256}\left(1398 - 210\zeta(2) - 486\zeta(3) - 371\zeta(4) - 252\zeta(5) + 104\zeta(2)\zeta(3) -122\zeta(6) + 80\zeta(3)^2 + 32\zeta(7) - 32\zeta(2)\zeta(5) + 16\zeta(3)\zeta(4)\right) \quad (659)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k^2(k+1)(k+2)^5} = \frac{1}{384}(6222 - 594\zeta(2) - 2790\zeta(3) - 1467\zeta(4) - 852\zeta(5) +312\zeta(2)\zeta(3) - 440\zeta(6) + 288\zeta(3)^2 + 96\zeta(7) - 96\zeta(2)\zeta(5) +48\zeta(3)\zeta(4)) \quad (660)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k(k+1)^2(k+2)^5} = \frac{1}{192}(8826 - 402\zeta(2) - 3822\zeta(3) - 2949\zeta(4) - 828\zeta(5) +264\zeta(2)\zeta(3) - 514\zeta(6) + 336\zeta(3)^2 + 96\zeta(7) - 96\zeta(2)\zeta(5) +48\zeta(3)\zeta(4)) \quad (661)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{(k+1)^3(k+2)^5} = \frac{1}{8}\left(1008 - 440\zeta(3) - 410\zeta(4) - 76\zeta(5) + 24\zeta(2)\zeta(3) - 49\zeta(6) +32\zeta(3)^2 + 8\zeta(7) - 8\zeta(2)\zeta(5) + 4\zeta(3)\zeta(4)\right) \quad (662)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k^2(k+2)^6} = \frac{1}{128}(-1524 + 282\zeta(2) + 498\zeta(3) + 272\zeta(4) + 424\zeta(5) - 160\zeta(2)\zeta(3) +165\zeta(6) - 88\zeta(3)^2 + 160\zeta(7) - 32\zeta(2)\zeta(5) - 80\zeta(3)\zeta(4) - 112\zeta(8) +64\zeta(3)\zeta(5) + 32M(2,6)) \quad (663)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k(k+1)(k+2)^6} = \frac{1}{384}(15366 - 2286\zeta(2) - 5778\zeta(3) - 3099\zeta(4) - 3396\zeta(5) +1272\zeta(2)\zeta(3) - 1430\zeta(6) + 816\zeta(3)^2 - 864\zeta(7) + 96\zeta(2)\zeta(5) +528\zeta(3)\zeta(4) + 672\zeta(8) - 384\zeta(3)\zeta(5) - 192M(2,6)) \quad (664)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^2}{(k+1)^2(k+2)^6} = \frac{1}{8}\left(-1008+112\zeta(2)+400\zeta(3)+252\zeta(4)+176\zeta(5)-64\zeta(2)\zeta(3)\right.$$
$$+81\zeta(6)-48\zeta(3)^2+32\zeta(7)-24\zeta(3)\zeta(4)-28\zeta(8)+16\zeta(3)\zeta(5)$$
$$\left.+8M(2,6)\right) \tag{665}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^2}{k(k+2)^7} = \frac{1}{768}\left(16890-3090\zeta(2)-4590\zeta(3)-2757\zeta(4)-4476\zeta(5)+1416\zeta(2)\zeta(3)\right.$$
$$-2042\zeta(6)+720\zeta(3)^2-3744\zeta(7)+1056\zeta(2)\zeta(5)+1200\zeta(3)\zeta(4)-288\zeta(8)$$
$$+384\zeta(3)\zeta(5)-192M(2,6)-64\zeta(9)+576\zeta(3)\zeta(6)+192\zeta(4)\zeta(5)$$
$$\left.-384\zeta(2)\zeta(7)-128\zeta(3)^3\right) \tag{666}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^2}{(k+1)(k+2)^7} = \frac{1}{24}\left(2016-336\zeta(2)-648\zeta(3)-366\zeta(4)-492\zeta(5)+168\zeta(2)\zeta(3)\right.$$
$$-217\zeta(6)+96\zeta(3)^2-288\zeta(7)+72\zeta(2)\zeta(5)+108\zeta(3)\zeta(4)+24\zeta(8)-24M(2,6)$$
$$\left.-4\zeta(9)+36\zeta(3)\zeta(6)+12\zeta(4)\zeta(5)-24\zeta(2)\zeta(7)-8\zeta(3)^3\right) \tag{667}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^2}{(k+2)^8} = \frac{1}{2}\left(72-12\zeta(2)-16\zeta(3)-11\zeta(4)-16\zeta(5)+4\zeta(2)\zeta(3)-9\zeta(6)\right.$$
$$+2\zeta(3)^2-16\zeta(7)+4\zeta(2)\zeta(5)+4\zeta(3)\zeta(4)-7\zeta(8)+4\zeta(3)\zeta(5)$$
$$-16\zeta(9)+4\zeta(3)\zeta(6)+4\zeta(4)\zeta(5)+4\zeta(2)\zeta(7)+9\zeta(10)-4\zeta(3)\zeta(7)$$
$$\left.-2\zeta(5)^2-2M(2,8)\right) \tag{668}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^3}{k^7} = \frac{1}{160}\left(-1661\zeta(10)+1280\zeta(3)\zeta(7)+80\zeta(3)^2\zeta(4)-560\zeta(2)\zeta(3)\zeta(5)\right.$$
$$\left.+720\zeta(5)^2+520M(2,8)\right) \tag{669}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^3}{k^6(k+1)} = \frac{1}{96}\left(960\zeta(4)-960\zeta(5)-96\zeta(2)\zeta(3)+558\zeta(6)-240\zeta(3)^2\right.$$
$$-1386\zeta(7)-192\zeta(2)\zeta(5)+1224\zeta(3)\zeta(4)-595\zeta(8)-120\zeta(2)\zeta(3)^2$$
$$+576\zeta(3)\zeta(5)+264M(2,6)-2084\zeta(9)+1164\zeta(3)\zeta(6)+1224\zeta(4)\zeta(5)$$
$$\left.-288\zeta(2)\zeta(7)-192\zeta(3)^3\right) \tag{670}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^3}{k^5(k+1)^2} = \frac{1}{96}\left(4800\zeta(4)-4560\zeta(5)-480\zeta(2)\zeta(3)+1674\zeta(6)-720\zeta(3)^2\right.$$
$$-2772\zeta(7)-384\zeta(2)\zeta(5)+2448\zeta(3)\zeta(4)-595\zeta(8)-120\zeta(2)\zeta(3)^2$$
$$\left.+576\zeta(3)\zeta(5)+264M(2,6)\right) \tag{671}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^3}{k^4(k+1)^3} = \frac{1}{16}\left(-1600\zeta(4)+1440\zeta(5)+160\zeta(2)\zeta(3)-312\zeta(6)+152\zeta(3)^2\right.$$
$$\left.+231\zeta(7)+32\zeta(2)\zeta(5)-204\zeta(3)\zeta(4)\right) \tag{672}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^3}{k^3(k+1)^4} = \frac{1}{16}\big(1600\zeta(4) - 1360\zeta(5) - 160\zeta(2)\zeta(3) + 192\zeta(6) - 136\zeta(3)^2$$
$$-119\zeta(7) - 32\zeta(2)\zeta(5) + 132\zeta(3)\zeta(4)) \quad (673)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^3}{k^2(k+1)^5} = \frac{1}{96}\big(-4800\zeta(4) + 3840\zeta(5) + 480\zeta(2)\zeta(3) - 594\zeta(6) + 576\zeta(3)^2$$
$$+1428\zeta(7) + 384\zeta(2)\zeta(5) - 1584\zeta(3)\zeta(4) - 43\zeta(8) - 120\zeta(2)\zeta(3)^2$$
$$+288\zeta(3)\zeta(5) - 24M(2,6)) \quad (674)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^3}{k(k+1)^6} = \frac{1}{96}\big(960\zeta(4) - 720\zeta(5) - 96\zeta(2)\zeta(3) + 198\zeta(6) - 192\zeta(3)^2$$
$$-714\zeta(7) - 192\zeta(2)\zeta(5) + 792\zeta(3)\zeta(4) + 43\zeta(8) + 120\zeta(2)\zeta(3)^2$$
$$-288\zeta(3)\zeta(5) + 24M(2,6) - 788\zeta(9) + 444\zeta(3)\zeta(6) + 792\zeta(4)\zeta(5)$$
$$-288\zeta(2)\zeta(7) - 96\zeta(3)^3\big) \quad (675)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^3}{(k+1)^7} = \frac{1}{160}\big(501\zeta(10) - 800\zeta(3)\zeta(7) - 80\zeta(3)^2\zeta(4) + 560\zeta(2)\zeta(3)\zeta(5)$$
$$-480\zeta(5)^2 - 40M(2,8)\big) \quad (676)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^3}{k^6(k+2)} = \frac{1}{768}(12 + 24\zeta(2) + 48\zeta(3) + 120\zeta(4) - 240\zeta(5) - 24\zeta(2)\zeta(3)$$
$$+279\zeta(6) - 120\zeta(3)^2 - 1386\zeta(7) - 192\zeta(2)\zeta(5) + 1224\zeta(3)\zeta(4) - 1190\zeta(8)$$
$$-240\zeta(2)\zeta(3)^2 + 1152\zeta(3)\zeta(5) + 528M(2,6) - 8336\zeta(9) + 4656\zeta(3)\zeta(6)$$
$$+4896\zeta(4)\zeta(5) - 1152\zeta(2)\zeta(7) - 768\zeta(3)^3\big) \quad (677)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^3}{k^5(k+1)(k+2)} = \frac{1}{384}(12 + 24\zeta(2) + 48\zeta(3) - 3720\zeta(4) + 3600\zeta(5) + 360\zeta(2)\zeta(3)$$
$$-1953\zeta(6) + 840\zeta(3)^2 + 4158\zeta(7) + 576\zeta(2)\zeta(5) - 3672\zeta(3)\zeta(4) + 1190\zeta(8)$$
$$+240\zeta(2)\zeta(3)^2 - 1152\zeta(3)\zeta(5) - 528M(2,6)\big) \quad (678)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^3}{k^4(k+1)^2(k+2)} = \frac{1}{64}(4 + 8\zeta(2) + 16\zeta(3) + 1960\zeta(4) - 1840\zeta(5) - 200\zeta(2)\zeta(3)$$
$$+465\zeta(6) - 200\zeta(3)^2 - 462\zeta(7) - 64\zeta(2)\zeta(5) + 408\zeta(3)\zeta(4)\big) \quad (679)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^3}{k^3(k+1)^3(k+2)} = \frac{1}{32}(4 + 8\zeta(2) + 16\zeta(3) - 1240\zeta(4) + 1040\zeta(5) + 120\zeta(2)\zeta(3)$$
$$-159\zeta(6) + 104\zeta(3)^2\big) \quad (680)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^3}{k^2(k+1)^4(k+2)} = \frac{1}{16}(-4 - 8\zeta(2) - 16\zeta(3) - 360\zeta(4) + 320\zeta(5) + 40\zeta(2)\zeta(3)$$
$$-33\zeta(6) + 32\zeta(3)^2 + 119\zeta(7) + 32\zeta(2)\zeta(5) - 132\zeta(3)\zeta(4)\big) \quad (681)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^3}{k(k+1)^5(k+2)} = \frac{1}{96}\big(48+96\zeta(2)+192\zeta(3)-480\zeta(4)-198\zeta(6)+192\zeta(3)^2$$
$$-43\zeta(8)-120\zeta(2)\zeta(3)^2+288\zeta(3)\zeta(5)-24M(2,6)\big) \quad (682)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^3}{(k+1)^6(k+2)} = \frac{1}{96}\big(-96-192\zeta(2)-384\zeta(3)+720\zeta(5)+96\zeta(2)\zeta(3)+198\zeta(6)$$
$$-192\zeta(3)^2+714\zeta(7)+192\zeta(2)\zeta(5)-792\zeta(3)\zeta(4)+43\zeta(8)$$
$$+120\zeta(2)\zeta(3)^2-288\zeta(3)\zeta(5)+24M(2,6)+788\zeta(9)-444\zeta(3)\zeta(6)$$
$$-792\zeta(4)\zeta(5)+288\zeta(2)\zeta(7)+96\zeta(3)^3\big) \quad (683)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^3}{k^5(k+2)^2} = \frac{1}{768}\big(156+192\zeta(2)+264\zeta(3)+402\zeta(4)-1140\zeta(5)-120\zeta(2)\zeta(3)$$
$$+837\zeta(6)-360\zeta(3)^2-2772\zeta(7)-384\zeta(2)\zeta(5)+2448\zeta(3)\zeta(4)-1190\zeta(8)$$
$$-240\zeta(2)\zeta(3)^2+1152\zeta(3)\zeta(5)+528M(2,6)\big) \quad (684)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^3}{k^4(k+1)(k+2)^2} = \frac{1}{64}\big(28+36\zeta(2)+52\zeta(3)-553\zeta(4)+410\zeta(5)+40\zeta(2)\zeta(3)$$
$$-186\zeta(6)+80\zeta(3)^2+231\zeta(7)+32\zeta(2)\zeta(5)-204\zeta(3)\zeta(4)\big) \quad (685)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^3}{k^3(k+1)^2(k+2)^2} = \frac{1}{64}\big(60+80\zeta(2)+120\zeta(3)+854\zeta(4)-1020\zeta(5)$$
$$-120\zeta(2)\zeta(3)+93\zeta(6)-40\zeta(3)^2\big) \quad (686)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^3}{k^2(k+1)^3(k+2)^2} = \frac{1}{16}\big(32+44\zeta(2)+68\zeta(3)-193\zeta(4)+10\zeta(5)-33\zeta(6)$$
$$+32\zeta(3)^2\big) \quad (687)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^3}{k(k+1)^4(k+2)^2} = \frac{1}{16}\big(68+96\zeta(2)+152\zeta(3)-26\zeta(4)-300\zeta(5)-40\zeta(2)\zeta(3)$$
$$-33\zeta(6)+32\zeta(3)^2-119\zeta(7)-32\zeta(2)\zeta(5)+132\zeta(3)\zeta(4)\big) \quad (688)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^3}{(k+1)^5(k+2)^2} = \frac{1}{96}\big(864+1248\zeta(2)+2016\zeta(3)-792\zeta(4)-3600\zeta(5)$$
$$-480\zeta(2)\zeta(3)-594\zeta(6)+576\zeta(3)^2-1428\zeta(7)-384\zeta(2)\zeta(5)$$
$$+1584\zeta(3)\zeta(4)-43\zeta(8)-120\zeta(2)\zeta(3)^2+288\zeta(3)\zeta(5)-24M(2,6)\big) \quad (689)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^3}{k^4(k+2)^3} = \frac{1}{128}\big(-164-112\zeta(2)-72\zeta(3)+4\zeta(4)+324\zeta(5)+64\zeta(2)\zeta(3)$$
$$-156\zeta(6)+76\zeta(3)^2+231\zeta(7)+32\zeta(2)\zeta(5)-204\zeta(3)\zeta(4)\big) \quad (690)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^3}{k^3(k+1)(k+2)^3} = \frac{1}{64}\big(-192-148\zeta(2)-124\zeta(3)+557\zeta(4)-86\zeta(5)+24\zeta(2)\zeta(3)$$
$$+30\zeta(6)-4\zeta(3)^2\big) \quad (691)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^3}{k^2(k+1)^2(k+2)^3}=\frac{1}{64}\big(444+376\zeta(2)+368\zeta(3)-260\zeta(4)-848\zeta(5)$$
$$-168\zeta(2)\zeta(3)+33\zeta(6)-32\zeta(3)^2\big) \quad (692)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^3}{k(k+1)^3(k+2)^3}=\frac{1}{32}\big(-508-464\zeta(2)-504\zeta(3)+646\zeta(4)+828\zeta(5)$$
$$+168\zeta(2)\zeta(3)+33\zeta(6)-32\zeta(3)^2\big) \quad (693)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^3}{(k+1)^4(k+2)^3}=\frac{1}{16}\big(576+560\zeta(2)+656\zeta(3)-672\zeta(4)-1128\zeta(5)$$
$$-208\zeta(2)\zeta(3)-66\zeta(6)+64\zeta(3)^2-119\zeta(7)-32\zeta(2)\zeta(5)$$
$$+132\zeta(3)\zeta(4)\big) \quad (694)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^3}{k^3(k+2)^4}=\frac{1}{128}\big(676+216\zeta(2)-96\zeta(3)-386\zeta(4)-256\zeta(5)-112\zeta(2)\zeta(3)$$
$$+22\zeta(6)-20\zeta(3)^2-119\zeta(7)-32\zeta(2)\zeta(5)+132\zeta(3)\zeta(4)\big) \quad (695)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^3}{k^2(k+1)(k+2)^4}=\frac{1}{64}\big(868+364\zeta(2)+28\zeta(3)-943\zeta(4)-170\zeta(5)$$
$$-136\zeta(2)\zeta(3)-8\zeta(6)-16\zeta(3)^2-119\zeta(7)-32\zeta(2)\zeta(5)$$
$$+132\zeta(3)\zeta(4)\big) \quad (696)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^3}{k(k+1)^2(k+2)^4}=\frac{1}{64}\big(2180+1104\zeta(2)+424\zeta(3)-2146\zeta(4)-1188\zeta(5)$$
$$-440\zeta(2)\zeta(3)+17\zeta(6)-64\zeta(3)^2-238\zeta(7)-64\zeta(2)\zeta(5)$$
$$+264\zeta(3)\zeta(4)\big) \quad (697)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^3}{(k+1)^3(k+2)^4}=\frac{1}{16}\big(1344+784\zeta(2)+464\zeta(3)-1396\zeta(4)-1008\zeta(5)$$
$$-304\zeta(2)\zeta(3)-8\zeta(6)-16\zeta(3)^2-119\zeta(7)-32\zeta(2)\zeta(5)$$
$$+132\zeta(3)\zeta(4)\big) \quad (698)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^3}{k^2(k+2)^5}=\frac{1}{768}\big(12444+1224\zeta(2)-4320\zeta(3)-6144\zeta(4)-2232\zeta(5)$$
$$+24\zeta(2)\zeta(3)-1911\zeta(6)+1152\zeta(3)^2-852\zeta(7)-960\zeta(2)\zeta(5)$$
$$+1872\zeta(3)\zeta(4)+86\zeta(8)+240\zeta(2)\zeta(3)^2-576\zeta(3)\zeta(5)+48M(2,6)\big) \quad (699)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^3}{k(k+1)(k+2)^5}=\frac{1}{384}\big(-17652-3408\zeta(2)+4152\zeta(3)+11802\zeta(4)+3252\zeta(5)$$
$$+792\zeta(2)\zeta(3)+1959\zeta(6)-1056\zeta(3)^2+1566\zeta(7)+1152\zeta(2)\zeta(5)$$
$$-2664\zeta(3)\zeta(4)-86\zeta(8)-240\zeta(2)\zeta(3)^2+576\zeta(3)\zeta(5)-48M(2,6)\big) \quad (700)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^3}{(k+1)^2(k+2)^5}=\frac{1}{96}\big(12096+3360\zeta(2)-1440\zeta(3)-9120\zeta(4)-3408\zeta(5)$$
$$-1056\zeta(2)\zeta(3)-954\zeta(6)+432\zeta(3)^2-1140\zeta(7)-672\zeta(2)\zeta(5)$$
$$+1728\zeta(3)\zeta(4)+43\zeta(8)+120\zeta(2)\zeta(3)^2-288\zeta(3)\zeta(5)+24M(2,6)\big) \tag{701}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^3}{k(k+2)^6}=\frac{1}{768}\big(30732-720\zeta(2)-11592\zeta(3)-10758\zeta(4)-8076\zeta(5)$$
$$+2712\zeta(2)\zeta(3)-5553\zeta(6)+3360\zeta(3)^2-2442\zeta(7)-768\zeta(2)\zeta(5)$$
$$+2808\zeta(3)\zeta(4)+4118\zeta(8)+240\zeta(2)\zeta(3)^2-2880\zeta(3)\zeta(5)-1104M(2,6)$$
$$-3152\zeta(9)+1776\zeta(3)\zeta(6)+3168\zeta(4)\zeta(5)-1152\zeta(2)\zeta(7)-384\zeta(3)^3\big) \tag{702}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^3}{(k+1)(k+2)^6}=\frac{1}{96}\big(12096+672\zeta(2)-3936\zeta(3)-5640\zeta(4)-2832\zeta(5)$$
$$+480\zeta(2)\zeta(3)-1878\zeta(6)+1104\zeta(3)^2-1002\zeta(7)-480\zeta(2)\zeta(5)$$
$$+1368\zeta(3)\zeta(4)+1051\zeta(8)+120\zeta(2)\zeta(3)^2-864\zeta(3)\zeta(5)-264M(2,6)$$
$$-788\zeta(9)+444\zeta(3)\zeta(6)+792\zeta(4)\zeta(5)-288\zeta(2)\zeta(7)-96\zeta(3)^3\big) \tag{703}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^3}{(k+2)^7}=\frac{1}{160}\big(13440-1120\zeta(2)-4640\zeta(3)-3520\zeta(4)-4080\zeta(5)+1440\zeta(2)\zeta(3)$$
$$-2300\zeta(6)+1200\zeta(3)^2-2560\zeta(7)+480\zeta(2)\zeta(5)+1200\zeta(3)\zeta(4)+1080\zeta(8)$$
$$-480\zeta(3)\zeta(5)-480M(2,6)-80\zeta(9)+720\zeta(3)\zeta(6)+240\zeta(4)\zeta(5)$$
$$-480\zeta(2)\zeta(7)-160\zeta(3)^3+501\zeta(10)-800\zeta(3)\zeta(7)-80\zeta(3)^2\zeta(4)$$
$$+560\zeta(2)\zeta(3)\zeta(5)-480\zeta(5)^2-40M(2,8)\big) \tag{704}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^4}{k^6}=\frac{1}{640}\big(-68823\zeta(10)+60000\zeta(3)\zeta(7)+1000\zeta(3)^2\zeta(4)$$
$$-21680\zeta(2)\zeta(3)\zeta(5)+23560\zeta(5)^2+12120M(2,8)+1280\zeta(2)M(2,6)\big) \tag{705}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^4}{k^5(k+1)}=\frac{1}{144}\big(4320\zeta(5)+864\zeta(2)\zeta(3)-5874\zeta(6)-432\zeta(3)^2+3330\zeta(7)$$
$$+720\zeta(2)\zeta(5)-3096\zeta(3)\zeta(4)+14833\zeta(8)+4032\zeta(2)\zeta(3)^2$$
$$-16704\zeta(3)\zeta(5)-3744M(2,6)+5232\zeta(9)-3348\zeta(3)\zeta(6)-3096\zeta(4)\zeta(5)$$
$$+1008\zeta(2)\zeta(7)+480\zeta(3)^3\big) \tag{706}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^4}{k^4(k+1)^2}=\frac{1}{144}\big(-17280\zeta(5)-3456\zeta(2)\zeta(3)+22776\zeta(6)+1728\zeta(3)^2$$
$$-6660\zeta(7)-1440\zeta(2)\zeta(5)+6192\zeta(3)\zeta(4)-14833\zeta(8)-4032\zeta(2)\zeta(3)^2$$
$$+16704\zeta(3)\zeta(5)+3744M(2,6)\big) \tag{707}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^4}{k^3(k+1)^3} = \frac{1}{4}\big(720\zeta(5)+144\zeta(2)\zeta(3)-919\zeta(6)-72\zeta(3)^2+147\zeta(7)$$
$$+40\zeta(2)\zeta(5)-160\zeta(3)\zeta(4)\big) \tag{708}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^4}{k^2(k+1)^4} = \frac{1}{144}\big(-17280\zeta(5)-3456\zeta(2)\zeta(3)+21336\zeta(6)+1728\zeta(3)^2$$
$$-3924\zeta(7)-1440\zeta(2)\zeta(5)+5328\zeta(3)\zeta(4)-12415\zeta(8)-3312\zeta(2)\zeta(3)^2$$
$$+13824\zeta(3)\zeta(5)+3024M(2,6)\big) \tag{709}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^4}{k(k+1)^5} = \frac{1}{144}\big(4320\zeta(5)+864\zeta(2)\zeta(3)-5154\zeta(6)-432\zeta(3)^2+1962\zeta(7)$$
$$+720\zeta(2)\zeta(5)-2664\zeta(3)\zeta(4)+12415\zeta(8)+3312\zeta(2)\zeta(3)^2$$
$$-13824\zeta(3)\zeta(5)-3024M(2,6)+2088\zeta(9)-1188\zeta(3)\zeta(6)-2664\zeta(4)\zeta(5)$$
$$+1008\zeta(2)\zeta(7)+384\zeta(3)^3\big) \tag{710}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^4}{(k+1)^6} = \frac{1}{640}\big(-48647\zeta(10)+42080\zeta(3)\zeta(7)-280\zeta(3)^2\zeta(4)$$
$$-12720\zeta(2)\zeta(3)\zeta(5)+13320\zeta(5)^2+7640M(2,8)+1280\zeta(2)M(2,6)\big) \tag{711}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^4}{k^5(k+2)} = \frac{1}{1152}\big(36+108\zeta(2)+396\zeta(3)+666\zeta(4)+1080\zeta(5)+216\zeta(2)\zeta(3)$$
$$-2937\zeta(6)-216\zeta(3)^2+3330\zeta(7)+720\zeta(2)\zeta(5)-3096\zeta(3)\zeta(4)+29666\zeta(8)$$
$$+8064\zeta(2)\zeta(3)^2-33408\zeta(3)\zeta(5)-7488M(2,6)+20928\zeta(9)-13392\zeta(3)\zeta(6)$$
$$-12384\zeta(4)\zeta(5)+4032\zeta(2)\zeta(7)+1920\zeta(3)^3\big) \tag{712}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^4}{k^4(k+1)(k+2)} = \frac{1}{576}\big(36+108\zeta(2)+396\zeta(3)+666\zeta(4)-16200\zeta(5)-3240\zeta(2)\zeta(3)$$
$$+20559\zeta(6)+1512\zeta(3)^2-9990\zeta(7)-2160\zeta(2)\zeta(5)+9288\zeta(3)\zeta(4)-29666\zeta(8)$$
$$-8064\zeta(2)\zeta(3)^2+33408\zeta(3)\zeta(5)+7488M(2,6)\big) \tag{713}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^4}{k^3(k+1)^2(k+2)} = \frac{1}{32}\big(4+12\zeta(2)+44\zeta(3)+74\zeta(4)+2040\zeta(5)+408\zeta(2)\zeta(3)$$
$$-2777\zeta(6)-216\zeta(3)^2+370\zeta(7)+80\zeta(2)\zeta(5)-344\zeta(3)\zeta(4)\big) \tag{714}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^4}{k^2(k+1)^3(k+2)} = \frac{1}{16}\big(4+12\zeta(2)+44\zeta(3)+74\zeta(4)-840\zeta(5)-168\zeta(2)\zeta(3)$$
$$+899\zeta(6)+72\zeta(3)^2-218\zeta(7)-80\zeta(2)\zeta(5)+296\zeta(3)\zeta(4)\big) \tag{715}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^4}{k(k+1)^4(k+2)} = \frac{1}{144}\big(72+216\zeta(2)+792\zeta(3)+1332\zeta(4)+2160\zeta(5)+432\zeta(2)\zeta(3)$$
$$-5154\zeta(6)-432\zeta(3)^2+12415\zeta(8)+3312\zeta(2)\zeta(3)^2-13824\zeta(3)\zeta(5)$$
$$-3024M(2,6)\big) \tag{716}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^4}{(k+1)^5(k+2)} = \frac{1}{144}\big(144 + 432\zeta(2) + 1584\zeta(3) + 2664\zeta(4) - 5154\zeta(6) - 432\zeta(3)^2$$
$$-1962\zeta(7) - 720\zeta(2)\zeta(5) + 2664\zeta(3)\zeta(4) + 12415\zeta(8) + 3312\zeta(2)\zeta(3)^2$$
$$-13824\zeta(3)\zeta(5) - 3024M(2,6) - 2088\zeta(9) + 1188\zeta(3)\zeta(6) + 2664\zeta(4)\zeta(5)$$
$$-1008\zeta(2)\zeta(7) - 384\zeta(3)^3\big) \qquad (717)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^4}{k^4(k+2)^2} = \frac{1}{576}(-252 - 504\zeta(2) - 1440\zeta(3) - 1404\zeta(4) - 1080\zeta(5)$$
$$-288\zeta(2)\zeta(3) + 5694\zeta(6) + 432\zeta(3)^2 - 3330\zeta(7) - 720\zeta(2)\zeta(5)$$
$$+3096\zeta(3)\zeta(4) - 14833\zeta(8) - 4032\zeta(2)\zeta(3)^2 + 16704\zeta(3)\zeta(5)$$
$$+3744M(2,6)) \qquad (718)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^4}{k^3(k+1)(k+2)^2} = \frac{1}{64}(-60 - 124\zeta(2) - 364\zeta(3) - 386\zeta(4) + 1560\zeta(5)$$
$$+296\zeta(2)\zeta(3) - 1019\zeta(6) - 72\zeta(3)^2 + 370\zeta(7) + 80\zeta(2)\zeta(5)$$
$$-344\zeta(3)\zeta(4)) \qquad (719)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^4}{k^2(k+1)^2(k+2)^2} = \frac{1}{16}(-32 - 68\zeta(2) - 204\zeta(3) - 230\zeta(4) - 240\zeta(5)$$
$$-56\zeta(2)\zeta(3) + 879\zeta(6) + 72\zeta(3)^2\big) \qquad (720)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^4}{k(k+1)^3(k+2)^2} = \frac{1}{16}(-68 - 148\zeta(2) - 452\zeta(3) - 534\zeta(4) + 360\zeta(5) + 56\zeta(2)\zeta(3)$$
$$+859\zeta(6) + 72\zeta(3)^2 + 218\zeta(7) + 80\zeta(2)\zeta(5) - 296\zeta(3)\zeta(4)\big) \qquad (721)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^4}{(k+1)^4(k+2)^2} = \frac{1}{144}(-1296 - 2880\zeta(2) - 8928\zeta(3) - 10944\zeta(4) + 4320\zeta(5)$$
$$+576\zeta(2)\zeta(3) + 20616\zeta(6) + 1728\zeta(3)^2 + 3924\zeta(7) + 1440\zeta(2)\zeta(5)$$
$$-5328\zeta(3)\zeta(4) - 12415\zeta(8) - 3312\zeta(2)\zeta(3)^2 + 13824\zeta(3)\zeta(5)$$
$$+3024M(2,6)) \qquad (722)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^4}{k^3(k+2)^3} = \frac{1}{64}(192 + 244\zeta(2) + 508\zeta(3) + 122\zeta(4) - 168\zeta(5) - 56\zeta(2)\zeta(3)$$
$$-853\zeta(6) - 136\zeta(3)^2 + 294\zeta(7) + 80\zeta(2)\zeta(5) - 320\zeta(3)\zeta(4)\big) \qquad (723)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^4}{k^2(k+1)(k+2)^3} = \frac{1}{64}(444 + 612\zeta(2) + 1380\zeta(3) + 630\zeta(4) - 1896\zeta(5)$$
$$-408\zeta(2)\zeta(3) - 687\zeta(6) - 200\zeta(3)^2 + 218\zeta(7) + 80\zeta(2)\zeta(5)$$
$$-296\zeta(3)\zeta(4)) \qquad (724)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^4}{k(k+1)^2(k+2)^3} = \frac{1}{32}\left(508+748\zeta(2)+1788\zeta(3)+1090\zeta(4)-1416\zeta(5)\right.$$
$$-296\zeta(2)\zeta(3)-2445\zeta(6)-344\zeta(3)^2+218\zeta(7)+80\zeta(2)\zeta(5)$$
$$\left.-296\zeta(3)\zeta(4)\right) \tag{725}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^4}{(k+1)^3(k+2)^3} = \frac{1}{2}\left(72+112\zeta(2)+280\zeta(3)+203\zeta(4)-222\zeta(5)-44\zeta(2)\zeta(3)\right.$$
$$\left.-413\zeta(6)-52\zeta(3)^2\right) \tag{726}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^4}{k^2(k+2)^4} = \frac{1}{576}\left(7812+5976\zeta(2)+7920\zeta(3)-5724\zeta(4)-6984\zeta(5)\right.$$
$$-3168\zeta(2)\zeta(3)-4290\zeta(6)-1872\zeta(3)^2-2322\zeta(7)-432\zeta(2)\zeta(5)$$
$$+2088\zeta(3)\zeta(4)+12415\zeta(8)+3312\zeta(2)\zeta(3)^2-13824\zeta(3)\zeta(5)$$
$$\left.-3024M(2,6)\right) \tag{727}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^4}{k(k+1)(k+2)^4} = \frac{1}{576}\left(19620+17460\zeta(2)+28260\zeta(3)-5778\zeta(4)-31032\zeta(5)\right.$$
$$-10008\zeta(2)\zeta(3)-14763\zeta(6)-5544\zeta(3)^2-2682\zeta(7)-144\zeta(2)\zeta(5)$$
$$+1512\zeta(3)\zeta(4)+24830\zeta(8)+6624\zeta(2)\zeta(3)^2-27648\zeta(3)\zeta(5)$$
$$\left.-6048M(2,6)\right) \tag{728}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^4}{(k+1)^2(k+2)^4} = \frac{1}{144}\left(12096+12096\zeta(2)+22176\zeta(3)+2016\zeta(4)-21888\zeta(5)\right.$$
$$-6336\zeta(2)\zeta(3)-18384\zeta(6)-4320\zeta(3)^2-360\zeta(7)+288\zeta(2)\zeta(5)$$
$$\left.-576\zeta(3)\zeta(4)+12415\zeta(8)+3312\zeta(2)\zeta(3)^2-13824\zeta(3)\zeta(5)-3024M(2,6)\right) \tag{729}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^4}{k(k+2)^5} = \frac{1}{1152}\left(52956+22860\zeta(2)+12060\zeta(3)-44622\zeta(4)-25992\zeta(5)\right.$$
$$-10728\zeta(2)\zeta(3)-9309\zeta(6)+1512\zeta(3)^2-20286\zeta(7)-9648\zeta(2)\zeta(5)$$
$$+27576\zeta(3)\zeta(4)+25862\zeta(8)+9504\zeta(2)\zeta(3)^2-34560\zeta(3)\zeta(5)-5472M(2,6)$$
$$+8352\zeta(9)-4752\zeta(3)\zeta(6)-10656\zeta(4)\zeta(5)+4032\zeta(2)\zeta(7)$$
$$\left.+1536\zeta(3)^3\right) \tag{730}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^4}{(k+1)(k+2)^5} = \frac{1}{144}\left(18144+10080\zeta(2)+10080\zeta(3)-12600\zeta(4)-14256\zeta(5)\right.$$
$$-5184\zeta(2)\zeta(3)-6018\zeta(6)-1008\zeta(3)^2-5742\zeta(7)-2448\zeta(2)\zeta(5)$$
$$+7272\zeta(3)\zeta(4)+12673\zeta(8)+4032\zeta(2)\zeta(3)^2-15552\zeta(3)\zeta(5)-2880M(2,6)$$
$$\left.+2088\zeta(9)-1188\zeta(3)\zeta(6)-2664\zeta(4)\zeta(5)+1008\zeta(2)\zeta(7)+384\zeta(3)^3\right) \tag{731}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^4}{(k+2)^6} = \frac{1}{1920}\big(241920+53760\zeta(2)-26880\zeta(3)-161280\zeta(4)-88320\zeta(5)$$
$$-7680\zeta(2)\zeta(3)-56640\zeta(6)+30720\zeta(3)^2-57120\zeta(7)-30720\zeta(2)\zeta(5)$$
$$+82560\zeta(3)\zeta(4)+43760\zeta(8)+9600\zeta(2)\zeta(3)^2-46080\zeta(3)\zeta(5)-9600M(2,6)$$
$$-63040\zeta(9)+35520\zeta(3)\zeta(6)+63360\zeta(4)\zeta(5)-23040\zeta(2)\zeta(7)-7680\zeta(3)^3$$
$$+145941\zeta(10)-126240\zeta(3)\zeta(7)+840\zeta(3)^2\zeta(4)+38160\zeta(2)\zeta(3)\zeta(5)$$
$$-39960\zeta(5)^2-22920M(2,8)-3840\zeta(2)M(2,6)\big) \tag{732}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^5}{k^5} = \frac{1}{256}\big(64433\zeta(10)-57760\zeta(3)\zeta(7)+360\zeta(3)^2\zeta(4)$$
$$+20560\zeta(2)\zeta(3)\zeta(5)-22648\zeta(5)^2-10920M(2,8)-1280\zeta(2)M(2,6)\big) \tag{733}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^5}{k^4(k+1)} = \frac{1}{288}\big(51408\zeta(6)+6480\zeta(3)^2-36918\zeta(7)-8208\zeta(2)\zeta(5)$$
$$-9504\zeta(3)\zeta(4)-67811\zeta(8)-19080\zeta(2)\zeta(3)^2+78768\zeta(3)\zeta(5)+16920M(2,6)$$
$$-37768\zeta(9)+58740\zeta(3)\zeta(6)-19008\zeta(4)\zeta(5)-9540\zeta(2)\zeta(7)$$
$$+1440\zeta(3)^3\big) \tag{734}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^5}{k^3(k+1)^2} = \frac{1}{288}\big(154224\zeta(6)+19440\zeta(3)^2-107226\zeta(7)-24624\zeta(2)\zeta(5)$$
$$-28512\zeta(3)\zeta(4)-67811\zeta(8)-19080\zeta(2)\zeta(3)^2+78768\zeta(3)\zeta(5)$$
$$+16920M(2,6)) \tag{735}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^5}{k^2(k+1)^3} = \frac{1}{288}\big(154224\zeta(6)+19440\zeta(3)^2-103698\zeta(7)-24624\zeta(2)\zeta(5)$$
$$-28512\zeta(3)\zeta(4)-65621\zeta(8)-17640\zeta(2)\zeta(3)^2+72432\zeta(3)\zeta(5)$$
$$+15480M(2,6)) \tag{736}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^5}{k(k+1)^4} = \frac{1}{288}\big(51408\zeta(6)+6480\zeta(3)^2-33390\zeta(7)-8208\zeta(2)\zeta(5)$$
$$-9504\zeta(3)\zeta(4)-65621\zeta(8)-17640\zeta(2)\zeta(3)^2+72432\zeta(3)\zeta(5)+15480M(2,6)$$
$$-28480\zeta(9)+51540\zeta(3)\zeta(6)-19008\zeta(4)\zeta(5)-9540\zeta(2)\zeta(7)$$
$$+1440\zeta(3)^3\big) \tag{737}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^5}{(k+1)^5} = \frac{1}{256}\big(49901\zeta(10)-43040\zeta(3)\zeta(7)-1080\zeta(3)^2\zeta(4)$$
$$+13840\zeta(2)\zeta(3)\zeta(5)-13592\zeta(5)^2-7560M(2,8)-1280\zeta(2)M(2,6)\big) \tag{738}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^5}{k^4(k+2)} = \frac{1}{1152}\big(-72-288\zeta(2)-1512\zeta(3)-4518\zeta(4)-5112\zeta(5)-1080\zeta(2)\zeta(3)$$
$$-12852\zeta(6)-1620\zeta(3)^2+18459\zeta(7)+4104\zeta(2)\zeta(5)+4752\zeta(3)\zeta(4)$$
$$+67811\zeta(8)+19080\zeta(2)\zeta(3)^2-78768\zeta(3)\zeta(5)-16920M(2,6)+75536\zeta(9)$$
$$-117480\zeta(3)\zeta(6)+38016\zeta(4)\zeta(5)+19080\zeta(2)\zeta(7)-2880\zeta(3)^3\big) \tag{739}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^5}{k^3(k+1)(k+2)} = \frac{1}{576}\big(-72 - 288\zeta(2) - 1512\zeta(3) - 4518\zeta(4) - 5112\zeta(5) -1080\zeta(2)\zeta(3) + 89964\zeta(6) + 11340\zeta(3)^2 - 55377\zeta(7) - 12312\zeta(2)\zeta(5) -14256\zeta(3)\zeta(4) - 67811\zeta(8) - 19080\zeta(2)\zeta(3)^2 + 78768\zeta(3)\zeta(5) +16920M(2,6)\big) \quad (740)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^5}{k^2(k+1)^2(k+2)} = \frac{1}{32}\big(-8 - 32\zeta(2) - 168\zeta(3) - 502\zeta(4) - 568\zeta(5) - 120\zeta(2)\zeta(3) -7140\zeta(6) - 900\zeta(3)^2 + 5761\zeta(7) + 1368\zeta(2)\zeta(5) + 1584\zeta(3)\zeta(4)\big) \quad (741)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^5}{k(k+1)^3(k+2)} = \frac{1}{288}\big(-144 - 576\zeta(2) - 3024\zeta(3) - 9036\zeta(4) - 10224\zeta(5) -2160\zeta(2)\zeta(3) + 25704\zeta(6) + 3240\zeta(3)^2 - 65621\zeta(8) - 17640\zeta(2)\zeta(3)^2 +72432\zeta(3)\zeta(5) + 15480M(2,6)\big) \quad (742)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^5}{(k+1)^4(k+2)} = \frac{1}{288}\big(288 + 1152\zeta(2) + 6048\zeta(3) + 18072\zeta(4) + 20448\zeta(5) +4320\zeta(2)\zeta(3) - 33390\zeta(7) - 8208\zeta(2)\zeta(5) - 9504\zeta(3)\zeta(4) + 65621\zeta(8) +17640\zeta(2)\zeta(3)^2 - 72432\zeta(3)\zeta(5) - 15480M(2,6) - 28480\zeta(9) + 51540\zeta(3)\zeta(6) -19008\zeta(4)\zeta(5) - 9540\zeta(2)\zeta(7) + 1440\zeta(3)^3\big) \quad (743)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^5}{k^3(k+2)^2} = \frac{1}{1152}\big(1080 + 3024\zeta(2) + 12888\zeta(3) + 27666\zeta(4) + 14760\zeta(5) +3960\zeta(2)\zeta(3) + 12786\zeta(6) + 2700\zeta(3)^2 - 53613\zeta(7) - 12312\zeta(2)\zeta(5) -14256\zeta(3)\zeta(4) - 67811\zeta(8) - 19080\zeta(2)\zeta(3)^2 + 78768\zeta(3)\zeta(5) +16920M(2,6)\big) \quad (744)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^5}{k^2(k+1)(k+2)^2} = \frac{1}{96}\big(192 + 552\zeta(2) + 2400\zeta(3) + 5364\zeta(4) + 3312\zeta(5) +840\zeta(2)\zeta(3) - 12863\zeta(6) - 1440\zeta(3)^2 + 294\zeta(7)\big) \quad (745)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^5}{k(k+1)^2(k+2)^2} = \frac{1}{96}\big(408 + 1200\zeta(2) + 5304\zeta(3) + 12234\zeta(4) + 8328\zeta(5) +2040\zeta(2)\zeta(3) - 4306\zeta(6) - 180\zeta(3)^2 - 16695\zeta(7) - 4104\zeta(2)\zeta(5) -4752\zeta(3)\zeta(4)\big) \quad (746)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^5}{(k+1)^3(k+2)^2} = \frac{1}{288}\big(2592 + 7776\zeta(2) + 34848\zeta(3) + 82440\zeta(4) + 60192\zeta(5) +14400\zeta(2)\zeta(3) - 51540\zeta(6) - 4320\zeta(3)^2 - 100170\zeta(7) - 24624\zeta(2)\zeta(5) -28512\zeta(3)\zeta(4) + 65621\zeta(8) + 17640\zeta(2)\zeta(3)^2 - 72432\zeta(3)\zeta(5) -15480M(2,6)\big) \quad (747)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^5}{k^2(k+2)^3} = \frac{1}{1152} (7992 + 15264\zeta(2) + 51480\zeta(3) + 71874\zeta(4) - 4248\zeta(5)$$
$$+360\zeta(2)\zeta(3) - 58584\zeta(6) - 9540\zeta(3)^2 - 32229\zeta(7) - 5112\zeta(2)\zeta(5)$$
$$-40896\zeta(3)\zeta(4) - 65621\zeta(8) - 17640\zeta(2)\zeta(3)^2 + 72432\zeta(3)\zeta(5)$$
$$+15480M(2,6)) \tag{748}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^5}{k(k+1)(k+2)^3} = \frac{1}{576} (-9144 - 18576\zeta(2) - 65880\zeta(3) - 104058\zeta(4) - 15624\zeta(5)$$
$$-5400\zeta(2)\zeta(3) + 135762\zeta(6) + 18180\zeta(3)^2 + 30465\zeta(7) + 5112\zeta(2)\zeta(5)$$
$$+40896\zeta(3)\zeta(4) + 65621\zeta(8) + 17640\zeta(2)\zeta(3)^2 - 72432\zeta(3)\zeta(5)$$
$$-15480M(2,6)) \tag{749}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^5}{(k+1)^2(k+2)^3} = \frac{1}{288} (10368 + 22176\zeta(2) + 81792\zeta(3) + 140760\zeta(4) + 40608\zeta(5)$$
$$+11520\zeta(2)\zeta(3) - 148680\zeta(6) - 18720\zeta(3)^2 - 80550\zeta(7) - 17424\zeta(2)\zeta(5)$$
$$-55152\zeta(3)\zeta(4) - 65621\zeta(8) - 17640\zeta(2)\zeta(3)^2 + 72432\zeta(3)\zeta(5)$$
$$+15480M(2,6)) \tag{750}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^5}{k(k+2)^4} = \frac{1}{1152} (39240 + 49968\zeta(2) + 129384\zeta(3) + 94806\zeta(4) - 82440\zeta(5)$$
$$-25560\zeta(2)\zeta(3) - 146718\zeta(6) - 36540\zeta(3)^2 - 675\zeta(7) + 5976\zeta(2)\zeta(5)$$
$$-37152\zeta(3)\zeta(4) + 182679\zeta(8) + 48600\zeta(2)\zeta(3)^2 - 204048\zeta(3)\zeta(5) - 45000M(2,6)$$
$$-56960\zeta(9) + 103080\zeta(3)\zeta(6) - 38016\zeta(4)\zeta(5) - 19080\zeta(2)\zeta(7)$$
$$+2880\zeta(3)^3\big) \tag{751}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^5}{(k+1)(k+2)^4} = \frac{1}{288} (24192 + 34272\zeta(2) + 97632\zeta(3) + 99432\zeta(4) - 33408\zeta(5)$$
$$-10080\zeta(2)\zeta(3) - 141240\zeta(6) - 27360\zeta(3)^2 - 15570\zeta(7) + 432\zeta(2)\zeta(5)$$
$$-39024\zeta(3)\zeta(4) + 58529\zeta(8) + 15480\zeta(2)\zeta(3)^2 - 65808\zeta(3)\zeta(5) - 14760M(2,6)$$
$$-28480\zeta(9) + 51540\zeta(3)\zeta(6) - 19008\zeta(4)\zeta(5) - 9540\zeta(2)\zeta(7)$$
$$+1440\zeta(3)^3\big) \tag{752}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^5}{(k+2)^5} = \frac{1}{2304} (290304 + 241920\zeta(2) + 460800\zeta(3) + 48960\zeta(4) - 414720\zeta(5)$$
$$-149760\zeta(2)\zeta(3) - 384960\zeta(6) - 97920\zeta(3)^2 - 162720\zeta(7) - 57600\zeta(2)\zeta(5)$$
$$+178560\zeta(3)\zeta(4) + 1003520\zeta(8) + 293760\zeta(2)\zeta(3)^2 - 1175040\zeta(3)\zeta(5)$$
$$-236160M(2,6) + 167040\zeta(9) - 95040\zeta(3)\zeta(6) - 213120\zeta(4)\zeta(5) + 80640\zeta(2)\zeta(7)$$
$$+30720\zeta(3)^3 - 449109\zeta(10) + 387360\zeta(3)\zeta(7) + 9720\zeta(3)^2\zeta(4)$$
$$-124560\zeta(2)\zeta(3)\zeta(5) + 122328\zeta(5)^2 + 68040M(2,8) + 11520\zeta(2)M(2,6)\big) \tag{753}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^6}{k^4} = \frac{1}{128}\big(-271367\zeta(10)+176560\zeta(3)\zeta(7)-84648\zeta(3)^2\zeta(4)$$
$$-400\zeta(2)\zeta(3)\zeta(5)+121688\zeta(5)^2+34376M(2,8)+15040\zeta(2)M(2,6)\big) \quad (754)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^6}{k^3(k+1)} = \frac{1}{24}\big(15456\zeta(7)+3480\zeta(2)\zeta(5)+7128\zeta(3)\zeta(4)-17529\zeta(8)$$
$$+984\zeta(2)\zeta(3)^2-11688\zeta(3)\zeta(5)-1368M(2,6)+7474\zeta(9)-13122\zeta(3)\zeta(6)$$
$$+6048\zeta(4)\zeta(5)+1953\zeta(2)\zeta(7)-544\zeta(3)^3\big) \quad (755)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^6}{k^2(k+1)^2} = \frac{1}{6}\big(-7728\zeta(7)-1740\zeta(2)\zeta(5)-3564\zeta(3)\zeta(4)+8639\zeta(8)$$
$$-477\zeta(2)\zeta(3)^2+5754\zeta(3)\zeta(5)+669M(2,6)\big) \quad (756)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^6}{k(k+1)^3} = \frac{1}{24}\big(15456\zeta(7)+3480\zeta(2)\zeta(5)+7128\zeta(3)\zeta(4)-17027\zeta(8)$$
$$+924\zeta(2)\zeta(3)^2-11328\zeta(3)\zeta(5)-1308M(2,6)+6146\zeta(9)-12582\zeta(3)\zeta(6)$$
$$+5832\zeta(4)\zeta(5)+1953\zeta(2)\zeta(7)-536\zeta(3)^3\big) \quad (757)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^6}{(k+1)^4} = \frac{1}{128}\big(-259945\zeta(10)+163568\zeta(3)\zeta(7)-81848\zeta(3)^2\zeta(4)$$
$$+5200\zeta(2)\zeta(3)\zeta(5)+113288\zeta(5)^2+31576M(2,8)+15040\zeta(2)M(2,6)\big) \quad (758)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^6}{k^3(k+2)} = \frac{1}{96}\big(12+60\zeta(2)+408\zeta(3)+1713\zeta(4)+3426\zeta(5)+732\zeta(2)\zeta(3)$$
$$+6291\zeta(6)+804\zeta(3)^2+7728\zeta(7)+1740\zeta(2)\zeta(5)+3564\zeta(3)\zeta(4)-17529\zeta(8)$$
$$+984\zeta(2)\zeta(3)^2-11688\zeta(3)\zeta(5)-1368M(2,6)+14948\zeta(9)-26244\zeta(3)\zeta(6)$$
$$+12096\zeta(4)\zeta(5)+3906\zeta(2)\zeta(7)-1088\zeta(3)^3\big) \quad (759)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^6}{k^2(k+1)(k+2)} = \frac{1}{16}\big(4+20\zeta(2)+136\zeta(3)+571\zeta(4)+1142\zeta(5)+244\zeta(2)\zeta(3)$$
$$+2097\zeta(6)+268\zeta(3)^2-7728\zeta(7)-1740\zeta(2)\zeta(5)-3564\zeta(3)\zeta(4)+5843\zeta(8)$$
$$-328\zeta(2)\zeta(3)^2+3896\zeta(3)\zeta(5)+456M(2,6)\big) \quad (760)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^6}{k(k+1)^2(k+2)} = \frac{1}{24}\big(12+60\zeta(2)+408\zeta(3)+1713\zeta(4)+3426\zeta(5)+732\zeta(2)\zeta(3)$$
$$+6291\zeta(6)+804\zeta(3)^2+7728\zeta(7)+1740\zeta(2)\zeta(5)+3564\zeta(3)\zeta(4)-17027\zeta(8)$$
$$+924\zeta(2)\zeta(3)^2-11328\zeta(3)\zeta(5)-1308M(2,6)\big) \quad (761)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^6}{(k+1)^3(k+2)} = \frac{1}{24}\big(24+120\zeta(2)+816\zeta(3)+3426\zeta(4)+6852\zeta(5)+1464\zeta(2)\zeta(3)$$
$$+12582\zeta(6)+1608\zeta(3)^2-17027\zeta(8)+924\zeta(2)\zeta(3)^2-11328\zeta(3)\zeta(5)$$
$$-1308M(2,6)-6146\zeta(9)+12582\zeta(3)\zeta(6)-5832\zeta(4)\zeta(5)-1953\zeta(2)\zeta(7)$$
$$+536\zeta(3)^3\big) \quad (762)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^6}{k^2(k+2)^2} = \frac{1}{96}\left(192+696\zeta(2)+3936\zeta(3)+12714\zeta(4)+16956\zeta(5)\right.$$
$$+3912\zeta(2)\zeta(3)+12279\zeta(6)+2136\zeta(3)^2-1239\zeta(7)-624\zeta(2)\zeta(5)$$
$$+2376\zeta(3)\zeta(4)-34556\zeta(8)+1908\zeta(2)\zeta(3)^2-23016\zeta(3)\zeta(5)$$
$$\left.-2676M(2,6)\right) \quad (763)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^6}{k(k+1)(k+2)^2} = \frac{1}{48}\left(204+756\zeta(2)+4344\zeta(3)+14427\zeta(4)+20382\zeta(5)\right.$$
$$+4644\zeta(2)\zeta(3)+18570\zeta(6)+2940\zeta(3)^2-24423\zeta(7)-5844\zeta(2)\zeta(5)$$
$$\left.-8316\zeta(3)\zeta(4)-17027\zeta(8)+924\zeta(2)\zeta(3)^2-11328\zeta(3)\zeta(5)-1308M(2,6)\right) \quad (764)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^6}{(k+1)^2(k+2)^2} = \frac{1}{24}\left(216+816\zeta(2)+4752\zeta(3)+16140\zeta(4)+23808\zeta(5)\right.$$
$$+5376\zeta(2)\zeta(3)+24861\zeta(6)+3744\zeta(3)^2-16695\zeta(7)-4104\zeta(2)\zeta(5)$$
$$-4752\zeta(3)\zeta(4)-34054\zeta(8)+1848\zeta(2)\zeta(3)^2-22656\zeta(3)\zeta(5)$$
$$\left.-2616M(2,6)\right) \quad (765)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^6}{k(k+2)^3} = \frac{1}{96}\left(1524+3948\zeta(2)+18360\zeta(3)+44121\zeta(4)+34122\zeta(5)\right.$$
$$+8364\zeta(2)\zeta(3)-18408\zeta(6)-1692\zeta(3)^2-32547\zeta(7)-6972\zeta(2)\zeta(5)$$
$$-24012\zeta(3)\zeta(4)-82648\zeta(8)-16716\zeta(2)\zeta(3)^2+61104\zeta(3)\zeta(5)+14172M(2,6)$$
$$+12292\zeta(9)-25164\zeta(3)\zeta(6)+11664\zeta(4)\zeta(5)+3906\zeta(2)\zeta(7)$$
$$\left.-1072\zeta(3)^3\right) \quad (766)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^6}{(k+1)(k+2)^3} = \frac{1}{48}\left(1728+4704\zeta(2)+22704\zeta(3)+58548\zeta(4)+54504\zeta(5)\right.$$
$$+13008\zeta(2)\zeta(3)+162\zeta(6)+1248\zeta(3)^2-56970\zeta(7)-12816\zeta(2)\zeta(5)$$
$$-32328\zeta(3)\zeta(4)-99675\zeta(8)-15792\zeta(2)\zeta(3)^2+49776\zeta(3)\zeta(5)+12864M(2,6)$$
$$+12292\zeta(9)-25164\zeta(3)\zeta(6)+11664\zeta(4)\zeta(5)+3906\zeta(2)\zeta(7)$$
$$\left.-1072\zeta(3)^3\right) \quad (767)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^6}{(k+2)^4} = \frac{1}{384}\left(-32256-59136\zeta(2)-224256\zeta(3)-386496\zeta(4)-122880\zeta(5)\right.$$
$$-26880\zeta(2)\zeta(3)+378528\zeta(6)+66432\zeta(3)^2+167280\zeta(7)+23424\zeta(2)\zeta(5)$$
$$+225792\zeta(3)\zeta(4)+28368\zeta(8)+8640\zeta(2)\zeta(3)^2-26496\zeta(3)\zeta(5)-2880M(2,6)$$
$$+227840\zeta(9)-412320\zeta(3)\zeta(6)+152064\zeta(4)\zeta(5)+76320\zeta(2)\zeta(7)-11520\zeta(3)^3$$
$$-779835\zeta(10)+490704\zeta(3)\zeta(7)-245544\zeta(3)^2\zeta(4)+15600\zeta(2)\zeta(3)\zeta(5)$$
$$\left.+339864\zeta(5)^2+94728M(2,8)+45120\zeta(2)M(2,6)\right) \quad (768)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^7}{k^3} = \frac{1}{2560}\big(16614991\zeta(10) - 10315520\zeta(3)\zeta(7) + 5879160\zeta(3)^2\zeta(4) -705040\zeta(2)\zeta(3)\zeta(5) - 7710760\zeta(5)^2 - 2021880M(2,8) - 1008000\zeta(2)M(2,6)\big) \tag{769}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^7}{k^2(k+1)} = \frac{1}{72}\big(479096\zeta(8) + 12096\zeta(2)\zeta(3)^2 + 109620\zeta(3)\zeta(5) - 276341\zeta(9) -88665\zeta(3)\zeta(6) - 143163\zeta(4)\zeta(5) - 59166\zeta(2)\zeta(7) - 4032\zeta(3)^3\big) \tag{770}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^7}{k(k+1)^2} = \frac{1}{72}\big(479096\zeta(8) + 12096\zeta(2)\zeta(3)^2 + 109620\zeta(3)\zeta(5) - 269402\zeta(9) -88665\zeta(3)\zeta(6) - 141273\zeta(4)\zeta(5) - 59166\zeta(2)\zeta(7) - 4032\zeta(3)^3\big) \tag{771}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^7}{(k+1)^3} = \frac{1}{2560}\big(16597239\zeta(10) - 9974400\zeta(3)\zeta(7) + 5800760\zeta(3)^2\zeta(4) -834960\zeta(2)\zeta(3)\zeta(5) - 7473640\zeta(5)^2 - 1956920M(2,8) - 1008000\zeta(2)M(2,6)\big) \tag{772}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^7}{k^2(k+2)} = \frac{1}{576}(-144 - 864\zeta(2) - 7200\zeta(3) - 38664\zeta(4) - 108504\zeta(5) -23184\zeta(2)\zeta(3) - 352887\zeta(6) - 45864\zeta(3)^2 - 319554\zeta(7) - 73080\zeta(2)\zeta(5) -148932\zeta(3)\zeta(4) - 958192\zeta(8) - 24192\zeta(2)\zeta(3)^2 - 219240\zeta(3)\zeta(5) +1105364\zeta(9) + 354660\zeta(3)\zeta(6) + 572652\zeta(4)\zeta(5) + 236664\zeta(2)\zeta(7) +16128\zeta(3)^3\big) \tag{773}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^7}{k(k+1)(k+2)} = \frac{1}{288}(-144 - 864\zeta(2) - 7200\zeta(3) - 38664\zeta(4) - 108504\zeta(5) -23184\zeta(2)\zeta(3) - 352887\zeta(6) - 45864\zeta(3)^2 - 319554\zeta(7) - 73080\zeta(2)\zeta(5) -148932\zeta(3)\zeta(4) + 958192\zeta(8) + 24192\zeta(2)\zeta(3)^2 + 219240\zeta(3)\zeta(5)\big) \tag{774}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^7}{(k+1)^2(k+2)} = \frac{1}{144}(144 + 864\zeta(2) + 7200\zeta(3) + 38664\zeta(4) + 108504\zeta(5) +23184\zeta(2)\zeta(3) + 352887\zeta(6) + 45864\zeta(3)^2 + 319554\zeta(7) + 73080\zeta(2)\zeta(5) +148932\zeta(3)\zeta(4) - 538804\zeta(9) - 177330\zeta(3)\zeta(6) - 282546\zeta(4)\zeta(5) -118332\zeta(2)\zeta(7) - 8064\zeta(3)^3\big) \tag{775}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^7}{k(k+2)^2} = \frac{1}{576}(2448 + 10944\zeta(2) + 77184\zeta(3) + 331344\zeta(4) + 683928\zeta(5) +152208\zeta(2)\zeta(3) + 1403655\zeta(6) + 199080\zeta(3)^2 + 257472\zeta(7) + 46872\zeta(2)\zeta(5) +247212\zeta(3)\zeta(4) - 472076\zeta(8) + 101808\zeta(2)\zeta(3)^2 - 732312\zeta(3)\zeta(5) -109872M(2,6) - 1077608\zeta(9) - 354660\zeta(3)\zeta(6) - 565092\zeta(4)\zeta(5) - 236664\zeta(2)\zeta(7) -16128\zeta(3)^3\big) \tag{776}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^7}{(k+1)(k+2)^2} = \frac{1}{144}\left(1296+5904\zeta(2)+42192\zeta(3)+185004\zeta(4)+396216\zeta(5)\right.$$
$$+87696\zeta(2)\zeta(3)+878271\zeta(6)+122472\zeta(3)^2+288513\zeta(7)+59976\zeta(2)\zeta(5)$$
$$+198072\zeta(3)\zeta(4)-715134\zeta(8)+38808\zeta(2)\zeta(3)^2-475776\zeta(3)\zeta(5)-54936M(2,6)$$
$$-538804\zeta(9)-177330\zeta(3)\zeta(6)-282546\zeta(4)\zeta(5)-118332\zeta(2)\zeta(7)$$
$$\left.-8064\zeta(3)^3\right) \quad (777)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^7}{(k+2)^3} = \frac{1}{7680}\left(-276480-913920\zeta(2)-5429760\zeta(3)-18389760\zeta(4)-26899200\zeta(5)\right.$$
$$-6182400\zeta(2)\zeta(3)-28153920\zeta(6)-4381440\zeta(3)^2+16691520\zeta(7)$$
$$+3951360\zeta(2)\zeta(5)+7674240\zeta(3)\zeta(4)+74888240\zeta(8)+7808640\zeta(2)\zeta(3)^2$$
$$-15187200\zeta(3)\zeta(5)-5738880M(2,6)-13767040\zeta(9)+28183680\zeta(3)\zeta(6)$$
$$-13063680\zeta(4)\zeta(5)-4374720\zeta(2)\zeta(7)+1200640\zeta(3)^3+49791717\zeta(10)$$
$$-29923200\zeta(3)\zeta(7)+17402280\zeta(3)^2\zeta(4)-2504880\zeta(2)\zeta(3)\zeta(5)$$
$$\left.-22420920\zeta(5)^2-5870760M(2,8)-3024000\zeta(2)M(2,6)\right) \quad (778)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^8}{k^2} = \frac{1}{480}\left(18741581\zeta(10)+6689520\zeta(3)\zeta(7)-524640\zeta(3)^2\zeta(4)\right.$$
$$\left.+1452480\zeta(2)\zeta(3)\zeta(5)+4247040\zeta(5)^2+485280M(2,8)+299520\zeta(2)M(2,6)\right) \quad (779)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^8}{k(k+1)} = \frac{1}{6}\left(166700\zeta(9)+88665\zeta(3)\zeta(6)+80400\zeta(4)\zeta(5)+35091\zeta(2)\zeta(7)\right.$$
$$\left.+4032\zeta(3)^3\right) \quad (780)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^8}{(k+1)^2} = \frac{1}{240}\left(9295879\zeta(10)+3314520\zeta(3)\zeta(7)-258540\zeta(3)^2\zeta(4)\right.$$
$$\left.+733800\zeta(2)\zeta(3)\zeta(5)+2098980\zeta(5)^2+238860M(2,8)+149760\zeta(2)M(2,6)\right) \quad (781)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^8}{k(k+2)} = \frac{1}{144}\left(72+504\zeta(2)+4968\zeta(3)+32400\zeta(4)+116280\zeta(5)+24768\zeta(2)\zeta(3)\right.$$
$$+530346\zeta(6)+69264\zeta(3)^2+849654\zeta(7)+193104\zeta(2)\zeta(5)+404280\zeta(3)\zeta(4)$$
$$+1906367\zeta(8)+48384\zeta(2)\zeta(3)^2+436896\zeta(3)\zeta(5)+2000400\zeta(9)$$
$$\left.+1063980\zeta(3)\zeta(6)+964800\zeta(4)\zeta(5)+421092\zeta(2)\zeta(7)+48384\zeta(3)^3\right) \quad (782)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^8}{(k+1)(k+2)} = \frac{1}{72}\left(72+504\zeta(2)+4968\zeta(3)+32400\zeta(4)+116280\zeta(5)+24768\zeta(2)\zeta(3)\right.$$
$$+530346\zeta(6)+69264\zeta(3)^2+849654\zeta(7)+193104\zeta(2)\zeta(5)+404280\zeta(3)\zeta(4)$$
$$\left.+1906367\zeta(8)+48384\zeta(2)\zeta(3)^2+436896\zeta(3)\zeta(5)\right) \quad (783)$$

$$
\begin{aligned}
\sum_{k=1}^{\infty}\frac{H(k)^8}{(k+2)^2} = \frac{1}{720}\,&(6480+34560\zeta(2)+292320\zeta(3)+1564560\zeta(4)+4348800\zeta(5)\\
&+950400\zeta(2)\zeta(3)+14106480\zeta(6)+1926720\zeta(3)^2+12318480\zeta(7)+2712960\zeta(2)\zeta(5)\\
&+6755040\zeta(3)\zeta(4)+4760990\zeta(8)+1260000\zeta(2)\zeta(3)^2-5146560\zeta(3)\zeta(5)\\
&-1098720M(2,6)-21552160\zeta(9)-7093200\zeta(3)\zeta(6)-11301840\zeta(4)\zeta(5)\\
&-4733280\zeta(2)\zeta(7)-322560\zeta(3)^3-27887637\zeta(10)-9943560\zeta(3)\zeta(7)\\
&+775620\zeta(3)^2\zeta(4)-2201400\zeta(2)\zeta(3)\zeta(5)-6296940\zeta(5)^2-716580M(2,8)\\
&-449280\zeta(2)M(2,6)) \qquad (784)
\end{aligned}
$$

Formulas for order $r = m + n + p + q = 11$:

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^{10}} = 6\zeta(11) - \zeta(2)\zeta(9) - \zeta(3)\zeta(8) - \zeta(4)\zeta(7) - \zeta(5)\zeta(6) \tag{785}$$

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{H(k)}{k^9(k+1)} = \frac{1}{4}\big(&4\zeta(2) - 8\zeta(3) + 5\zeta(4) - 12\zeta(5) + 4\zeta(2)\zeta(3) + 7\zeta(6) - 2\zeta(3)^2 \\ &-16\zeta(7) + 4\zeta(2)\zeta(5) + 4\zeta(3)\zeta(4) + 9\zeta(8) - 4\zeta(3)\zeta(5) - 20\zeta(9) \\ &+4\zeta(3)\zeta(6) + 4\zeta(4)\zeta(5) + 4\zeta(2)\zeta(7) + 11\zeta(10) - 4\zeta(3)\zeta(7) \\ &-2\zeta(5)^2\big)\end{aligned} \tag{786}$$

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{H(k)}{k^8(k+1)^2} = \frac{1}{2}\,(&16\zeta(2) - 30\zeta(3) + 15\zeta(4) - 30\zeta(5) + 10\zeta(2)\zeta(3) + 14\zeta(6) \\ &-4\zeta(3)^2 - 24\zeta(7) + 6\zeta(2)\zeta(5) + 6\zeta(3)\zeta(4) + 9\zeta(8) - 4\zeta(3)\zeta(5) \\ &-10\zeta(9) + 2\zeta(3)\zeta(6) + 2\zeta(4)\zeta(5) + 2\zeta(2)\zeta(7))\end{aligned} \tag{787}$$

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{H(k)}{k^7(k+1)^3} = \frac{1}{4}\,(&112\zeta(2) - 196\zeta(3) + 74\zeta(4) - 120\zeta(5) + 40\zeta(2)\zeta(3) + 42\zeta(6) \\ &-12\zeta(3)^2 - 48\zeta(7) + 12\zeta(2)\zeta(5) + 12\zeta(3)\zeta(4) + 9\zeta(8) - 4\zeta(3)\zeta(5)\big)\end{aligned} \tag{788}$$

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{H(k)}{k^6(k+1)^4} = \frac{1}{2}\,(&112\zeta(2) - 182\zeta(3) + 47\zeta(4) - 64\zeta(5) + 22\zeta(2)\zeta(3) + 14\zeta(6) \\ &-4\zeta(3)^2 - 8\zeta(7) + 2\zeta(2)\zeta(5) + 2\zeta(3)\zeta(4)\big)\end{aligned} \tag{789}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^5(k+1)^5} = 70\zeta(2) - 105\zeta(3) + 15\zeta(4) - 25\zeta(5) + 10\zeta(2)\zeta(3) + \zeta(6) \tag{790}$$

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{H(k)}{k^4(k+1)^6} = \frac{1}{2}\,(&112\zeta(2) - 154\zeta(3) + 5\zeta(4) - 46\zeta(5) + 22\zeta(2)\zeta(3) - 6\zeta(6) \\ &+4\zeta(3)^2 - 6\zeta(7) + 2\zeta(2)\zeta(5) + 2\zeta(3)\zeta(4)\big)\end{aligned} \tag{791}$$

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{H(k)}{k^3(k+1)^7} = \frac{1}{4}\,(&112\zeta(2) - 140\zeta(3) - 10\zeta(4) - 80\zeta(5) + 40\zeta(2)\zeta(3) - 18\zeta(6) \\ &+12\zeta(3)^2 - 36\zeta(7) + 12\zeta(2)\zeta(5) + 12\zeta(3)\zeta(4) - 5\zeta(8) + 4\zeta(3)\zeta(5)\big)\end{aligned} \tag{792}$$

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{H(k)}{k^2(k+1)^8} = \frac{1}{2}\,(&16\zeta(2) - 18\zeta(3) - 3\zeta(4) - 20\zeta(5) + 10\zeta(2)\zeta(3) - 6\zeta(6) \\ &+4\zeta(3)^2 - 18\zeta(7) + 6\zeta(2)\zeta(5) + 6\zeta(3)\zeta(4) - 5\zeta(8) + 4\zeta(3)\zeta(5) - 8\zeta(9) \\ &+2\zeta(3)\zeta(6) + 2\zeta(4)\zeta(5) + 2\zeta(2)\zeta(7))\end{aligned} \tag{793}$$

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{H(k)}{k(k+1)^9} = \frac{1}{4}\big(&4\zeta(2) - 4\zeta(3) - \zeta(4) - 8\zeta(5) + 4\zeta(2)\zeta(3) - 3\zeta(6) + 2\zeta(3)^2 \\ &-12\zeta(7) + 4\zeta(2)\zeta(5) + 4\zeta(3)\zeta(4) - 5\zeta(8) + 4\zeta(3)\zeta(5) - 16\zeta(9) \\ &+4\zeta(3)\zeta(6) + 4\zeta(4)\zeta(5) + 4\zeta(2)\zeta(7) - 7\zeta(10) + 4\zeta(3)\zeta(7) \\ &+2\zeta(5)^2\big)\end{aligned} \tag{794}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{(k+1)^{10}} = 5\zeta(11) - \zeta(2)\zeta(9) - \zeta(3)\zeta(8) - \zeta(4)\zeta(7) - \zeta(5)\zeta(6) \tag{795}$$

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{H(k)}{k^9(k+2)} = \frac{1}{512} & (1 + \zeta(2) - 4\zeta(3) + 5\zeta(4) - 24\zeta(5) + 8\zeta(2)\zeta(3) + 28\zeta(6) \\ & -8\zeta(3)^2 - 128\zeta(7) + 32\zeta(2)\zeta(5) + 32\zeta(3)\zeta(4) + 144\zeta(8) - 64\zeta(3)\zeta(5) \\ & -640\zeta(9) + 128\zeta(3)\zeta(6) + 128\zeta(4)\zeta(5) + 128\zeta(2)\zeta(7) + 704\zeta(10) \\ & -256\zeta(3)\zeta(7) - 128\zeta(5)^2\big)\end{aligned} \tag{796}$$

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{H(k)}{k^8(k+1)(k+2)} = \frac{1}{256} & (1 - 255\zeta(2) + 508\zeta(3) - 315\zeta(4) + 744\zeta(5) - 248\zeta(2)\zeta(3) \\ & -420\zeta(6) + 120\zeta(3)^2 + 896\zeta(7) - 224\zeta(2)\zeta(5) - 224\zeta(3)\zeta(4) - 432\zeta(8) \\ & +192\zeta(3)\zeta(5) + 640\zeta(9) - 128\zeta(3)\zeta(6) - 128\zeta(4)\zeta(5) \\ & -128\zeta(2)\zeta(7))\end{aligned} \tag{797}$$

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{H(k)}{k^7(k+1)^2(k+2)} = \frac{1}{128} & (1 + 769\zeta(2) - 1412\zeta(3) + 645\zeta(4) - 1176\zeta(5) + 392\zeta(2)\zeta(3) \\ & +476\zeta(6) - 136\zeta(3)^2 - 640\zeta(7) + 160\zeta(2)\zeta(5) + 160\zeta(3)\zeta(4) + 144\zeta(8) \\ & -64\zeta(3)\zeta(5))\end{aligned} \tag{798}$$

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{H(k)}{k^6(k+1)^3(k+2)} = \frac{1}{64} & (1 - 1023\zeta(2) + 1724\zeta(3) - 539\zeta(4) + 744\zeta(5) - 248\zeta(2)\zeta(3) \\ & -196\zeta(6) + 56\zeta(3)^2 + 128\zeta(7) - 32\zeta(2)\zeta(5) - 32\zeta(3)\zeta(4)\big)\end{aligned} \tag{799}$$

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{H(k)}{k^5(k+1)^4(k+2)} = \frac{1}{32} & (1 + 769\zeta(2) - 1188\zeta(3) + 213\zeta(4) - 280\zeta(5) + 104\zeta(2)\zeta(3) \\ & +28\zeta(6) - 8\zeta(3)^2\big)\end{aligned} \tag{800}$$

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{H(k)}{k^4(k+1)^5(k+2)} = \frac{1}{16} & (1 - 351\zeta(2) + 492\zeta(3) - 27\zeta(4) + 120\zeta(5) - 56\zeta(2)\zeta(3) \\ & +12\zeta(6) - 8\zeta(3)^2\big)\end{aligned} \tag{801}$$

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{H(k)}{k^3(k+1)^6(k+2)} = \frac{1}{8} & (1 + 97\zeta(2) - 124\zeta(3) - 7\zeta(4) - 64\zeta(5) + 32\zeta(2)\zeta(3) \\ & -12\zeta(6) + 8\zeta(3)^2 - 24\zeta(7) + 8\zeta(2)\zeta(5) + 8\zeta(3)\zeta(4)\big)\end{aligned} \tag{802}$$

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{H(k)}{k^2(k+1)^7(k+2)} = \frac{1}{4} & (1 - 15\zeta(2) + 16\zeta(3) + 3\zeta(4) + 16\zeta(5) - 8\zeta(2)\zeta(3) + 6\zeta(6) \\ & -4\zeta(3)^2 + 12\zeta(7) - 4\zeta(2)\zeta(5) - 4\zeta(3)\zeta(4) + 5\zeta(8) - 4\zeta(3)\zeta(5)\big)\end{aligned} \tag{803}$$

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{H(k)}{k(k+1)^8(k+2)} = \frac{1}{2} & (1 + \zeta(2) - 2\zeta(3) - 4\zeta(5) + 2\zeta(2)\zeta(3) - 6\zeta(7) + 2\zeta(2)\zeta(5) \\ & +2\zeta(3)\zeta(4) - 8\zeta(9) + 2\zeta(3)\zeta(6) + 2\zeta(4)\zeta(5) + 2\zeta(2)\zeta(7))\end{aligned} \tag{804}$$

$$\sum_{k=1}^{\infty}\frac{H(k)}{(k+1)^9(k+2)} = \frac{1}{4}\big(4-4\zeta(3)+\zeta(4)-8\zeta(5)+4\zeta(2)\zeta(3)+3\zeta(6)-2\zeta(3)^2 -12\zeta(7)+4\zeta(2)\zeta(5)+4\zeta(3)\zeta(4)+5\zeta(8)-4\zeta(3)\zeta(5)-16\zeta(9) +4\zeta(3)\zeta(6)+4\zeta(4)\zeta(5)+4\zeta(2)\zeta(7)+7\zeta(10)-4\zeta(3)\zeta(7) -2\zeta(5)^2\big) \quad (805)$$

$$\sum_{k=1}^{\infty}\frac{H(k)}{k^8(k+2)^2} = \frac{1}{256}(-6-3\zeta(2)+15\zeta(3)-15\zeta(4)+60\zeta(5)-20\zeta(2)\zeta(3)-56\zeta(6) +16\zeta(3)^2+192\zeta(7)-48\zeta(2)\zeta(5)-48\zeta(3)\zeta(4)-144\zeta(8)+64\zeta(3)\zeta(5) +320\zeta(9)-64\zeta(3)\zeta(6)-64\zeta(4)\zeta(5)-64\zeta(2)\zeta(7)) \quad (806)$$

$$\sum_{k=1}^{\infty}\frac{H(k)}{k^7(k+1)(k+2)^2} = \frac{1}{256}(13-249\zeta(2)+478\zeta(3)-285\zeta(4)+624\zeta(5)-208\zeta(2)\zeta(3) -308\zeta(6)+88\zeta(3)^2+512\zeta(7)-128\zeta(2)\zeta(5)-128\zeta(3)\zeta(4)-144\zeta(8) +64\zeta(3)\zeta(5)) \quad (807)$$

$$\sum_{k=1}^{\infty}\frac{H(k)}{k^6(k+1)^2(k+2)^2} = \frac{1}{64}\big(-7-260\zeta(2)+467\zeta(3)-180\zeta(4)+276\zeta(5)-92\zeta(2)\zeta(3) -84\zeta(6)+24\zeta(3)^2+64\zeta(7)-16\zeta(2)\zeta(5)-16\zeta(3)\zeta(4)\big) \quad (808)$$

$$\sum_{k=1}^{\infty}\frac{H(k)}{k^5(k+1)^3(k+2)^2} = \frac{1}{64}\big(-15+503\zeta(2)-790\zeta(3)+179\zeta(4)-192\zeta(5)+64\zeta(2)\zeta(3) +28\zeta(6)-8\zeta(3)^2\big) \quad (809)$$

$$\sum_{k=1}^{\infty}\frac{H(k)}{k^4(k+1)^4(k+2)^2} = \frac{1}{16}(-8-133\zeta(2)+199\zeta(3)-17\zeta(4)+44\zeta(5) -20\zeta(2)\zeta(3)) \quad (810)$$

$$\sum_{k=1}^{\infty}\frac{H(k)}{k^3(k+1)^5(k+2)^2} = \frac{1}{16}\big(-17+85\zeta(2)-94\zeta(3)-7\zeta(4)-32\zeta(5)+16\zeta(2)\zeta(3) -12\zeta(6)+8\zeta(3)^2\big) \quad (811)$$

$$\sum_{k=1}^{\infty}\frac{H(k)}{k^2(k+1)^6(k+2)^2} = \frac{1}{4}(-9-6\zeta(2)+15\zeta(3)+16\zeta(5)-8\zeta(2)\zeta(3)+12\zeta(7) -4\zeta(2)\zeta(5)-4\zeta(3)\zeta(4)) \quad (812)$$

$$\sum_{k=1}^{\infty}\frac{H(k)}{k(k+1)^7(k+2)^2} = \frac{1}{4}\big(-19+3\zeta(2)+14\zeta(3)-3\zeta(4)+16\zeta(5)-8\zeta(2)\zeta(3)-6\zeta(6) +4\zeta(3)^2+12\zeta(7)-4\zeta(2)\zeta(5)-4\zeta(3)\zeta(4)-5\zeta(8)+4\zeta(3)\zeta(5)\big) \quad (813)$$

$$\sum_{k=1}^{\infty}\frac{H(k)}{(k+1)^8(k+2)^2} = \frac{1}{2}(20-2\zeta(2)-16\zeta(3)+3\zeta(4)-20\zeta(5)+10\zeta(2)\zeta(3)+6\zeta(6) -4\zeta(3)^2-18\zeta(7)+6\zeta(2)\zeta(5)+6\zeta(3)\zeta(4)+5\zeta(8)-4\zeta(3)\zeta(5)-8\zeta(9) +2\zeta(3)\zeta(6)+2\zeta(4)\zeta(5)+2\zeta(2)\zeta(7)) \quad (814)$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^7(k+2)^3} = \frac{1}{256} \big(34 + 5\zeta(2) - 51\zeta(3) + 37\zeta(4) - 120\zeta(5) + 40\zeta(2)\zeta(3) + 84\zeta(6) - 24\zeta(3)^2 - 192\zeta(7) + 48\zeta(2)\zeta(5) + 48\zeta(3)\zeta(4) + 72\zeta(8) - 32\zeta(3)\zeta(5)\big) \tag{815}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^6(k+1)(k+2)^3} = \frac{1}{256} \big(81 - 239\zeta(2) + 376\zeta(3) - 211\zeta(4) + 384\zeta(5) - 128\zeta(2)\zeta(3) - 140\zeta(6) + 40\zeta(3)^2 + 128\zeta(7) - 32\zeta(2)\zeta(5) - 32\zeta(3)\zeta(4)\big) \tag{816}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^5(k+1)^2(k+2)^3} = \frac{1}{128} \big(95 + 281\zeta(2) - 558\zeta(3) + 149\zeta(4) - 168\zeta(5) + 56\zeta(2)\zeta(3) + 28\zeta(6) - 8\zeta(3)^2\big) \tag{817}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^4(k+1)^3(k+2)^3} = \frac{1}{32} \big(55 - 111\zeta(2) + 116\zeta(3) - 15\zeta(4) + 12\zeta(5) - 4\zeta(2)\zeta(3)\big) \tag{818}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^3(k+1)^4(k+2)^3} = \frac{1}{16} \big(63 + 22\zeta(2) - 83\zeta(3) + 2\zeta(4) - 32\zeta(5) + 16\zeta(2)\zeta(3)\big) \tag{819}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^2(k+1)^5(k+2)^3} = \frac{1}{16} \big(143 - 41\zeta(2) - 72\zeta(3) + 11\zeta(4) - 32\zeta(5) + 16\zeta(2)\zeta(3) + 12\zeta(6) - 8\zeta(3)^2\big) \tag{820}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k(k+1)^6(k+2)^3} = \frac{1}{8} \big(161 - 29\zeta(2) - 102\zeta(3) + 11\zeta(4) - 64\zeta(5) + 32\zeta(2)\zeta(3) + 12\zeta(6) - 8\zeta(3)^2 - 24\zeta(7) + 8\zeta(2)\zeta(5) + 8\zeta(3)\zeta(4)\big) \tag{821}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{(k+1)^7(k+2)^3} = \frac{1}{4} \big(180 - 32\zeta(2) - 116\zeta(3) + 14\zeta(4) - 80\zeta(5) + 40\zeta(2)\zeta(3) + 18\zeta(6) - 12\zeta(3)^2 - 36\zeta(7) + 12\zeta(2)\zeta(5) + 12\zeta(3)\zeta(4) + 5\zeta(8) - 4\zeta(3)\zeta(5)\big) \tag{822}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^6(k+2)^4} = \frac{1}{256} \big(122 - 9\zeta(2) - 107\zeta(3) + 43\zeta(4) - 128\zeta(5) + 44\zeta(2)\zeta(3) + 56\zeta(6) - 16\zeta(3)^2 - 64\zeta(7) + 16\zeta(2)\zeta(5) + 16\zeta(3)\zeta(4)\big) \tag{823}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^5(k+1)(k+2)^4} = \frac{1}{256} \big(325 - 257\zeta(2) + 162\zeta(3) - 125\zeta(4) + 128\zeta(5) - 40\zeta(2)\zeta(3) - 28\zeta(6) + 8\zeta(3)^2\big) \tag{824}$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^4(k+1)^2(k+2)^4} = \frac{1}{32}\left(-105 - 6\zeta(2) + 99\zeta(3) - 6\zeta(4) + 10\zeta(5) - 4\zeta(2)\zeta(3)\right) \quad (825)$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^3(k+1)^3(k+2)^4} = \frac{1}{32}\left(265 - 99\zeta(2) - 82\zeta(3) - 3\zeta(4) - 8\zeta(5) + 4\zeta(2)\zeta(3)\right) \quad (826)$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^2(k+1)^4(k+2)^4} = \frac{1}{16}\left(-328 + 77\zeta(2) + 165\zeta(3) + \zeta(4) + 40\zeta(5) - 20\zeta(2)\zeta(3)\right) \quad (827)$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k(k+1)^5(k+2)^4} = \frac{1}{16}\left(799 - 195\zeta(2) - 402\zeta(3) + 9\zeta(4) - 112\zeta(5) + 56\zeta(2)\zeta(3) + 12\zeta(6) - 8\zeta(3)^2\right) \quad (828)$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{(k+1)^6(k+2)^4} = \frac{1}{2}\left(-240 + 56\zeta(2) + 126\zeta(3) - 5\zeta(4) + 44\zeta(5) - 22\zeta(2)\zeta(3) - 6\zeta(6) + 4\zeta(3)^2 + 6\zeta(7) - 2\zeta(2)\zeta(5) - 2\zeta(3)\zeta(4)\right) \quad (829)$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^5(k+2)^5} = \frac{1}{256}\left(315 - 58\zeta(2) - 163\zeta(3) + 2\zeta(4) - 108\zeta(5) + 40\zeta(2)\zeta(3) + 8\zeta(6)\right) \quad (830)$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^4(k+1)(k+2)^5} = \frac{1}{256}\left(955 - 373\zeta(2) - 164\zeta(3) - 121\zeta(4) - 88\zeta(5) + 40\zeta(2)\zeta(3) - 12\zeta(6) + 8\zeta(3)^2\right) \quad (831)$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^3(k+1)^2(k+2)^5} = \frac{1}{128}\left(1375 - 349\zeta(2) - 560\zeta(3) - 97\zeta(4) - 128\zeta(5) + 56\zeta(2)\zeta(3) - 12\zeta(6) + 8\zeta(3)^2\right) \quad (832)$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^2(k+1)^3(k+2)^5} = \frac{1}{64}\left(1905 - 547\zeta(2) - 724\zeta(3) - 103\zeta(4) - 144\zeta(5) + 64\zeta(2)\zeta(3) - 12\zeta(6) + 8\zeta(3)^2\right) \quad (833)$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k(k+1)^4(k+2)^5} = \frac{1}{32}\left(2561 - 701\zeta(2) - 1054\zeta(3) - 105\zeta(4) - 224\zeta(5) + 104\zeta(2)\zeta(3) - 12\zeta(6) + 8\zeta(3)^2\right) \quad (834)$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{(k+1)^5(k+2)^5} = 210 - 56\zeta(2) - 91\zeta(3) - 6\zeta(4) - 21\zeta(5) + 10\zeta(2)\zeta(3) \quad (835)$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^4(k+2)^6} = \frac{1}{256} \big(634 - 135\zeta(2) - 205\zeta(3) - 83\zeta(4) - 140\zeta(5) + 44\zeta(2)\zeta(3) - 40\zeta(6) + 16\zeta(3)^2 - 48\zeta(7) + 16\zeta(2)\zeta(5) + 16\zeta(3)\zeta(4)\big) \quad (836)$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^3(k+1)(k+2)^6} = \frac{1}{256} \big(2223 - 643\zeta(2) - 574\zeta(3) - 287\zeta(4) - 368\zeta(5) + 128\zeta(2)\zeta(3) - 92\zeta(6) + 40\zeta(3)^2 - 96\zeta(7) + 32\zeta(2)\zeta(5) + 32\zeta(3)\zeta(4)\big) \quad (837)$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^2(k+1)^2(k+2)^6} = \frac{1}{64} (1799 - 496\zeta(2) - 567\zeta(3) - 192\zeta(4) - 248\zeta(5) + 92\zeta(2)\zeta(3) - 52\zeta(6) + 24\zeta(3)^2 - 48\zeta(7) + 16\zeta(2)\zeta(5) + 16\zeta(3)\zeta(4)) \quad (838)$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k(k+1)^3(k+2)^6} = \frac{1}{64} (-5503 + 1539\zeta(2) + 1858\zeta(3) + 487\zeta(4) + 640\zeta(5) - 248\zeta(2)\zeta(3) + 116\zeta(6) - 56\zeta(3)^2 + 96\zeta(7) - 32\zeta(2)\zeta(5) - 32\zeta(3)\zeta(4)) \quad (839)$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{(k+1)^4(k+2)^6} = \frac{1}{2} \big(504 - 140\zeta(2) - 182\zeta(3) - 37\zeta(4) - 54\zeta(5) + 22\zeta(2)\zeta(3) - 8\zeta(6) + 4\zeta(3)^2 - 6\zeta(7) + 2\zeta(2)\zeta(5) + 2\zeta(3)\zeta(4)\big) \quad (840)$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^3(k+2)^7} = \frac{1}{256} (1058 - 205\zeta(2) - 233\zeta(3) - 173\zeta(4) - 208\zeta(5) + 40\zeta(2)\zeta(3) - 116\zeta(6) + 24\zeta(3)^2 - 176\zeta(7) + 48\zeta(2)\zeta(5) + 48\zeta(3)\zeta(4) - 40\zeta(8) + 32\zeta(3)\zeta(5)) \quad (841)$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^2(k+1)(k+2)^7} = \frac{1}{256} (4339 - 1053\zeta(2) - 1040\zeta(3) - 633\zeta(4) - 784\zeta(5) + 208\zeta(2)\zeta(3) - 324\zeta(6) + 88\zeta(3)^2 - 448\zeta(7) + 128\zeta(2)\zeta(5) + 128\zeta(3)\zeta(4) - 80\zeta(8) + 64\zeta(3)\zeta(5)) \quad (842)$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k(k+1)^2(k+2)^7} = \frac{1}{128} (7937 - 2045\zeta(2) - 2174\zeta(3) - 1017\zeta(4) - 1280\zeta(5) + 392\zeta(2)\zeta(3) - 428\zeta(6) + 136\zeta(3)^2 - 544\zeta(7) + 160\zeta(2)\zeta(5) + 160\zeta(3)\zeta(4) - 80\zeta(8) + 64\zeta(3)\zeta(5)) \quad (843)$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{(k+1)^3(k+2)^7} = \frac{1}{4} (840 - 224\zeta(2) - 252\zeta(3) - 94\zeta(4) - 120\zeta(5) + 40\zeta(2)\zeta(3) - 34\zeta(6) + 12\zeta(3)^2 - 40\zeta(7) + 12\zeta(2)\zeta(5) + 12\zeta(3)\zeta(4) - 5\zeta(8) + 4\zeta(3)\zeta(5)) \quad (844)$$

$$\sum_{k=1}^{\infty}\frac{H(k)}{k^2(k+2)^8} = \frac{1}{256}(1542-243\zeta(2)-249\zeta(3)-231\zeta(4)-248\zeta(5)+20\zeta(2)\zeta(3)$$
$$-200\zeta(6)+16\zeta(3)^2-272\zeta(7)+48\zeta(2)\zeta(5)+48\zeta(3)\zeta(4)-144\zeta(8)$$
$$+64\zeta(3)\zeta(5)-256\zeta(9)+64\zeta(3)\zeta(6)+64\zeta(4)\zeta(5)+64\zeta(2)\zeta(7)) \quad (845)$$

$$\sum_{k=1}^{\infty}\frac{H(k)}{k(k+1)(k+2)^8} = \frac{1}{256}(-7423+1539\zeta(2)+1538\zeta(3)+1095\zeta(4)+1280\zeta(5)-248\zeta(2)\zeta(3)$$
$$+724\zeta(6)-120\zeta(3)^2+992\zeta(7)-224\zeta(2)\zeta(5)-224\zeta(3)\zeta(4)+368\zeta(8)$$
$$-192\zeta(3)\zeta(5)+512\zeta(9)-128\zeta(3)\zeta(6)-128\zeta(4)\zeta(5)$$
$$-128\zeta(2)\zeta(7)) \quad (846)$$

$$\sum_{k=1}^{\infty}\frac{H(k)}{(k+1)^2(k+2)^8} = \frac{1}{2}(-240+56\zeta(2)+58\zeta(3)+33\zeta(4)+40\zeta(5)-10\zeta(2)\zeta(3)$$
$$+18\zeta(6)-4\zeta(3)^2+24\zeta(7)-6\zeta(2)\zeta(5)-6\zeta(3)\zeta(4)+7\zeta(8)$$
$$-4\zeta(3)\zeta(5)+8\zeta(9)-2\zeta(3)\zeta(6)-2\zeta(4)\zeta(5)-2\zeta(2)\zeta(7)) \quad (847)$$

$$\sum_{k=1}^{\infty}\frac{H(k)}{k(k+2)^9} = \frac{1}{512}(4097-509\zeta(2)-510\zeta(3)-505\zeta(4)-512\zeta(5)+8\zeta(2)\zeta(3)$$
$$-492\zeta(6)+8\zeta(3)^2-544\zeta(7)+32\zeta(2)\zeta(5)+32\zeta(3)\zeta(4)-464\zeta(8)$$
$$+64\zeta(3)\zeta(5)-768\zeta(9)+128\zeta(3)\zeta(6)+128\zeta(4)\zeta(5)+128\zeta(2)\zeta(7)$$
$$-448\zeta(10)+256\zeta(3)\zeta(7)+128\zeta(5)^2\big) \quad (848)$$

$$\sum_{k=1}^{\infty}\frac{H(k)}{(k+1)(k+2)^9} = \frac{1}{4}(180-32\zeta(2)-32\zeta(3)-25\zeta(4)-28\zeta(5)+4\zeta(2)\zeta(3)-19\zeta(6)$$
$$+2\zeta(3)^2-24\zeta(7)+4\zeta(2)\zeta(5)+4\zeta(3)\zeta(4)-13\zeta(8)+4\zeta(3)\zeta(5)$$
$$-20\zeta(9)+4\zeta(3)\zeta(6)+4\zeta(4)\zeta(5)+4\zeta(2)\zeta(7)-7\zeta(10)+4\zeta(3)\zeta(7)$$
$$+2\zeta(5)^2\big) \quad (849)$$

$$\sum_{k=1}^{\infty}\frac{H(k)}{(k+2)^{10}} = 10-\zeta(2)-\zeta(3)-\zeta(4)-\zeta(5)-\zeta(6)-\zeta(7)-\zeta(8)-\zeta(9)-\zeta(10)-5\zeta(11)$$
$$+\zeta(2)\zeta(9)+\zeta(3)\zeta(8)+\zeta(4)\zeta(7)+\zeta(5)\zeta(6) \quad (850)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^2}{k^9} = \frac{1}{2}(26\zeta(11)-2\zeta(2)\zeta(9)-9\zeta(3)\zeta(8)-5\zeta(4)\zeta(7)-7\zeta(5)\zeta(6)$$
$$+2\zeta(3)^2\zeta(5)\big) \quad (851)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^2}{k^8(k+1)} = \frac{1}{24}\big(-72\zeta(3)+102\zeta(4)-84\zeta(5)+24\zeta(2)\zeta(3)+97\zeta(6)-48\zeta(3)^2$$
$$-144\zeta(7)+24\zeta(2)\zeta(5)+60\zeta(3)\zeta(4)+24M(2,6)-220\zeta(9)+84\zeta(3)\zeta(6)$$
$$+60\zeta(4)\zeta(5)+24\zeta(2)\zeta(7)-8\zeta(3)^3+24M(2,8)\big) \quad (852)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k^7(k+1)^2} = \frac{1}{12}\big(252\zeta(3) - 339\zeta(4) + 210\zeta(5) - 60\zeta(2)\zeta(3) - 194\zeta(6) + 96\zeta(3)^2 + 216\zeta(7) - 36\zeta(2)\zeta(5) - 90\zeta(3)\zeta(4) - 24M(2,6) + 110\zeta(9) - 42\zeta(3)\zeta(6) - 30\zeta(4)\zeta(5) - 12\zeta(2)\zeta(7) + 4\zeta(3)^3\big) \tag{853}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k^6(k+1)^3} = \frac{1}{4}\big(-252\zeta(3) + 321\zeta(4) - 146\zeta(5) + 44\zeta(2)\zeta(3) + 97\zeta(6) - 48\zeta(3)^2 - 72\zeta(7) + 12\zeta(2)\zeta(5) + 30\zeta(3)\zeta(4) + 4M(2,6)\big) \tag{854}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k^5(k+1)^4} = \frac{1}{24}\big(2520\zeta(3) - 3030\zeta(4) + 1020\zeta(5) - 360\zeta(2)\zeta(3) - 425\zeta(6) + 216\zeta(3)^2 + 144\zeta(7) - 24\zeta(2)\zeta(5) - 60\zeta(3)\zeta(4)\big) \tag{855}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k^4(k+1)^5} = \frac{1}{24}\big(2520\zeta(3) - 2850\zeta(4) + 780\zeta(5) - 360\zeta(2)\zeta(3) - 245\zeta(6) + 144\zeta(3)^2 + 24\zeta(7) - 24\zeta(2)\zeta(5) + 12\zeta(3)\zeta(4)\big) \tag{856}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k^3(k+1)^6} = \frac{1}{4}\big(252\zeta(3) - 267\zeta(4) + 74\zeta(5) - 44\zeta(2)\zeta(3) - 37\zeta(6) + 24\zeta(3)^2 + 12\zeta(7) - 12\zeta(2)\zeta(5) + 6\zeta(3)\zeta(4) + 14\zeta(8) - 8\zeta(3)\zeta(5) - 4M(2,6)\big) \tag{857}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k^2(k+1)^7} = \frac{1}{12}\big(-252\zeta(3) + 249\zeta(4) - 90\zeta(5) + 60\zeta(2)\zeta(3) + 74\zeta(6) - 48\zeta(3)^2 - 36\zeta(7) + 36\zeta(2)\zeta(5) - 18\zeta(3)\zeta(4) - 84\zeta(8) + 48\zeta(3)\zeta(5) + 24M(2,6) + 2\zeta(9) - 18\zeta(3)\zeta(6) - 6\zeta(4)\zeta(5) + 12\zeta(2)\zeta(7) + 4\zeta(3)^3\big) \tag{858}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k(k+1)^8} = \frac{1}{24}\big(72\zeta(3) - 66\zeta(4) + 36\zeta(5) - 24\zeta(2)\zeta(3) - 37\zeta(6) + 24\zeta(3)^2 + 24\zeta(7) - 24\zeta(2)\zeta(5) + 12\zeta(3)\zeta(4) + 84\zeta(8) - 48\zeta(3)\zeta(5) - 24M(2,6) - 4\zeta(9) + 36\zeta(3)\zeta(6) + 12\zeta(4)\zeta(5) - 24\zeta(2)\zeta(7) - 8\zeta(3)^3 + 108\zeta(10) - 48\zeta(3)\zeta(7) - 24\zeta(5)^2 - 24M(2,8)\big) \tag{859}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{(k+1)^9} = \frac{1}{2}\big(4\zeta(11) + 2\zeta(2)\zeta(9) - 5\zeta(3)\zeta(8) - \zeta(4)\zeta(7) - 3\zeta(5)\zeta(6) + 2\zeta(3)^2\zeta(5)\big) \tag{860}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k^8(k+2)} = \frac{1}{1536}\big(-6 - 6\zeta(2) - 18\zeta(3) + 51\zeta(4) - 84\zeta(5) + 24\zeta(2)\zeta(3) + 194\zeta(6) - 96\zeta(3)^2 - 576\zeta(7) + 96\zeta(2)\zeta(5) + 240\zeta(3)\zeta(4) + 192M(2,6) - 3520\zeta(9) + 1344\zeta(3)\zeta(6) + 960\zeta(4)\zeta(5) + 384\zeta(2)\zeta(7) - 128\zeta(3)^3 + 768M(2,8)\big) \tag{861}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k^7(k+1)(k+2)} = \frac{1}{768}\,(6+6\zeta(2)-2286\zeta(3)+3213\zeta(4)-2604\zeta(5)+744\zeta(2)\zeta(3)$$
$$+2910\zeta(6)-1440\zeta(3)^2-4032\zeta(7)+672\zeta(2)\zeta(5)+1680\zeta(3)\zeta(4)+576M(2,6)$$
$$-3520\zeta(9)+1344\zeta(3)\zeta(6)+960\zeta(4)\zeta(5)+384\zeta(2)\zeta(7)-128\zeta(3)^3\big) \qquad (862)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k^6(k+1)^2(k+2)} = \frac{1}{384}\,(6+6\zeta(2)+5778\zeta(3)-7635\zeta(4)+4116\zeta(5)$$
$$-1176\zeta(2)\zeta(3)-3298\zeta(6)+1632\zeta(3)^2+2880\zeta(7)-480\zeta(2)\zeta(5)$$
$$-1200\zeta(3)\zeta(4)-192M(2,6)) \qquad (863)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k^5(k+1)^3(k+2)} = \frac{1}{192}\,(-6-6\zeta(2)+6318\zeta(3)-7773\zeta(4)+2892\zeta(5)$$
$$-936\zeta(2)\zeta(3)-1358\zeta(6)+672\zeta(3)^2+576\zeta(7)-96\zeta(2)\zeta(5)$$
$$-240\zeta(3)\zeta(4)) \qquad (864)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k^4(k+1)^4(k+2)} = \frac{1}{32}\,(2+2\zeta(2)+1254\zeta(3)-1449\zeta(4)+396\zeta(5)-168\zeta(2)\zeta(3)$$
$$-114\zeta(6)+64\zeta(3)^2\big) \qquad (865)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k^3(k+1)^5(k+2)} = \frac{1}{48}\,(6+6\zeta(2)-1278\zeta(3)+1353\zeta(4)-372\zeta(5)+216\zeta(2)\zeta(3)$$
$$+148\zeta(6)-96\zeta(3)^2-48\zeta(7)+48\zeta(2)\zeta(5)-24\zeta(3)\zeta(4)\big) \qquad (866)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k^2(k+1)^6(k+2)} = \frac{1}{24}\,(6+6\zeta(2)+234\zeta(3)-249\zeta(4)+72\zeta(5)-48\zeta(2)\zeta(3)$$
$$-74\zeta(6)+48\zeta(3)^2+24\zeta(7)-24\zeta(2)\zeta(5)+12\zeta(3)\zeta(4)+84\zeta(8)$$
$$-48\zeta(3)\zeta(5)-24M(2,6)) \qquad (867)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k(k+1)^7(k+2)} = \frac{1}{6}\,(3+3\zeta(2)-9\zeta(3)-9\zeta(5)+6\zeta(2)\zeta(3)-6\zeta(7)$$
$$+6\zeta(2)\zeta(5)-3\zeta(3)\zeta(4)+\zeta(9)-9\zeta(3)\zeta(6)-3\zeta(4)\zeta(5)$$
$$+6\zeta(2)\zeta(7)+2\zeta(3)^3\big) \qquad (868)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{(k+1)^8(k+2)} = \frac{1}{24}\,(24+24\zeta(2)-66\zeta(4)-36\zeta(5)+24\zeta(2)\zeta(3)-37\zeta(6)$$
$$+24\zeta(3)^2-24\zeta(7)+24\zeta(2)\zeta(5)-12\zeta(3)\zeta(4)+84\zeta(8)-48\zeta(3)\zeta(5)$$
$$-24M(2,6)+4\zeta(9)-36\zeta(3)\zeta(6)-12\zeta(4)\zeta(5)+24\zeta(2)\zeta(7)+8\zeta(3)^3$$
$$+108\zeta(10)-48\zeta(3)\zeta(7)-24\zeta(5)^2-24M(2,8)\big) \qquad (869)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k^7(k+2)^2} = \frac{1}{1536}\,(78+42\zeta(2)+102\zeta(3)-339\zeta(4)+420\zeta(5)-120\zeta(2)\zeta(3)$$
$$-776\zeta(6)+384\zeta(3)^2+1728\zeta(7)-288\zeta(2)\zeta(5)-720\zeta(3)\zeta(4)-384M(2,6)$$
$$+3520\zeta(9)-1344\zeta(3)\zeta(6)-960\zeta(4)\zeta(5)-384\zeta(2)\zeta(7)+128\zeta(3)^3\big) \qquad (870)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^2}{k^6(k+1)(k+2)^2} = \frac{1}{384}\,(42+24\zeta(2)-1092\zeta(3)+1437\zeta(4)-1092\zeta(5) +312\zeta(2)\zeta(3)+1067\zeta(6)-528\zeta(3)^2-1152\zeta(7)+192\zeta(2)\zeta(5) +480\zeta(3)\zeta(4)+96M(2,6)) \tag{871}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^2}{k^5(k+1)^2(k+2)^2} = \frac{1}{128}\,(30+18\zeta(2)+1198\zeta(3)-1587\zeta(4)+644\zeta(5) -184\zeta(2)\zeta(3)-388\zeta(6)+192\zeta(3)^2+192\zeta(7)-32\zeta(2)\zeta(5) -80\zeta(3)\zeta(4)) \tag{872}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^2}{k^4(k+1)^3(k+2)^2} = \frac{1}{96}\,(48+30\zeta(2)-1362\zeta(3)+1506\zeta(4)-480\zeta(5) +192\zeta(2)\zeta(3)+97\zeta(6)-48\zeta(3)^2) \tag{873}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^2}{k^3(k+1)^4(k+2)^2} = \frac{1}{96}\,(102+66\zeta(2)+1038\zeta(3)-1335\zeta(4)+228\zeta(5) -120\zeta(2)\zeta(3)-148\zeta(6)+96\zeta(3)^2) \tag{874}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^2}{k^2(k+1)^5(k+2)^2} = \frac{1}{8}\,(18+12\zeta(2)-40\zeta(3)+3\zeta(4)-24\zeta(5)+16\zeta(2)\zeta(3) -8\zeta(7)+8\zeta(2)\zeta(5)-4\zeta(3)\zeta(4)) \tag{875}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^2}{k(k+1)^6(k+2)^2} = \frac{1}{24}\,(114+78\zeta(2)-6\zeta(3)-231\zeta(4)-72\zeta(5)+48\zeta(2)\zeta(3) -74\zeta(6)+48\zeta(3)^2-24\zeta(7)+24\zeta(2)\zeta(5)-12\zeta(3)\zeta(4)+84\zeta(8) -48\zeta(3)\zeta(5)-24M(2,6)) \tag{876}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^2}{(k+1)^7(k+2)^2} = \frac{1}{12}\,(120+84\zeta(2)-24\zeta(3)-231\zeta(4)-90\zeta(5)+60\zeta(2)\zeta(3) -74\zeta(6)+48\zeta(3)^2-36\zeta(7)+36\zeta(2)\zeta(5)-18\zeta(3)\zeta(4)+84\zeta(8) -48\zeta(3)\zeta(5)-24M(2,6)+2\zeta(9)-18\zeta(3)\zeta(6)-6\zeta(4)\zeta(5)+12\zeta(2)\zeta(7) +4\zeta(3)^3) \tag{877}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^2}{k^6(k+2)^3} = \frac{1}{512}\,(-162-34\zeta(2)-54\zeta(3)+325\zeta(4)-292\zeta(5)+88\zeta(2)\zeta(3) +388\zeta(6)-192\zeta(3)^2-576\zeta(7)+96\zeta(2)\zeta(5)+240\zeta(3)\zeta(4)+64M(2,6)) \tag{878}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^2}{k^5(k+1)(k+2)^3} = \frac{1}{768}\,(-570-150\zeta(2)+2022\zeta(3)-1899\zeta(4)+1308\zeta(5) -360\zeta(2)\zeta(3)-970\zeta(6)+480\zeta(3)^2+576\zeta(7)-96\zeta(2)\zeta(5) -240\zeta(3)\zeta(4)) \tag{879}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k^4(k+1)^2(k+2)^3} = \frac{1}{192}\left(330 + 102\zeta(2) + 786\zeta(3) - 1431\zeta(4) + 312\zeta(5)\right.$$
$$\left. -96\zeta(2)\zeta(3) - 97\zeta(6) + 48\zeta(3)^2\right) \quad (880)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k^3(k+1)^3(k+2)^3} = \frac{1}{32}\left(126 + 44\zeta(2) - 192\zeta(3) + 25\zeta(4) - 56\zeta(5)\right.$$
$$\left. +32\zeta(2)\zeta(3)\right) \quad (881)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k^2(k+1)^4(k+2)^3} = \frac{1}{96}\left(-858 - 330\zeta(2) + 114\zeta(3) + 1185\zeta(4) + 108\zeta(5)\right.$$
$$\left. -72\zeta(2)\zeta(3) + 148\zeta(6) - 96\zeta(3)^2\right) \quad (882)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k(k+1)^5(k+2)^3} = \frac{1}{48}\left(966 + 402\zeta(2) - 354\zeta(3) - 1167\zeta(4) - 252\zeta(5)\right.$$
$$+168\zeta(2)\zeta(3) - 148\zeta(6) + 96\zeta(3)^2 - 48\zeta(7) + 48\zeta(2)\zeta(5)$$
$$\left. -24\zeta(3)\zeta(4)\right) \quad (883)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{(k+1)^6(k+2)^3} = \frac{1}{4}\left(180 + 80\zeta(2) - 60\zeta(3) - 233\zeta(4) - 54\zeta(5) + 36\zeta(2)\zeta(3)\right.$$
$$-37\zeta(6) + 24\zeta(3)^2 - 12\zeta(7) + 12\zeta(2)\zeta(5) - 6\zeta(3)\zeta(4) + 14\zeta(8)$$
$$\left. -8\zeta(3)\zeta(5) - 4M(2,6)\right) \quad (884)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k^5(k+2)^4} = \frac{1}{1536}\left(1950 - 6\zeta(2) - 282\zeta(3) - 1647\zeta(4) + 828\zeta(5) - 264\zeta(2)\zeta(3)\right.$$
$$\left. -850\zeta(6) + 432\zeta(3)^2 + 576\zeta(7) - 96\zeta(2)\zeta(5) - 240\zeta(3)\zeta(4)\right) \quad (885)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k^4(k+1)(k+2)^4} = \frac{1}{64}\left(210 + 12\zeta(2) - 192\zeta(3) + 21\zeta(4) - 40\zeta(5) + 8\zeta(2)\zeta(3)\right.$$
$$\left. +10\zeta(6) - 4\zeta(3)^2\right) \quad (886)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k^3(k+1)^2(k+2)^4} = \frac{1}{192}\left(1590 + 174\zeta(2) - 366\zeta(3) - 1305\zeta(4) + 72\zeta(5)\right.$$
$$\left. -48\zeta(2)\zeta(3) - 37\zeta(6) + 24\zeta(3)^2\right) \quad (887)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k^2(k+1)^3(k+2)^4} = \frac{1}{96}\left(1968 + 306\zeta(2) - 942\zeta(3) - 1230\zeta(4) - 96\zeta(5)\right.$$
$$\left. +48\zeta(2)\zeta(3) - 37\zeta(6) + 24\zeta(3)^2\right) \quad (888)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k(k+1)^4(k+2)^4} = \frac{1}{32}\left(1598 + 314\zeta(2) - 666\zeta(3) - 1215\zeta(4) - 100\zeta(5)\right.$$
$$\left. +56\zeta(2)\zeta(3) - 74\zeta(6) + 48\zeta(3)^2\right) \quad (889)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^2}{(k+1)^5(k+2)^4}=\frac{1}{24}\left(2880+672\zeta(2)-1176\zeta(3)-2406\zeta(4)-276\zeta(5)\right.$$
$$+168\zeta(2)\zeta(3)-185\zeta(6)+120\zeta(3)^2-24\zeta(7)+24\zeta(2)\zeta(5)$$
$$\left.-12\zeta(3)\zeta(4)\right) \quad (890)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^2}{k^4(k+2)^5}=\frac{1}{1536}\left(-5730+702\zeta(2)+1818\zeta(3)+2073\zeta(4)+468\zeta(5)\right.$$
$$-216\zeta(2)\zeta(3)+634\zeta(6)-384\zeta(3)^2-96\zeta(7)+96\zeta(2)\zeta(5)$$
$$\left.-48\zeta(3)\zeta(4)\right) \quad (891)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^2}{k^3(k+1)(k+2)^5}=\frac{1}{768}\left(8250-558\zeta(2)-4122\zeta(3)-1821\zeta(4)-948\zeta(5)\right.$$
$$+312\zeta(2)\zeta(3)-514\zeta(6)+336\zeta(3)^2+96\zeta(7)-96\zeta(2)\zeta(5)$$
$$\left.+48\zeta(3)\zeta(4)\right) \quad (892)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^2}{k^2(k+1)^2(k+2)^5}=\frac{1}{128}\left(-3810+70\zeta(2)+1618\zeta(3)+1477\zeta(4)+268\zeta(5)\right.$$
$$-72\zeta(2)\zeta(3)+196\zeta(6)-128\zeta(3)^2-32\zeta(7)+32\zeta(2)\zeta(5)$$
$$\left.-16\zeta(3)\zeta(4)\right) \quad (893)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^2}{k(k+1)^3(k+2)^5}=\frac{1}{192}\left(-15366-402\zeta(2)+6738\zeta(3)+6891\zeta(4)+996\zeta(5)\right.$$
$$-312\zeta(2)\zeta(3)+662\zeta(6)-432\zeta(3)^2-96\zeta(7)+96\zeta(2)\zeta(5)$$
$$\left.-48\zeta(3)\zeta(4)\right) \quad (894)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^2}{(k+1)^4(k+2)^5}=\frac{1}{24}\left(-5040-336\zeta(2)+2184\zeta(3)+2634\zeta(4)+324\zeta(5)\right.$$
$$-120\zeta(2)\zeta(3)+221\zeta(6)-144\zeta(3)^2-24\zeta(7)+24\zeta(2)\zeta(5)$$
$$\left.-12\zeta(3)\zeta(4)\right) \quad (895)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^2}{k^3(k+2)^6}=\frac{1}{512}\left(4446-774\zeta(2)-1482\zeta(3)-915\zeta(4)-1100\zeta(5)\right.$$
$$+424\zeta(2)\zeta(3)-452\zeta(6)+256\zeta(3)^2-288\zeta(7)+32\zeta(2)\zeta(5)+176\zeta(3)\zeta(4)$$
$$\left.+224\zeta(8)-128\zeta(3)\zeta(5)-64M(2,6)\right) \quad (896)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^2}{k^2(k+1)(k+2)^6}=\frac{1}{384}\left(10794-1440\zeta(2)-4284\zeta(3)-2283\zeta(4)-2124\zeta(5)\right.$$
$$+792\zeta(2)\zeta(3)-935\zeta(6)+552\zeta(3)^2-384\zeta(7)+288\zeta(3)\zeta(4)+336\zeta(8)$$
$$\left.-192\zeta(3)\zeta(5)-96M(2,6)\right) \quad (897)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^2}{k(k+1)^2(k+2)^6} = \frac{1}{384}\big(33018 - 3090\zeta(2) - 13422\zeta(3) - 8997\zeta(4) - 5052\zeta(5)$$
$$+1800\zeta(2)\zeta(3) - 2458\zeta(6) + 1488\zeta(3)^2 - 672\zeta(7) - 96\zeta(2)\zeta(5)$$
$$+624\zeta(3)\zeta(4) + 672\zeta(8) - 384\zeta(3)\zeta(5) - 192M(2,6)\big) \quad (898)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^2}{(k+1)^3(k+2)^6} = \frac{1}{4}\big(1008 - 56\zeta(2) - 420\zeta(3) - 331\zeta(4) - 126\zeta(5) + 44\zeta(2)\zeta(3)$$
$$-65\zeta(6) + 40\zeta(3)^2 - 12\zeta(7) - 4\zeta(2)\zeta(5) + 14\zeta(3)\zeta(4) + 14\zeta(8)$$
$$-8\zeta(3)\zeta(5) - 4M(2,6)\big) \quad (899)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^2}{k^2(k+2)^7} = \frac{1}{1536}\big(26034 - 4782\zeta(2) - 7578\zeta(3) - 4389\zeta(4) - 7020\zeta(5)$$
$$+2376\zeta(2)\zeta(3) - 3032\zeta(6) + 1248\zeta(3)^2 - 4704\zeta(7) + 1248\zeta(2)\zeta(5)$$
$$+1680\zeta(3)\zeta(4) + 384\zeta(8) - 384M(2,6) - 64\zeta(9) + 576\zeta(3)\zeta(6) + 192\zeta(4)\zeta(5)$$
$$-384\zeta(2)\zeta(7) - 128\zeta(3)^3\big) \quad (900)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^2}{k(k+1)(k+2)^7} = \frac{1}{768}\big(47622 - 7662\zeta(2) - 16146\zeta(3) - 8955\zeta(4) - 11268\zeta(5)$$
$$+3960\zeta(2)\zeta(3) - 4902\zeta(6) + 2352\zeta(3)^2 - 5472\zeta(7) + 1248\zeta(2)\zeta(5)$$
$$+2256\zeta(3)\zeta(4) + 1056\zeta(8) - 384\zeta(3)\zeta(5) - 576M(2,6) - 64\zeta(9) + 576\zeta(3)\zeta(6)$$
$$+192\zeta(4)\zeta(5) - 384\zeta(2)\zeta(7) - 128\zeta(3)^3\big) \quad (901)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^2}{(k+1)^2(k+2)^7} = \frac{1}{12}\big(2520 - 336\zeta(2) - 924\zeta(3) - 561\zeta(4) - 510\zeta(5)$$
$$+180\zeta(2)\zeta(3) - 230\zeta(6) + 120\zeta(3)^2 - 192\zeta(7) + 36\zeta(2)\zeta(5) + 90\zeta(3)\zeta(4)$$
$$+54\zeta(8) - 24\zeta(3)\zeta(5) - 24M(2,6) - 2\zeta(9) + 18\zeta(3)\zeta(6) + 6\zeta(4)\zeta(5)$$
$$-12\zeta(2)\zeta(7) - 4\zeta(3)^3\big) \quad (902)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^2}{k(k+2)^8} = \frac{1}{1536}\big(44538 - 7698\zeta(2) - 10734\zeta(3) - 6981\zeta(4) - 10620\zeta(5)$$
$$+2952\zeta(2)\zeta(3) - 5498\zeta(6) + 1488\zeta(3)^2 - 9888\zeta(7) + 2592\zeta(2)\zeta(5)$$
$$+2736\zeta(3)\zeta(4) - 2976\zeta(8) + 1920\zeta(3)\zeta(5) - 192M(2,6) - 6208\zeta(9)$$
$$+2112\zeta(3)\zeta(6) + 1728\zeta(4)\zeta(5) + 1152\zeta(2)\zeta(7) - 128\zeta(3)^3 + 3456\zeta(10)$$
$$-1536\zeta(3)\zeta(7) - 768\zeta(5)^2 - 768M(2,8)\big) \quad (903)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^2}{(k+1)(k+2)^8} = \frac{1}{24}\big(2880 - 480\zeta(2) - 840\zeta(3) - 498\zeta(4) - 684\zeta(5) + 216\zeta(2)\zeta(3)$$
$$-325\zeta(6) + 120\zeta(3)^2 - 480\zeta(7) + 120\zeta(2)\zeta(5) + 156\zeta(3)\zeta(4) - 60\zeta(8)$$
$$+48\zeta(3)\zeta(5) - 24M(2,6) - 196\zeta(9) + 84\zeta(3)\zeta(6) + 60\zeta(4)\zeta(5) + 24\zeta(2)\zeta(7)$$
$$-8\zeta(3)^3 + 108\zeta(10) - 48\zeta(3)\zeta(7) - 24\zeta(5)^2 - 24M(2,8)\big) \quad (904)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^2}{(k+2)^9} = \frac{1}{2}\left(-90+14\zeta(2)+18\zeta(3)+13\zeta(4)+18\zeta(5)-4\zeta(2)\zeta(3)+11\zeta(6)\right.$$
$$-2\zeta(3)^2+18\zeta(7)-4\zeta(2)\zeta(5)-4\zeta(3)\zeta(4)+9\zeta(8)-4\zeta(3)\zeta(5)$$
$$+18\zeta(9)-4\zeta(3)\zeta(6)-4\zeta(4)\zeta(5)-4\zeta(2)\zeta(7)+7\zeta(10)-4\zeta(3)\zeta(7)$$
$$-2\zeta(5)^2+4\zeta(11)+2\zeta(2)\zeta(9)-5\zeta(3)\zeta(8)-\zeta(4)\zeta(7)-3\zeta(5)\zeta(6)$$
$$\left.+2\zeta(3)^2\zeta(5)\right) \tag{905}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^3}{k^8} = M(3,8) \tag{906}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^3}{k^7(k+1)} = \frac{1}{480}\left(4800\zeta(4)-4800\zeta(5)-480\zeta(2)\zeta(3)+2790\zeta(6)-1200\zeta(3)^2\right.$$
$$-6930\zeta(7)-960\zeta(2)\zeta(5)+6120\zeta(3)\zeta(4)-2975\zeta(8)-600\zeta(2)\zeta(3)^2$$
$$+2880\zeta(3)\zeta(5)+1320M(2,6)-10420\zeta(9)+5820\zeta(3)\zeta(6)+6120\zeta(4)\zeta(5)$$
$$-1440\zeta(2)\zeta(7)-960\zeta(3)^3-4983\zeta(10)+3840\zeta(3)\zeta(7)+240\zeta(3)^2\zeta(4)$$
$$\left.-1680\zeta(2)\zeta(3)\zeta(5)+2160\zeta(5)^2+1560M(2,8)\right) \tag{907}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^3}{k^6(k+1)^2} = \frac{1}{48}\left(2880\zeta(4)-2760\zeta(5)-288\zeta(2)\zeta(3)+1116\zeta(6)-480\zeta(3)^2\right.$$
$$-2079\zeta(7)-288\zeta(2)\zeta(5)+1836\zeta(3)\zeta(4)-595\zeta(8)-120\zeta(2)\zeta(3)^2$$
$$+576\zeta(3)\zeta(5)+264M(2,6)-1042\zeta(9)+582\zeta(3)\zeta(6)+612\zeta(4)\zeta(5)$$
$$\left.-144\zeta(2)\zeta(7)-96\zeta(3)^3\right) \tag{908}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^3}{k^5(k+1)^3} = \frac{1}{96}\left(14400\zeta(4)-13200\zeta(5)-1440\zeta(2)\zeta(3)+3546\zeta(6)\right.$$
$$-1632\zeta(3)^2-4158\zeta(7)-576\zeta(2)\zeta(5)+3672\zeta(3)\zeta(4)-595\zeta(8)$$
$$\left.-120\zeta(2)\zeta(3)^2+576\zeta(3)\zeta(5)+264M(2,6)\right) \tag{909}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^3}{k^4(k+1)^4} = \frac{1}{8}\left(1600\zeta(4)-1400\zeta(5)-160\zeta(2)\zeta(3)+252\zeta(6)-144\zeta(3)^2\right.$$
$$\left.-175\zeta(7)-32\zeta(2)\zeta(5)+168\zeta(3)\zeta(4)\right) \tag{910}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^3}{k^3(k+1)^5} = \frac{1}{96}\left(14400\zeta(4)-12000\zeta(5)-1440\zeta(2)\zeta(3)+1746\zeta(6)-1392\zeta(3)^2\right.$$
$$-2142\zeta(7)-576\zeta(2)\zeta(5)+2376\zeta(3)\zeta(4)+43\zeta(8)+120\zeta(2)\zeta(3)^2$$
$$\left.-288\zeta(3)\zeta(5)+24M(2,6)\right) \tag{911}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^3}{k^2(k+1)^6} = \frac{1}{48}\left(-2880\zeta(4)+2280\zeta(5)+288\zeta(2)\zeta(3)-396\zeta(6)+384\zeta(3)^2\right.$$
$$+1071\zeta(7)+288\zeta(2)\zeta(5)-1188\zeta(3)\zeta(4)-43\zeta(8)-120\zeta(2)\zeta(3)^2$$
$$+288\zeta(3)\zeta(5)-24M(2,6)+394\zeta(9)-222\zeta(3)\zeta(6)-396\zeta(4)\zeta(5)$$
$$\left.+144\zeta(2)\zeta(7)+48\zeta(3)^3\right) \tag{912}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^3}{k(k+1)^7}=\frac{1}{480}\big(4800\zeta(4)-3600\zeta(5)-480\zeta(2)\zeta(3)+990\zeta(6)-960\zeta(3)^2$$
$$-3570\zeta(7)-960\zeta(2)\zeta(5)+3960\zeta(3)\zeta(4)+215\zeta(8)+600\zeta(2)\zeta(3)^2$$
$$-1440\zeta(3)\zeta(5)+120M(2,6)-3940\zeta(9)+2220\zeta(3)\zeta(6)+3960\zeta(4)\zeta(5)$$
$$-1440\zeta(2)\zeta(7)-480\zeta(3)^3+1503\zeta(10)-2400\zeta(3)\zeta(7)-240\zeta(3)^2\zeta(4)$$
$$+1680\zeta(2)\zeta(3)\zeta(5)-1440\zeta(5)^2-120M(2,8)\big) \quad (913)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^3}{(k+1)^8}=\frac{1}{2}\,(44\zeta(11)-21\zeta(3)\zeta(8)-9\zeta(4)\zeta(7)-15\zeta(5)\zeta(6)$$
$$+6\zeta(3)^2\zeta(5)-2M(3,8)\big) \quad (914)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^3}{k^7(k+2)}=\frac{1}{7680}\,(60+120\zeta(2)+240\zeta(3)+600\zeta(4)-1200\zeta(5)-120\zeta(2)\zeta(3)$$
$$+1395\zeta(6)-600\zeta(3)^2-6930\zeta(7)-960\zeta(2)\zeta(5)+6120\zeta(3)\zeta(4)-5950\zeta(8)$$
$$-1200\zeta(2)\zeta(3)^2+5760\zeta(3)\zeta(5)+2640M(2,6)-41680\zeta(9)+23280\zeta(3)\zeta(6)$$
$$+24480\zeta(4)\zeta(5)-5760\zeta(2)\zeta(7)-3840\zeta(3)^3-39864\zeta(10)+30720\zeta(3)\zeta(7)$$
$$+1920\zeta(3)^2\zeta(4)-13440\zeta(2)\zeta(3)\zeta(5)+17280\zeta(5)^2+12480M(2,8)\big) \quad (915)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^3}{k^6(k+1)(k+2)}=\frac{1}{768}\,(12+24\zeta(2)+48\zeta(3)-7560\zeta(4)+7440\zeta(5)+744\zeta(2)\zeta(3)$$
$$-4185\zeta(6)+1800\zeta(3)^2+9702\zeta(7)+1344\zeta(2)\zeta(5)-8568\zeta(3)\zeta(4)+3570\zeta(8)$$
$$+720\zeta(2)\zeta(3)^2-3456\zeta(3)\zeta(5)-1584M(2,6)+8336\zeta(9)-4656\zeta(3)\zeta(6)$$
$$-4896\zeta(4)\zeta(5)+1152\zeta(2)\zeta(7)+768\zeta(3)^3\big) \quad (916)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^3}{k^5(k+1)^2(k+2)}=\frac{1}{384}\,(12+24\zeta(2)+48\zeta(3)+15480\zeta(4)-14640\zeta(5)$$
$$-1560\zeta(2)\zeta(3)+4743\zeta(6)-2040\zeta(3)^2-6930\zeta(7)-960\zeta(2)\zeta(5)$$
$$+6120\zeta(3)\zeta(4)-1190\zeta(8)-240\zeta(2)\zeta(3)^2+1152\zeta(3)\zeta(5)+528M(2,6)\big)$$
$$(917)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^3}{k^4(k+1)^3(k+2)}=\frac{1}{64}\,(4+8\zeta(2)+16\zeta(3)-4440\zeta(4)+3920\zeta(5)+440\zeta(2)\zeta(3)$$
$$-783\zeta(6)+408\zeta(3)^2+462\zeta(7)+64\zeta(2)\zeta(5)-408\zeta(3)\zeta(4)\big) \quad (918)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^3}{k^3(k+1)^4(k+2)}=\frac{1}{32}\,(4+8\zeta(2)+16\zeta(3)+1960\zeta(4)-1680\zeta(5)-200\zeta(2)\zeta(3)$$
$$+225\zeta(6)-168\zeta(3)^2-238\zeta(7)-64\zeta(2)\zeta(5)+264\zeta(3)\zeta(4)\big) \quad (919)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^3}{k^2(k+1)^5(k+2)}=\frac{1}{96}\,(24+48\zeta(2)+96\zeta(3)-2640\zeta(4)+1920\zeta(5)$$
$$+240\zeta(2)\zeta(3)-396\zeta(6)+384\zeta(3)^2+714\zeta(7)+192\zeta(2)\zeta(5)-792\zeta(3)\zeta(4)$$
$$-43\zeta(8)-120\zeta(2)\zeta(3)^2+288\zeta(3)\zeta(5)-24M(2,6)\big) \quad (920)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^3}{k(k+1)^6(k+2)} = \frac{1}{48}\big(24 + 48\zeta(2) + 96\zeta(3) + 240\zeta(4) - 360\zeta(5) - 48\zeta(2)\zeta(3)$$
$$-357\zeta(7) - 96\zeta(2)\zeta(5) + 396\zeta(3)\zeta(4) - 394\zeta(9) + 222\zeta(3)\zeta(6)$$
$$+396\zeta(4)\zeta(5) - 144\zeta(2)\zeta(7) - 48\zeta(3)^3\big) \quad (921)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^3}{(k+1)^7(k+2)} = \frac{1}{480}\big(480 + 960\zeta(2) + 1920\zeta(3) - 3600\zeta(5) - 480\zeta(2)\zeta(3) - 990\zeta(6)$$
$$+960\zeta(3)^2 - 3570\zeta(7) - 960\zeta(2)\zeta(5) + 3960\zeta(3)\zeta(4) - 215\zeta(8)$$
$$-600\zeta(2)\zeta(3)^2 + 1440\zeta(3)\zeta(5) - 120M(2,6) - 3940\zeta(9) + 2220\zeta(3)\zeta(6)$$
$$+3960\zeta(4)\zeta(5) - 1440\zeta(2)\zeta(7) - 480\zeta(3)^3 - 1503\zeta(10) + 2400\zeta(3)\zeta(7)$$
$$+240\zeta(3)^2\zeta(4) - 1680\zeta(2)\zeta(3)\zeta(5) + 1440\zeta(5)^2 + 120M(2,8)\big) \quad (922)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^3}{k^6(k+2)^2} = \frac{1}{768}\big(-84 - 108\zeta(2) - 156\zeta(3) - 261\zeta(4) + 690\zeta(5) + 72\zeta(2)\zeta(3)$$
$$-558\zeta(6) + 240\zeta(3)^2 + 2079\zeta(7) + 288\zeta(2)\zeta(5) - 1836\zeta(3)\zeta(4) + 1190\zeta(8)$$
$$+240\zeta(2)\zeta(3)^2 - 1152\zeta(3)\zeta(5) - 528M(2,6) + 4168\zeta(9) - 2328\zeta(3)\zeta(6)$$
$$-2448\zeta(4)\zeta(5) + 576\zeta(2)\zeta(7) + 384\zeta(3)^3\big) \quad (923)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^3}{k^5(k+1)(k+2)^2} = \frac{1}{768}\big(-180 - 240\zeta(2) - 360\zeta(3) + 7038\zeta(4) - 6060\zeta(5)$$
$$-600\zeta(2)\zeta(3) + 3069\zeta(6) - 1320\zeta(3)^2 - 5544\zeta(7) - 768\zeta(2)\zeta(5)$$
$$+4896\zeta(3)\zeta(4) - 1190\zeta(8) - 240\zeta(2)\zeta(3)^2 + 1152\zeta(3)\zeta(5) + 528M(2,6)\big) \quad (924)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^3}{k^4(k+1)^2(k+2)^2} = \frac{1}{64}\big(-32 - 44\zeta(2) - 68\zeta(3) - 1407\zeta(4) + 1430\zeta(5)$$
$$+160\zeta(2)\zeta(3) - 279\zeta(6) + 120\zeta(3)^2 + 231\zeta(7) + 32\zeta(2)\zeta(5)$$
$$-204\zeta(3)\zeta(4)\big) \quad (925)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^3}{k^3(k+1)^3(k+2)^2} = \frac{1}{64}\big(68 + 96\zeta(2) + 152\zeta(3) - 1626\zeta(4) + 1060\zeta(5)$$
$$+120\zeta(2)\zeta(3) - 225\zeta(6) + 168\zeta(3)^2\big) \quad (926)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^3}{k^2(k+1)^4(k+2)^2} = \frac{1}{16}\big(36 + 52\zeta(2) + 84\zeta(3) + 167\zeta(4) - 310\zeta(5)$$
$$-40\zeta(2)\zeta(3) - 119\zeta(7) - 32\zeta(2)\zeta(5) + 132\zeta(3)\zeta(4)\big) \quad (927)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^3}{k(k+1)^5(k+2)^2} = \frac{1}{96}\big(456 + 672\zeta(2) + 1104\zeta(3) - 636\zeta(4) - 1800\zeta(5)$$
$$-240\zeta(2)\zeta(3) - 396\zeta(6) + 384\zeta(3)^2 - 714\zeta(7) - 192\zeta(2)\zeta(5) + 792\zeta(3)\zeta(4)$$
$$-43\zeta(8) - 120\zeta(2)\zeta(3)^2 + 288\zeta(3)\zeta(5) - 24M(2,6)\big) \quad (928)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^3}{(k+1)^6(k+2)^2} = \frac{1}{48}\big(480+720\zeta(2)+1200\zeta(3)-396\zeta(4)-2160\zeta(5)$$
$$-288\zeta(2)\zeta(3)-396\zeta(6)+384\zeta(3)^2-1071\zeta(7)-288\zeta(2)\zeta(5)$$
$$+1188\zeta(3)\zeta(4)-43\zeta(8)-120\zeta(2)\zeta(3)^2+288\zeta(3)\zeta(5)-24M(2,6)-394\zeta(9)$$
$$+222\zeta(3)\zeta(6)+396\zeta(4)\zeta(5)-144\zeta(2)\zeta(7)-48\zeta(3)^3\big) \quad (929)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^3}{k^5(k+2)^3} = \frac{1}{1536}\big(1140+864\zeta(2)+696\zeta(3)+378\zeta(4)-3084\zeta(5)-504\zeta(2)\zeta(3)$$
$$+1773\zeta(6)-816\zeta(3)^2-4158\zeta(7)-576\zeta(2)\zeta(5)+3672\zeta(3)\zeta(4)-1190\zeta(8)$$
$$-240\zeta(2)\zeta(3)^2+1152\zeta(3)\zeta(5)+528M(2,6)\big) \quad (930)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^3}{k^4(k+1)(k+2)^3} = \frac{1}{128}\big(220+184\zeta(2)+176\zeta(3)-1110\zeta(4)+496\zeta(5)$$
$$+16\zeta(2)\zeta(3)-216\zeta(6)+84\zeta(3)^2+231\zeta(7)+32\zeta(2)\zeta(5)$$
$$-204\zeta(3)\zeta(4)\big) \quad (931)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^3}{k^3(k+1)^2(k+2)^3} = \frac{1}{64}\big(252+228\zeta(2)+244\zeta(3)+297\zeta(4)-934\zeta(5)$$
$$-144\zeta(2)\zeta(3)+63\zeta(6)-36\zeta(3)^2\big) \quad (932)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^3}{k^2(k+1)^3(k+2)^3} = \frac{1}{64}\big(572+552\zeta(2)+640\zeta(3)-1032\zeta(4)-808\zeta(5)$$
$$-168\zeta(2)\zeta(3)-99\zeta(6)+96\zeta(3)^2\big) \quad (933)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^3}{k(k+1)^4(k+2)^3} = \frac{1}{32}\big(644+656\zeta(2)+808\zeta(3)-698\zeta(4)-1428\zeta(5)$$
$$-248\zeta(2)\zeta(3)-99\zeta(6)+96\zeta(3)^2-238\zeta(7)-64\zeta(2)\zeta(5)$$
$$+264\zeta(3)\zeta(4)\big) \quad (934)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^3}{(k+1)^5(k+2)^3} = \frac{1}{96}\big(4320+4608\zeta(2)+5952\zeta(3)-4824\zeta(4)-10368\zeta(5)$$
$$-1728\zeta(2)\zeta(3)-990\zeta(6)+960\zeta(3)^2-2142\zeta(7)-576\zeta(2)\zeta(5)$$
$$+2376\zeta(3)\zeta(4)-43\zeta(8)-120\zeta(2)\zeta(3)^2+288\zeta(3)\zeta(5)-24M(2,6)\big) \quad (935)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^3}{k^4(k+2)^4} = \frac{1}{128}\big(420+164\zeta(2)-12\zeta(3)-195\zeta(4)-290\zeta(5)-88\zeta(2)\zeta(3)$$
$$+89\zeta(6)-48\zeta(3)^2-175\zeta(7)-32\zeta(2)\zeta(5)+168\zeta(3)\zeta(4)\big) \quad (936)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^3}{k^3(k+1)(k+2)^4} = \frac{1}{128}\big(-1060-512\zeta(2)-152\zeta(3)+1500\zeta(4)+84\zeta(5)$$
$$+160\zeta(2)\zeta(3)+38\zeta(6)+12\zeta(3)^2+119\zeta(7)+32\zeta(2)\zeta(5)$$
$$-132\zeta(3)\zeta(4)\big) \quad (937)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^3}{k^2(k+1)^2(k+2)^4} = \frac{1}{64}\,(1312+740\zeta(2)+396\zeta(3)-1203\zeta(4)-1018\zeta(5)$$
$$-304\zeta(2)\zeta(3)+25\zeta(6)-48\zeta(3)^2-119\zeta(7)-32\zeta(2)\zeta(5)$$
$$+132\zeta(3)\zeta(4)) \qquad (938)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^3}{k(k+1)^3(k+2)^4} = \frac{1}{64}\,(3196+2032\zeta(2)+1432\zeta(3)-3438\zeta(4)-2844\zeta(5)$$
$$-776\zeta(2)\zeta(3)-49\zeta(6)-238\zeta(7)-64\zeta(2)\zeta(5)+264\zeta(3)\zeta(4)) \qquad (939)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^3}{(k+1)^4(k+2)^4} = \frac{1}{8}\,(960+672\zeta(2)+560\zeta(3)-1034\zeta(4)-1068\zeta(5)$$
$$-256\zeta(2)\zeta(3)-37\zeta(6)+24\zeta(3)^2-119\zeta(7)-32\zeta(2)\zeta(5)$$
$$+132\zeta(3)\zeta(4)) \qquad (940)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^3}{k^3(k+2)^5} = \frac{1}{1536}\,(16500+2520\zeta(2)-4896\zeta(3)-8460\zeta(4)-3768\zeta(5)$$
$$-648\zeta(2)\zeta(3)-1779\zeta(6)+1032\zeta(3)^2-1566\zeta(7)-1152\zeta(2)\zeta(5)$$
$$+2664\zeta(3)\zeta(4)+86\zeta(8)+240\zeta(2)\zeta(3)^2-576\zeta(3)\zeta(5)+48M(2,6)\big) \qquad (941)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^3}{k^2(k+1)(k+2)^5} = \frac{1}{768}\,(22860+5592\zeta(2)-3984\zeta(3)-17460\zeta(4)-4272\zeta(5)$$
$$-1608\zeta(2)\zeta(3)-2007\zeta(6)+960\zeta(3)^2-2280\zeta(7)-1344\zeta(2)\zeta(5)$$
$$+3456\zeta(3)\zeta(4)+86\zeta(8)+240\zeta(2)\zeta(3)^2-576\zeta(3)\zeta(5)+48M(2,6)\big) \qquad (942)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^3}{k(k+1)^2(k+2)^5} = \frac{1}{384}\,(30732+10032\zeta(2)-1608\zeta(3)-24678\zeta(4)-10380\zeta(5)$$
$$-3432\zeta(2)\zeta(3)-1857\zeta(6)+672\zeta(3)^2-2994\zeta(7)-1536\zeta(2)\zeta(5)$$
$$+4248\zeta(3)\zeta(4)+86\zeta(8)+240\zeta(2)\zeta(3)^2-576\zeta(3)\zeta(5)+48M(2,6)\big) \qquad (943)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^3}{(k+1)^3(k+2)^5} = \frac{1}{96}\,(20160+8064\zeta(2)+1344\zeta(3)-17496\zeta(4)-9456\zeta(5)$$
$$-2880\zeta(2)\zeta(3)-1002\zeta(6)+336\zeta(3)^2-1854\zeta(7)-864\zeta(2)\zeta(5)$$
$$+2520\zeta(3)\zeta(4)+43\zeta(8)+120\zeta(2)\zeta(3)^2-288\zeta(3)\zeta(5)+24M(2,6)\big) \qquad (944)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^3}{k^2(k+2)^6} = \frac{1}{768}\,(21588+252\zeta(2)-7956\zeta(3)-8451\zeta(4)-5154\zeta(5)$$
$$+1368\zeta(2)\zeta(3)-3732\zeta(6)+2256\zeta(3)^2-1647\zeta(7)-864\zeta(2)\zeta(5)$$
$$+2340\zeta(3)\zeta(4)+2102\zeta(8)+240\zeta(2)\zeta(3)^2-1728\zeta(3)\zeta(5)-528M(2,6)$$
$$-1576\zeta(9)+888\zeta(3)\zeta(6)+1584\zeta(4)\zeta(5)-576\zeta(2)\zeta(7)-192\zeta(3)^3\big) \qquad (945)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^3}{k(k+1)(k+2)^6} = \frac{1}{768}\big(-66036 - 6096\zeta(2) + 19896\zeta(3) + 34362\zeta(4) + 14580\zeta(5)$$
$$-1128\zeta(2)\zeta(3) + 9471\zeta(6) - 5472\zeta(3)^2 + 5574\zeta(7) + 3072\zeta(2)\zeta(5)$$
$$-8136\zeta(3)\zeta(4) - 4290\zeta(8) - 720\zeta(2)\zeta(3)^2 + 4032\zeta(3)\zeta(5) + 1008M(2,6)$$
$$+3152\zeta(9) - 1776\zeta(3)\zeta(6) - 3168\zeta(4)\zeta(5) + 1152\zeta(2)\zeta(7) + 384\zeta(3)^3\big) \quad (946)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^3}{(k+1)^2(k+2)^6} = \frac{1}{48}\big(-12096 - 2016\zeta(2) + 2688\zeta(3) + 7380\zeta(4) + 3120\zeta(5)$$
$$+288\zeta(2)\zeta(3) + 1416\zeta(6) - 768\zeta(3)^2 + 1071\zeta(7) + 576\zeta(2)\zeta(5)$$
$$-1548\zeta(3)\zeta(4) - 547\zeta(8) - 120\zeta(2)\zeta(3)^2 + 576\zeta(3)\zeta(5) + 120M(2,6) + 394\zeta(9)$$
$$-222\zeta(3)\zeta(6) - 396\zeta(4)\zeta(5) + 144\zeta(2)\zeta(7) + 48\zeta(3)^3\big) \quad (947)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^3}{k(k+2)^7} = \frac{1}{7680}\big(476220 - 30480\zeta(2) - 169320\zeta(3) - 138270\zeta(4) - 138300\zeta(5)$$
$$+48120\zeta(2)\zeta(3) - 82965\zeta(6) + 45600\zeta(3)^2 - 73650\zeta(7) + 7680\zeta(2)\zeta(5)$$
$$+42840\zeta(3)\zeta(4) + 46510\zeta(8) + 1200\zeta(2)\zeta(3)^2 - 25920\zeta(3)\zeta(5) - 17040M(2,6)$$
$$-17680\zeta(9) + 26160\zeta(3)\zeta(6) + 21600\zeta(4)\zeta(5) - 17280\zeta(2)\zeta(7) - 5760\zeta(3)^3$$
$$+12024\zeta(10) - 19200\zeta(3)\zeta(7) - 1920\zeta(3)^2\zeta(4) + 13440\zeta(2)\zeta(3)\zeta(5)$$
$$-11520\zeta(5)^2 - 960M(2,8)\big) \quad (948)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^3}{(k+1)(k+2)^7} = \frac{1}{480}\big(100800 - 33600\zeta(3) - 38760\zeta(4) - 26400\zeta(5) + 6720\zeta(2)\zeta(3)$$
$$-16290\zeta(6) + 9120\zeta(3)^2 - 12690\zeta(7) - 960\zeta(2)\zeta(5) + 10440\zeta(3)\zeta(4) + 8495\zeta(8)$$
$$+600\zeta(2)\zeta(3)^2 - 5760\zeta(3)\zeta(5) - 2760M(2,6) - 4180\zeta(9) + 4380\zeta(3)\zeta(6)$$
$$+4680\zeta(4)\zeta(5) - 2880\zeta(2)\zeta(7) - 960\zeta(3)^3 + 1503\zeta(10) - 2400\zeta(3)\zeta(7)$$
$$-240\zeta(3)^2\zeta(4) + 1680\zeta(2)\zeta(3)\zeta(5) - 1440\zeta(5)^2 - 120M(2,8)\big) \quad (949)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^3}{(k+2)^8} = \frac{1}{8}\big(960 - 96\zeta(2) - 304\zeta(3) - 222\zeta(4) - 284\zeta(5) + 96\zeta(2)\zeta(3)$$
$$-157\zeta(6) + 72\zeta(3)^2 - 216\zeta(7) + 48\zeta(2)\zeta(5) + 84\zeta(3)\zeta(4) + 16\zeta(8) - 24M(2,6)$$
$$-100\zeta(9) + 60\zeta(3)\zeta(6) + 36\zeta(4)\zeta(5) - 8\zeta(3)^3 + 108\zeta(10) - 48\zeta(3)\zeta(7)$$
$$-24\zeta(5)^2 - 24M(2,8) + 176\zeta(11) - 84\zeta(3)\zeta(8) - 36\zeta(4)\zeta(7) - 60\zeta(5)\zeta(6)$$
$$+24\zeta(3)^2\zeta(5) - 8M(3,8)\big) \quad (950)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^4}{k^7} = \frac{1}{48}\big(-2877\zeta(11) - 272\zeta(2)\zeta(9) + 1190\zeta(3)\zeta(8) + 1212\zeta(4)\zeta(7)$$
$$+1018\zeta(5)\zeta(6) + 80\zeta(2)\zeta(3)^3 - 576\zeta(3)^2\zeta(5) + 176M(3,8)\big) \quad (951)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^4}{k^6(k+1)} = \frac{1}{5760}\big(172800\zeta(5)+34560\zeta(2)\zeta(3)-234960\zeta(6)-17280\zeta(3)^2$$
$$+133200\zeta(7)+28800\zeta(2)\zeta(5)-123840\zeta(3)\zeta(4)+593320\zeta(8)$$
$$+161280\zeta(2)\zeta(3)^2-668160\zeta(3)\zeta(5)-149760M(2,6)+209280\zeta(9)$$
$$-133920\zeta(3)\zeta(6)-123840\zeta(4)\zeta(5)+40320\zeta(2)\zeta(7)+19200\zeta(3)^3$$
$$+619407\zeta(10)-540000\zeta(3)\zeta(7)-9000\zeta(3)^2\zeta(4)+195120\zeta(2)\zeta(3)\zeta(5)$$
$$-212040\zeta(5)^2-109080M(2,8)-11520\zeta(2)M(2,6)\big) \quad (952)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^4}{k^5(k+1)^2} = \frac{1}{72}\big(10800\zeta(5)+2160\zeta(2)\zeta(3)-14325\zeta(6)-1080\zeta(3)^2+4995\zeta(7)$$
$$+1080\zeta(2)\zeta(5)-4644\zeta(3)\zeta(4)+14833\zeta(8)+4032\zeta(2)\zeta(3)^2$$
$$-16704\zeta(3)\zeta(5)-3744M(2,6)+2616\zeta(9)-1674\zeta(3)\zeta(6)-1548\zeta(4)\zeta(5)$$
$$+504\zeta(2)\zeta(7)+240\zeta(3)^3\big) \quad (953)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^4}{k^4(k+1)^3} = \frac{1}{144}\big(43200\zeta(5)+8640\zeta(2)\zeta(3)-55860\zeta(6)-4320\zeta(3)^2$$
$$+11952\zeta(7)+2880\zeta(2)\zeta(5)-11952\zeta(3)\zeta(4)+14833\zeta(8)+4032\zeta(2)\zeta(3)^2$$
$$-16704\zeta(3)\zeta(5)-3744M(2,6)) \quad (954)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^4}{k^3(k+1)^4} = \frac{1}{144}\big(43200\zeta(5)+8640\zeta(2)\zeta(3)-54420\zeta(6)-4320\zeta(3)^2$$
$$+9216\zeta(7)+2880\zeta(2)\zeta(5)-11088\zeta(3)\zeta(4)+12415\zeta(8)+3312\zeta(2)\zeta(3)^2$$
$$-13824\zeta(3)\zeta(5)-3024M(2,6)) \quad (955)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^4}{k^2(k+1)^5} = \frac{1}{72}\big(10800\zeta(5)+2160\zeta(2)\zeta(3)-13245\zeta(6)-1080\zeta(3)^2+2943\zeta(7)$$
$$+1080\zeta(2)\zeta(5)-3996\zeta(3)\zeta(4)+12415\zeta(8)+3312\zeta(2)\zeta(3)^2$$
$$-13824\zeta(3)\zeta(5)-3024M(2,6)+1044\zeta(9)-594\zeta(3)\zeta(6)-1332\zeta(4)\zeta(5)$$
$$+504\zeta(2)\zeta(7)+192\zeta(3)^3\big) \quad (956)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^4}{k(k+1)^6} = \frac{1}{5760}\big(172800\zeta(5)+34560\zeta(2)\zeta(3)-206160\zeta(6)-17280\zeta(3)^2$$
$$+78480\zeta(7)+28800\zeta(2)\zeta(5)-106560\zeta(3)\zeta(4)+496600\zeta(8)+132480\zeta(2)\zeta(3)^2$$
$$-552960\zeta(3)\zeta(5)-120960M(2,6)+83520\zeta(9)-47520\zeta(3)\zeta(6)-106560\zeta(4)\zeta(5)$$
$$+40320\zeta(2)\zeta(7)+15360\zeta(3)^3+437823\zeta(10)-378720\zeta(3)\zeta(7)$$
$$+2520\zeta(3)^2\zeta(4)+114480\zeta(2)\zeta(3)\zeta(5)-119880\zeta(5)^2-68760M(2,8)$$
$$-11520\zeta(2)M(2,6)) \quad (957)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^4}{(k+1)^7} = \frac{1}{48} \left(237\zeta(11) + 368\zeta(2)\zeta(9) - 86\zeta(3)\zeta(8) - 684\zeta(4)\zeta(7) \right.$$
$$\left. -202\zeta(5)\zeta(6) - 80\zeta(2)\zeta(3)^3 + 288\zeta(3)^2\zeta(5) + 16M(3,8)\right) \quad (958)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^4}{k^6(k+2)} = \frac{1}{11520} (180 + 540\zeta(2) + 1980\zeta(3) + 3330\zeta(4) + 5400\zeta(5) + 1080\zeta(2)\zeta(3)$$
$$-14685\zeta(6) - 1080\zeta(3)^2 + 16650\zeta(7) + 3600\zeta(2)\zeta(5) - 15480\zeta(3)\zeta(4)$$
$$+148330\zeta(8) + 40320\zeta(2)\zeta(3)^2 - 167040\zeta(3)\zeta(5) - 37440M(2,6) + 104640\zeta(9)$$
$$-66960\zeta(3)\zeta(6) - 61920\zeta(4)\zeta(5) + 20160\zeta(2)\zeta(7) + 9600\zeta(3)^3 + 619407\zeta(10)$$
$$-540000\zeta(3)\zeta(7) - 9000\zeta(3)^2\zeta(4) + 195120\zeta(2)\zeta(3)\zeta(5) - 212040\zeta(5)^2$$
$$-109080M(2,8) - 11520\zeta(2)M(2,6)) \quad (959)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^4}{k^5(k+1)(k+2)} = \frac{1}{384} (12 + 36\zeta(2) + 132\zeta(3) + 222\zeta(4) - 11160\zeta(5) - 2232\zeta(2)\zeta(3)$$
$$+14685\zeta(6) + 1080\zeta(3)^2 - 7770\zeta(7) - 1680\zeta(2)\zeta(5) + 7224\zeta(3)\zeta(4) - 29666\zeta(8)$$
$$-8064\zeta(2)\zeta(3)^2 + 33408\zeta(3)\zeta(5) + 7488M(2,6) - 6976\zeta(9) + 4464\zeta(3)\zeta(6)$$
$$+4128\zeta(4)\zeta(5) - 1344\zeta(2)\zeta(7) - 640\zeta(3)^3\big) \quad (960)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^4}{k^4(k+1)^2(k+2)} = \frac{1}{576} (-36 - 108\zeta(2) - 396\zeta(3) - 666\zeta(4) - 52920\zeta(5)$$
$$-10584\zeta(2)\zeta(3) + 70545\zeta(6) + 5400\zeta(3)^2 - 16650\zeta(7) - 3600\zeta(2)\zeta(5)$$
$$+15480\zeta(3)\zeta(4) - 29666\zeta(8) - 8064\zeta(2)\zeta(3)^2 + 33408\zeta(3)\zeta(5)$$
$$+7488M(2,6)) \quad (961)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^4}{k^3(k+1)^3(k+2)} = \frac{1}{32} (-4 - 12\zeta(2) - 44\zeta(3) - 74\zeta(4) + 3720\zeta(5) + 744\zeta(2)\zeta(3)$$
$$-4575\zeta(6) - 360\zeta(3)^2 + 806\zeta(7) + 240\zeta(2)\zeta(5) - 936\zeta(3)\zeta(4)\big) \quad (962)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^4}{k^2(k+1)^4(k+2)} = \frac{1}{144} (36 + 108\zeta(2) + 396\zeta(3) + 666\zeta(4) + 9720\zeta(5)$$
$$+1944\zeta(2)\zeta(3) - 13245\zeta(6) - 1080\zeta(3)^2 + 1962\zeta(7) + 720\zeta(2)\zeta(5)$$
$$-2664\zeta(3)\zeta(4) + 12415\zeta(8) + 3312\zeta(2)\zeta(3)^2 - 13824\zeta(3)\zeta(5)$$
$$-3024M(2,6)) \quad (963)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^4}{k(k+1)^5(k+2)} = \frac{1}{24} (12 + 36\zeta(2) + 132\zeta(3) + 222\zeta(4) - 360\zeta(5) - 72\zeta(2)\zeta(3)$$
$$-327\zeta(7) - 120\zeta(2)\zeta(5) + 444\zeta(3)\zeta(4) - 348\zeta(9) + 198\zeta(3)\zeta(6)$$
$$+444\zeta(4)\zeta(5) - 168\zeta(2)\zeta(7) - 64\zeta(3)^3\big) \quad (964)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^4}{(k+1)^6(k+2)} = \frac{1}{5760} \left(5760 + 17280\zeta(2) + 63360\zeta(3) + 106560\zeta(4) - 206160\zeta(6) \right.$$
$$-17280\zeta(3)^2 - 78480\zeta(7) - 28800\zeta(2)\zeta(5) + 106560\zeta(3)\zeta(4) + 496600\zeta(8)$$
$$+132480\zeta(2)\zeta(3)^2 - 552960\zeta(3)\zeta(5) - 120960M(2,6) - 83520\zeta(9) + 47520\zeta(3)\zeta(6)$$
$$+106560\zeta(4)\zeta(5) - 40320\zeta(2)\zeta(7) - 15360\zeta(3)^3 + 437823\zeta(10)$$
$$-378720\zeta(3)\zeta(7) + 2520\zeta(3)^2\zeta(4) + 114480\zeta(2)\zeta(3)\zeta(5) - 119880\zeta(5)^2$$
$$\left. -68760M(2,8) - 11520\zeta(2)M(2,6)\right) \quad (965)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^4}{k^5(k+2)^2} = \frac{1}{2304} \left(540 + 1116\zeta(2) + 3276\zeta(3) + 3474\zeta(4) + 3240\zeta(5) \right.$$
$$+792\zeta(2)\zeta(3) - 14325\zeta(6) - 1080\zeta(3)^2 + 9990\zeta(7) + 2160\zeta(2)\zeta(5)$$
$$-9288\zeta(3)\zeta(4) + 59332\zeta(8) + 16128\zeta(2)\zeta(3)^2 - 66816\zeta(3)\zeta(5) - 14976M(2,6)$$
$$+20928\zeta(9) - 13392\zeta(3)\zeta(6) - 12384\zeta(4)\zeta(5) + 4032\zeta(2)\zeta(7)$$
$$\left. +1920\zeta(3)^3\right) \quad (966)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^4}{k^4(k+1)(k+2)^2} = \frac{1}{576} \left(288 + 612\zeta(2) + 1836\zeta(3) + 2070\zeta(4) - 15120\zeta(5) \right.$$
$$-2952\zeta(2)\zeta(3) + 14865\zeta(6) + 1080\zeta(3)^2 - 6660\zeta(7) - 1440\zeta(2)\zeta(5)$$
$$+6192\zeta(3)\zeta(4) - 14833\zeta(8) - 4032\zeta(2)\zeta(3)^2 + 16704\zeta(3)\zeta(5)$$
$$\left. +3744M(2,6)\right) \quad (967)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^4}{k^3(k+1)^2(k+2)^2} = \frac{1}{64} \left(68 + 148\zeta(2) + 452\zeta(3) + 534\zeta(4) + 2520\zeta(5) \right.$$
$$+520\zeta(2)\zeta(3) - 4535\zeta(6) - 360\zeta(3)^2 + 370\zeta(7) + 80\zeta(2)\zeta(5)$$
$$\left. -344\zeta(3)\zeta(4)\right) \quad (968)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^4}{k^2(k+1)^3(k+2)^2} = \frac{1}{8} \left(18 + 40\zeta(2) + 124\zeta(3) + 152\zeta(4) - 300\zeta(5) \right.$$
$$\left. -56\zeta(2)\zeta(3) + 10\zeta(6) - 109\zeta(7) - 40\zeta(2)\zeta(5) + 148\zeta(3)\zeta(4)\right) \quad (969)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^4}{k(k+1)^4(k+2)^2} = \frac{1}{144} \left(684 + 1548\zeta(2) + 4860\zeta(3) + 6138\zeta(4) - 1080\zeta(5) \right.$$
$$-72\zeta(2)\zeta(3) - 12885\zeta(6) - 1080\zeta(3)^2 - 1962\zeta(7) - 720\zeta(2)\zeta(5)$$
$$+2664\zeta(3)\zeta(4) + 12415\zeta(8) + 3312\zeta(2)\zeta(3)^2 - 13824\zeta(3)\zeta(5)$$
$$\left. -3024M(2,6)\right) \quad (970)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^4}{(k+1)^5(k+2)^2} = \frac{1}{72} \left(720 + 1656\zeta(2) + 5256\zeta(3) + 6804\zeta(4) - 2160\zeta(5) \right.$$
$$-288\zeta(2)\zeta(3) - 12885\zeta(6) - 1080\zeta(3)^2 - 2943\zeta(7) - 1080\zeta(2)\zeta(5)$$
$$+3996\zeta(3)\zeta(4) + 12415\zeta(8) + 3312\zeta(2)\zeta(3)^2 - 13824\zeta(3)\zeta(5) - 3024M(2,6)$$
$$\left. -1044\zeta(9) + 594\zeta(3)\zeta(6) + 1332\zeta(4)\zeta(5) - 504\zeta(2)\zeta(7) - 192\zeta(3)^3\right) \quad (971)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^4}{k^4(k+2)^3} = \frac{1}{1152}\left(-1980-2700\zeta(2)-6012\zeta(3)-2502\zeta(4)+432\zeta(5)\right.$$
$$+216\zeta(2)\zeta(3)+13371\zeta(6)+1656\zeta(3)^2-5976\zeta(7)-1440\zeta(2)\zeta(5)$$
$$+5976\zeta(3)\zeta(4)-14833\zeta(8)-4032\zeta(2)\zeta(3)^2+16704\zeta(3)\zeta(5)$$
$$\left.+3744M(2,6)\right) \quad (972)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^4}{k^3(k+1)(k+2)^3} = \frac{1}{32}\left(-126-184\zeta(2)-436\zeta(3)-254\zeta(4)+864\zeta(5)\right.$$
$$\left.+176\zeta(2)\zeta(3)-83\zeta(6)+32\zeta(3)^2+38\zeta(7)-12\zeta(3)\zeta(4)\right) \quad (973)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^4}{k^2(k+1)^2(k+2)^3} = \frac{1}{64}\left(572+884\zeta(2)+2196\zeta(3)+1550\zeta(4)-936\zeta(5)\right.$$
$$-184\zeta(2)\zeta(3)-4203\zeta(6)-488\zeta(3)^2+218\zeta(7)+80\zeta(2)\zeta(5)$$
$$\left.-296\zeta(3)\zeta(4)\right) \quad (974)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^4}{k(k+1)^3(k+2)^3} = \frac{1}{32}\left(-644-1044\zeta(2)-2692\zeta(3)-2158\zeta(4)+2136\zeta(5)\right.$$
$$+408\zeta(2)\zeta(3)+4163\zeta(6)+488\zeta(3)^2+218\zeta(7)+80\zeta(2)\zeta(5)$$
$$\left.-296\zeta(3)\zeta(4)\right) \quad (975)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^4}{(k+1)^4(k+2)^3} = \frac{1}{144}\left(6480+10944\zeta(2)+29088\zeta(3)+25560\zeta(4)-20304\zeta(5)\right.$$
$$-3744\zeta(2)\zeta(3)-50352\zeta(6)-5472\zeta(3)^2-3924\zeta(7)-1440\zeta(2)\zeta(5)$$
$$+5328\zeta(3)\zeta(4)+12415\zeta(8)+3312\zeta(2)\zeta(3)^2-13824\zeta(3)\zeta(5)$$
$$\left.-3024M(2,6)\right) \quad (976)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^4}{k^3(k+2)^4} = \frac{1}{1152}\left(9540+8172\zeta(2)+12492\zeta(3)-4626\zeta(4)-8496\zeta(5)\right.$$
$$-3672\zeta(2)\zeta(3)-11967\zeta(6)-3096\zeta(3)^2+324\zeta(7)+288\zeta(2)\zeta(5)$$
$$\left.-792\zeta(3)\zeta(4)+12415\zeta(8)+3312\zeta(2)\zeta(3)^2-13824\zeta(3)\zeta(5)-3024M(2,6)\right) \quad (977)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^4}{k^2(k+1)(k+2)^4} = \frac{1}{576}\left(11808+11484\zeta(2)+20340\zeta(3)-54\zeta(4)-24048\zeta(5)\right.$$
$$-6840\zeta(2)\zeta(3)-10473\zeta(6)-3672\zeta(3)^2-360\zeta(7)+288\zeta(2)\zeta(5)$$
$$\left.-576\zeta(3)\zeta(4)+12415\zeta(8)+3312\zeta(2)\zeta(3)^2-13824\zeta(3)\zeta(5)-3024M(2,6)\right) \quad (978)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^4}{k(k+1)^2(k+2)^4} = \frac{1}{576}\left(28764+30924\zeta(2)+60444\zeta(3)+13842\zeta(4)-56520\zeta(5)\right.$$
$$-15336\zeta(2)\zeta(3)-58773\zeta(6)-11736\zeta(3)^2+1242\zeta(7)+1296\zeta(2)\zeta(5)$$
$$-3816\zeta(3)\zeta(4)+24830\zeta(8)+6624\zeta(2)\zeta(3)^2-27648\zeta(3)\zeta(5)$$
$$\left.-6048M(2,6)\right) \quad (979)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^4}{(k+1)^3(k+2)^4} = \frac{1}{144} \big(17280 + 20160\zeta(2) + 42336\zeta(3) + 16632\zeta(4) - 37872\zeta(5)$$
$$-9504\zeta(2)\zeta(3) - 48120\zeta(6) - 8064\zeta(3)^2 - 360\zeta(7) + 288\zeta(2)\zeta(5)$$
$$-576\zeta(3)\zeta(4) + 12415\zeta(8) + 3312\zeta(2)\zeta(3)^2 - 13824\zeta(3)\zeta(5) - 3024M(2,6)\big) \quad (980)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^4}{k^2(k+2)^5} = \frac{1}{2304} \big(68580 + 34812\zeta(2) + 27900\zeta(3) - 56070\zeta(4) - 39960\zeta(5)$$
$$-17064\zeta(2)\zeta(3) - 17889\zeta(6) - 2232\zeta(3)^2 - 24930\zeta(7) - 10512\zeta(2)\zeta(5)$$
$$+31752\zeta(3)\zeta(4) + 50692\zeta(8) + 16128\zeta(2)\zeta(3)^2 - 62208\zeta(3)\zeta(5) - 11520M(2,6)$$
$$+8352\zeta(9) - 4752\zeta(3)\zeta(6) - 10656\zeta(4)\zeta(5) + 4032\zeta(2)\zeta(7)$$
$$+1536\zeta(3)^3\big) \quad (981)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^4}{k(k+1)(k+2)^5} = \frac{1}{384} \big(30732 + 19260\zeta(2) + 22860\zeta(3) - 18726\zeta(4) - 29352\zeta(5)$$
$$-10248\zeta(2)\zeta(3) - 12945\zeta(6) - 3192\zeta(3)^2 - 8550\zeta(7) - 3312\zeta(2)\zeta(5)$$
$$+10200\zeta(3)\zeta(4) + 25174\zeta(8) + 7584\zeta(2)\zeta(3)^2 - 29952\zeta(3)\zeta(5) - 5856M(2,6)$$
$$+2784\zeta(9) - 1584\zeta(3)\zeta(6) - 3552\zeta(4)\zeta(5) + 1344\zeta(2)\zeta(7) + 512\zeta(3)^3\big) \quad (982)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^4}{(k+1)^2(k+2)^5} = \frac{1}{72} \big(15120 + 11088\zeta(2) + 16128\zeta(3) - 5292\zeta(4) - 18072\zeta(5)$$
$$-5760\zeta(2)\zeta(3) - 12201\zeta(6) - 2664\zeta(3)^2 - 3051\zeta(7) - 1080\zeta(2)\zeta(5)$$
$$+3348\zeta(3)\zeta(4) + 12544\zeta(8) + 3672\zeta(2)\zeta(3)^2 - 14688\zeta(3)\zeta(5) - 2952M(2,6)$$
$$+1044\zeta(9) - 594\zeta(3)\zeta(6) - 1332\zeta(4)\zeta(5) + 504\zeta(2)\zeta(7) + 192\zeta(3)^3\big) \quad (983)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^4}{k(k+2)^6} = \frac{1}{11520} \big(990540 + 275580\zeta(2) - 20340\zeta(3) - 706950\zeta(4) - 394920\zeta(5)$$
$$-76680\zeta(2)\zeta(3) - 216465\zeta(6) + 99720\zeta(3)^2 - 272790\zeta(7) - 140400\zeta(2)\zeta(5)$$
$$+385560\zeta(3)\zeta(4) + 260590\zeta(8) + 76320\zeta(2)\zeta(3)^2 - 311040\zeta(3)\zeta(5) - 56160M(2,6)$$
$$-147360\zeta(9) + 82800\zeta(3)\zeta(6) + 136800\zeta(4)\zeta(5) - 48960\zeta(2)\zeta(7) - 15360\zeta(3)^3$$
$$+437823\zeta(10) - 378720\zeta(3)\zeta(7) + 2520\zeta(3)^2\zeta(4) + 114480\zeta(2)\zeta(3)\zeta(5)$$
$$-119880\zeta(5)^2 - 68760M(2,8) - 11520\zeta(2)M(2,6)\big) \quad (984)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^4}{(k+1)(k+2)^6} = \frac{1}{5760} \big(1451520 + 564480\zeta(2) + 322560\zeta(3) - 987840\zeta(4) - 835200\zeta(5)$$
$$-230400\zeta(2)\zeta(3) - 410640\zeta(6) + 51840\zeta(3)^2 - 401040\zeta(7) - 190080\zeta(2)\zeta(5)$$
$$+538560\zeta(3)\zeta(4) + 638200\zeta(8) + 190080\zeta(2)\zeta(3)^2 - 760320\zeta(3)\zeta(5)$$
$$-144000M(2,6) - 105600\zeta(9) + 59040\zeta(3)\zeta(6) + 83520\zeta(4)\zeta(5) - 28800\zeta(2)\zeta(7)$$
$$-7680\zeta(3)^3 + 437823\zeta(10) - 378720\zeta(3)\zeta(7) + 2520\zeta(3)^2\zeta(4)$$
$$+114480\zeta(2)\zeta(3)\zeta(5) - 119880\zeta(5)^2 - 68760M(2,8) - 11520\zeta(2)M(2,6)\big) \quad (985)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^4}{(k+2)^7} = \frac{1}{240}(-50400-6720\zeta(2)+10080\zeta(3)+27720\zeta(4)+18000\zeta(5)$$
$$-1440\zeta(2)\zeta(3)+12180\zeta(6)-6720\zeta(3)^2+11700\zeta(7)+3360\zeta(2)\zeta(5)$$
$$-12960\zeta(3)\zeta(4)-9310\zeta(8)-1200\zeta(2)\zeta(3)^2+7680\zeta(3)\zeta(5)+2640M(2,6)$$
$$+8120\zeta(9)-6600\zeta(3)\zeta(6)-8640\zeta(4)\zeta(5)+4320\zeta(2)\zeta(7)+1440\zeta(3)^3$$
$$-3006\zeta(10)+4800\zeta(3)\zeta(7)+480\zeta(3)^2\zeta(4)-3360\zeta(2)\zeta(3)\zeta(5)$$
$$+2880\zeta(5)^2+240M(2,8)-1185\zeta(11)-1840\zeta(2)\zeta(9)+430\zeta(3)\zeta(8)$$
$$+3420\zeta(4)\zeta(7)+1010\zeta(5)\zeta(6)+400\zeta(2)\zeta(3)^3-1440\zeta(3)^2\zeta(5)$$
$$-80M(3,8)) \qquad (986)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^5}{k^6} = \frac{1}{576}(-781671\zeta(11)-88016\zeta(2)\zeta(9)+296660\zeta(3)\zeta(8)+411984\zeta(4)\zeta(7)$$
$$+220080\zeta(5)\zeta(6)+21120\zeta(2)\zeta(3)^3-141120\zeta(3)^2\zeta(5)+8640\zeta(3)M(2,6)$$
$$+27840M(3,8)) \qquad (987)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^5}{k^5(k+1)} = \frac{1}{2304}\big(411264\zeta(6)+51840\zeta(3)^2-295344\zeta(7)-65664\zeta(2)\zeta(5)$$
$$-76032\zeta(3)\zeta(4)-542488\zeta(8)-152640\zeta(2)\zeta(3)^2+630144\zeta(3)\zeta(5)+135360M(2,6)$$
$$-302144\zeta(9)+469920\zeta(3)\zeta(6)-152064\zeta(4)\zeta(5)-76320\zeta(2)\zeta(7)+11520\zeta(3)^3$$
$$-579897\zeta(10)+519840\zeta(3)\zeta(7)-3240\zeta(3)^2\zeta(4)-185040\zeta(2)\zeta(3)\zeta(5)$$
$$+203832\zeta(5)^2+98280M(2,8)+11520\zeta(2)M(2,6)\big) \qquad (988)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^5}{k^4(k+1)^2} = \frac{1}{144}\big(102816\zeta(6)+12960\zeta(3)^2-72072\zeta(7)-16416\zeta(2)\zeta(5)$$
$$-19008\zeta(3)\zeta(4)-67811\zeta(8)-19080\zeta(2)\zeta(3)^2+78768\zeta(3)\zeta(5)+16920M(2,6)$$
$$-18884\zeta(9)+29370\zeta(3)\zeta(6)-9504\zeta(4)\zeta(5)-4770\zeta(2)\zeta(7)$$
$$+720\zeta(3)^3\big) \qquad (989)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^5}{k^3(k+1)^3} = \frac{1}{72}\big(77112\zeta(6)+9720\zeta(3)^2-52731\zeta(7)-12312\zeta(2)\zeta(5)$$
$$-14256\zeta(3)\zeta(4)-33358\zeta(8)-9180\zeta(2)\zeta(3)^2+37800\zeta(3)\zeta(5)$$
$$+8100M(2,6)) \qquad (990)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^5}{k^2(k+1)^4} = \frac{1}{144}\big(-102816\zeta(6)-12960\zeta(3)^2+68544\zeta(7)+16416\zeta(2)\zeta(5)$$
$$+19008\zeta(3)\zeta(4)+65621\zeta(8)+17640\zeta(2)\zeta(3)^2-72432\zeta(3)\zeta(5)-15480M(2,6)$$
$$+14240\zeta(9)-25770\zeta(3)\zeta(6)+9504\zeta(4)\zeta(5)+4770\zeta(2)\zeta(7)$$
$$-720\zeta(3)^3\big) \qquad (991)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^5}{k(k+1)^5} = \frac{1}{2304}\big(411264\zeta(6)+51840\zeta(3)^2-267120\zeta(7)-65664\zeta(2)\zeta(5)$$
$$-76032\zeta(3)\zeta(4)-524968\zeta(8)-141120\zeta(2)\zeta(3)^2+579456\zeta(3)\zeta(5)+123840M(2,6)$$
$$-227840\zeta(9)+412320\zeta(3)\zeta(6)-152064\zeta(4)\zeta(5)-76320\zeta(2)\zeta(7)+11520\zeta(3)^3$$
$$-449109\zeta(10)+387360\zeta(3)\zeta(7)+9720\zeta(3)^2\zeta(4)-124560\zeta(2)\zeta(3)\zeta(5)$$
$$+122328\zeta(5)^2+68040M(2,8)+11520\zeta(2)M(2,6)\big) \quad (992)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^5}{(k+1)^6} = \frac{1}{576}\left(-667227\zeta(11)-68816\zeta(2)\zeta(9)+248300\zeta(3)\zeta(8)\right.$$
$$+350784\zeta(4)\zeta(7)+176280\zeta(5)\zeta(6)+16320\zeta(2)\zeta(3)^3-112320\zeta(3)^2\zeta(5)$$
$$\left.+8640\zeta(3)M(2,6)+23040M(3,8)\right) \quad (993)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^5}{k^5(k+2)} = \frac{1}{4608}\left(144+576\zeta(2)+3024\zeta(3)+9036\zeta(4)+10224\zeta(5)+2160\zeta(2)\zeta(3)\right.$$
$$+25704\zeta(6)+3240\zeta(3)^2-36918\zeta(7)-8208\zeta(2)\zeta(5)-9504\zeta(3)\zeta(4)$$
$$-135622\zeta(8)-38160\zeta(2)\zeta(3)^2+157536\zeta(3)\zeta(5)+33840M(2,6)-151072\zeta(9)$$
$$+234960\zeta(3)\zeta(6)-76032\zeta(4)\zeta(5)-38160\zeta(2)\zeta(7)+5760\zeta(3)^3-579897\zeta(10)$$
$$+519840\zeta(3)\zeta(7)-3240\zeta(3)^2\zeta(4)-185040\zeta(2)\zeta(3)\zeta(5)+203832\zeta(5)^2$$
$$\left.+98280M(2,8)+11520\zeta(2)M(2,6)\right) \quad (994)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^5}{k^4(k+1)(k+2)} = \frac{1}{1152}\left(72+288\zeta(2)+1512\zeta(3)+4518\zeta(4)+5112\zeta(5)\right.$$
$$+1080\zeta(2)\zeta(3)-192780\zeta(6)-24300\zeta(3)^2+129213\zeta(7)+28728\zeta(2)\zeta(5)$$
$$+33264\zeta(3)\zeta(4)+203433\zeta(8)+57240\zeta(2)\zeta(3)^2-236304\zeta(3)\zeta(5)-50760M(2,6)$$
$$+75536\zeta(9)-117480\zeta(3)\zeta(6)+38016\zeta(4)\zeta(5)+19080\zeta(2)\zeta(7)$$
$$\left.-2880\zeta(3)^3\right) \quad (995)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^5}{k^3(k+1)^2(k+2)} = \frac{1}{576}\left(72+288\zeta(2)+1512\zeta(3)+4518\zeta(4)+5112\zeta(5)\right.$$
$$+1080\zeta(2)\zeta(3)+218484\zeta(6)+27540\zeta(3)^2-159075\zeta(7)-36936\zeta(2)\zeta(5)$$
$$-42768\zeta(3)\zeta(4)-67811\zeta(8)-19080\zeta(2)\zeta(3)^2+78768\zeta(3)\zeta(5)$$
$$\left.+16920M(2,6)\right) \quad (996)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^5}{k^2(k+1)^3(k+2)} = \frac{1}{288}\left(72+288\zeta(2)+1512\zeta(3)+4518\zeta(4)+5112\zeta(5)\right.$$
$$+1080\zeta(2)\zeta(3)-89964\zeta(6)-11340\zeta(3)^2+51849\zeta(7)+12312\zeta(2)\zeta(5)$$
$$+14256\zeta(3)\zeta(4)+65621\zeta(8)+17640\zeta(2)\zeta(3)^2-72432\zeta(3)\zeta(5)$$
$$\left.-15480M(2,6)\right) \quad (997)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^5}{k(k+1)^4(k+2)} = \frac{1}{144}\big(72+288\zeta(2)+1512\zeta(3)+4518\zeta(4)+5112\zeta(5)$$
$$+1080\zeta(2)\zeta(3)+12852\zeta(6)+1620\zeta(3)^2-16695\zeta(7)-4104\zeta(2)\zeta(5)$$
$$-4752\zeta(3)\zeta(4)-14240\zeta(9)+25770\zeta(3)\zeta(6)-9504\zeta(4)\zeta(5)-4770\zeta(2)\zeta(7)$$
$$+720\zeta(3)^3\big) \tag{998}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^5}{(k+1)^5(k+2)} = \frac{1}{2304}\big(2304+9216\zeta(2)+48384\zeta(3)+144576\zeta(4)+163584\zeta(5)$$
$$+34560\zeta(2)\zeta(3)-267120\zeta(7)-65664\zeta(2)\zeta(5)-76032\zeta(3)\zeta(4)+524968\zeta(8)$$
$$+141120\zeta(2)\zeta(3)^2-579456\zeta(3)\zeta(5)-123840M(2,6)-227840\zeta(9)$$
$$+412320\zeta(3)\zeta(6)-152064\zeta(4)\zeta(5)-76320\zeta(2)\zeta(7)+11520\zeta(3)^3$$
$$+449109\zeta(10)-387360\zeta(3)\zeta(7)-9720\zeta(3)^2\zeta(4)+124560\zeta(2)\zeta(3)\zeta(5)$$
$$-122328\zeta(5)^2-68040M(2,8)-11520\zeta(2)M(2,6)\big) \tag{999}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^5}{k^4(k+2)^2} = \frac{1}{1152}\big(576+1656\zeta(2)+7200\zeta(3)+16092\zeta(4)+9936\zeta(5)$$
$$+2520\zeta(2)\zeta(3)+12819\zeta(6)+2160\zeta(3)^2-36036\zeta(7)-8208\zeta(2)\zeta(5)$$
$$-9504\zeta(3)\zeta(4)-67811\zeta(8)-19080\zeta(2)\zeta(3)^2+78768\zeta(3)\zeta(5)+16920M(2,6)$$
$$-37768\zeta(9)+58740\zeta(3)\zeta(6)-19008\zeta(4)\zeta(5)-9540\zeta(2)\zeta(7)$$
$$+1440\zeta(3)^3\big) \tag{1000}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^5}{k^3(k+1)(k+2)^2} = \frac{1}{1152}\big(-1224-3600\zeta(2)-15912\zeta(3)-36702\zeta(4)-24984\zeta(5)$$
$$-6120\zeta(2)\zeta(3)+167142\zeta(6)+19980\zeta(3)^2-57141\zeta(7)-12312\zeta(2)\zeta(5)$$
$$-14256\zeta(3)\zeta(4)-67811\zeta(8)-19080\zeta(2)\zeta(3)^2+78768\zeta(3)\zeta(5)$$
$$+16920M(2,6)) \tag{1001}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^5}{k^2(k+1)^2(k+2)^2} = \frac{1}{96}\big(-216-648\zeta(2)-2904\zeta(3)-6870\zeta(4)-5016\zeta(5)$$
$$-1200\zeta(2)\zeta(3)-8557\zeta(6)-1260\zeta(3)^2+16989\zeta(7)+4104\zeta(2)\zeta(5)$$
$$+4752\zeta(3)\zeta(4)) \tag{1002}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^5}{k(k+1)^3(k+2)^2} = \frac{1}{288}\big(-1368-4176\zeta(2)-18936\zeta(3)-45738\zeta(4)-35208\zeta(5)$$
$$-8280\zeta(2)\zeta(3)+38622\zeta(6)+3780\zeta(3)^2+50085\zeta(7)+12312\zeta(2)\zeta(5)$$
$$+14256\zeta(3)\zeta(4)-65621\zeta(8)-17640\zeta(2)\zeta(3)^2+72432\zeta(3)\zeta(5)$$
$$+15480M(2,6)) \tag{1003}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^5}{(k+1)^4(k+2)^2} = \frac{1}{144}\Big(1440+4464\zeta(2)+20448\zeta(3)+50256\zeta(4)+40320\zeta(5)$$
$$+9360\zeta(2)\zeta(3)-25770\zeta(6)-2160\zeta(3)^2-66780\zeta(7)-16416\zeta(2)\zeta(5)$$
$$-19008\zeta(3)\zeta(4)+65621\zeta(8)+17640\zeta(2)\zeta(3)^2-72432\zeta(3)\zeta(5)-15480M(2,6)$$
$$-14240\zeta(9)+25770\zeta(3)\zeta(6)-9504\zeta(4)\zeta(5)-4770\zeta(2)\zeta(7)$$
$$+720\zeta(3)^3\Big) \tag{1004}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^5}{k^3(k+2)^3} = \frac{1}{1152}\big(4536+9144\zeta(2)+32184\zeta(3)+49770\zeta(4)+5256\zeta(5)$$
$$+2160\zeta(2)\zeta(3)-22899\zeta(6)-3420\zeta(3)^2-42921\zeta(7)-8712\zeta(2)\zeta(5)$$
$$-27576\zeta(3)\zeta(4)-66716\zeta(8)-18360\zeta(2)\zeta(3)^2+75600\zeta(3)\zeta(5)$$
$$+16200M(2,6)\big) \tag{1005}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^5}{k^2(k+1)(k+2)^3} = \frac{1}{1152}\big(10296+21888\zeta(2)+80280\zeta(3)+136242\zeta(4)+35496\zeta(5)$$
$$+10440\zeta(2)\zeta(3)-212940\zeta(6)-26820\zeta(3)^2-28701\zeta(7)-5112\zeta(2)\zeta(5)$$
$$-40896\zeta(3)\zeta(4)-65621\zeta(8)-17640\zeta(2)\zeta(3)^2+72432\zeta(3)\zeta(5)$$
$$+15480M(2,6)\big) \tag{1006}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^5}{k(k+1)^2(k+2)^3} = \frac{1}{576}\big(11592+25776\zeta(2)+97704\zeta(3)+177462\zeta(4)+65592\zeta(5)$$
$$+17640\zeta(2)\zeta(3)-161598\zeta(6)-19260\zeta(3)^2-130635\zeta(7)-29736\zeta(2)\zeta(5)$$
$$-69408\zeta(3)\zeta(4)-65621\zeta(8)-17640\zeta(2)\zeta(3)^2+72432\zeta(3)\zeta(5)$$
$$+15480M(2,6)\big) \tag{1007}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^5}{(k+1)^3(k+2)^3} = \frac{1}{24}\big(1080+2496\zeta(2)+9720\zeta(3)+18600\zeta(4)+8400\zeta(5)$$
$$+2160\zeta(2)\zeta(3)-16685\zeta(6)-1920\zeta(3)^2-15060\zeta(7)-3504\zeta(2)\zeta(5)$$
$$-6972\zeta(3)\zeta(4)\big) \tag{1008}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^5}{k^2(k+2)^4} = \frac{1}{1152}\Big(-23616-32616\zeta(2)-90432\zeta(3)-83340\zeta(4)+43344\zeta(5)$$
$$+12600\zeta(2)\zeta(3)+102651\zeta(6)+23040\zeta(3)^2+16452\zeta(7)-432\zeta(2)\zeta(5)$$
$$+39024\zeta(3)\zeta(4)-58529\zeta(8)-15480\zeta(2)\zeta(3)^2+65808\zeta(3)\zeta(5)+14760M(2,6)$$
$$+28480\zeta(9)-51540\zeta(3)\zeta(6)+19008\zeta(4)\zeta(5)+9540\zeta(2)\zeta(7)$$
$$-1440\zeta(3)^3\Big) \tag{1009}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^5}{k(k+1)(k+2)^4} = \frac{1}{1152}\Big(57528+87120\zeta(2)+261144\zeta(3)+302922\zeta(4)-51192\zeta(5)$$
$$-14760\zeta(2)\zeta(3)-418242\zeta(6)-72900\zeta(3)^2-61605\zeta(7)-4248\zeta(2)\zeta(5)$$
$$-118944\zeta(3)\zeta(4)+51437\zeta(8)+13320\zeta(2)\zeta(3)^2-59184\zeta(3)\zeta(5)-14040M(2,6)$$
$$-56960\zeta(9)+103080\zeta(3)\zeta(6)-38016\zeta(4)\zeta(5)-19080\zeta(2)\zeta(7)$$
$$+2880\zeta(3)^3\Big) \tag{1010}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^5}{(k+1)^2(k+2)^4}=\frac{1}{72}\big(8640+14112\zeta(2)+44856\zeta(3)+60048\zeta(4)+1800\zeta(5)$$
$$+360\zeta(2)\zeta(3)-72480\zeta(6)-11520\zeta(3)^2-24030\zeta(7)-4248\zeta(2)\zeta(5)$$
$$-23544\zeta(3)\zeta(4)-1773\zeta(8)-540\zeta(2)\zeta(3)^2+1656\zeta(3)\zeta(5)+180M(2,6)$$
$$-7120\zeta(9)+12885\zeta(3)\zeta(6)-4752\zeta(4)\zeta(5)-2385\zeta(2)\zeta(7)+360\zeta(3)^3\big) \quad (1011)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^5}{k(k+2)^5}=\frac{1}{4608}\big(368784+341856\zeta(2)+719568\zeta(3)+238572\zeta(4)-579600\zeta(5)$$
$$-200880\zeta(2)\zeta(3)-678396\zeta(6)-171000\zeta(3)^2-164070\zeta(7)-45648\zeta(2)\zeta(5)$$
$$+104256\zeta(3)\zeta(4)+1368878\zeta(8)+390960\zeta(2)\zeta(3)^2-1583136\zeta(3)\zeta(5)$$
$$-326160M(2,6)+53120\zeta(9)+111120\zeta(3)\zeta(6)-289152\zeta(4)\zeta(5)+42480\zeta(2)\zeta(7)$$
$$+36480\zeta(3)^3-449109\zeta(10)+387360\zeta(3)\zeta(7)+9720\zeta(3)^2\zeta(4)$$
$$-124560\zeta(2)\zeta(3)\zeta(5)+122328\zeta(5)^2+68040M(2,8)+11520\zeta(2)M(2,6)\big) \quad (1012)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^5}{(k+1)(k+2)^5}=\frac{1}{2304}\big(483840+516096\zeta(2)+1241856\zeta(3)+844416\zeta(4)-681984\zeta(5)$$
$$-230400\zeta(2)\zeta(3)-1514880\zeta(6)-316800\zeta(3)^2-287280\zeta(7)-54144\zeta(2)\zeta(5)$$
$$-133632\zeta(3)\zeta(4)+1471752\zeta(8)+417600\zeta(2)\zeta(3)^2-1701504\zeta(3)\zeta(5)$$
$$-354240M(2,6)-60800\zeta(9)+317280\zeta(3)\zeta(6)-365184\zeta(4)\zeta(5)+4320\zeta(2)\zeta(7)$$
$$+42240\zeta(3)^3-449109\zeta(10)+387360\zeta(3)\zeta(7)+9720\zeta(3)^2\zeta(4)$$
$$-124560\zeta(2)\zeta(3)\zeta(5)+122328\zeta(5)^2+68040M(2,8)+11520\zeta(2)M(2,6)\big) \quad (1013)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^5}{(k+2)^6}=\frac{1}{1152}\big(-290304-177408\zeta(2)-266112\zeta(3)+87552\zeta(4)+298368\zeta(5)$$
$$+97920\zeta(2)\zeta(3)+236112\zeta(6)+28800\zeta(3)^2+161280\zeta(7)+69120\zeta(2)\zeta(5)$$
$$-201600\zeta(3)\zeta(4)-547240\zeta(8)-161280\zeta(2)\zeta(3)^2+645120\zeta(3)\zeta(5)$$
$$+126720M(2,6)+11040\zeta(9)-5760\zeta(3)\zeta(6)+11520\zeta(4)\zeta(5)-5760\zeta(2)\zeta(7)$$
$$-3840\zeta(3)^3-437823\zeta(10)+378720\zeta(3)\zeta(7)-2520\zeta(3)^2\zeta(4)$$
$$-114480\zeta(2)\zeta(3)\zeta(5)+119880\zeta(5)^2+68760M(2,8)+11520\zeta(2)M(2,6)-1334454\zeta(11)$$
$$-137632\zeta(2)\zeta(9)+496600\zeta(3)\zeta(8)+701568\zeta(4)\zeta(7)+352560\zeta(5)\zeta(6)$$
$$+32640\zeta(2)\zeta(3)^3-224640\zeta(3)^2\zeta(5)+17280\zeta(3)M(2,6)+46080M(3,8)\big) \quad (1014)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^6}{k^5}=\frac{1}{192}\big(734643\zeta(11)+83472\zeta(2)\zeta(9)-271244\zeta(3)\zeta(8)-395088\zeta(4)\zeta(7)$$
$$-205424\zeta(5)\zeta(6)-19360\zeta(2)\zeta(3)^3+130176\zeta(3)^2\zeta(5)-9120\zeta(3)M(2,6)$$
$$-25600M(3,8)\big) \quad (1015)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^6}{k^4(k+1)}=\frac{1}{384}\left(-247296\zeta(7)-55680\zeta(2)\zeta(5)-114048\zeta(3)\zeta(4)+280464\zeta(8)\right.$$
$$-15744\zeta(2)\zeta(3)^2+187008\zeta(3)\zeta(5)+21888M(2,6)-119584\zeta(9)+209952\zeta(3)\zeta(6)$$
$$-96768\zeta(4)\zeta(5)-31248\zeta(2)\zeta(7)+8704\zeta(3)^3-814101\zeta(10)+529680\zeta(3)\zeta(7)$$
$$-253944\zeta(3)^2\zeta(4)-1200\zeta(2)\zeta(3)\zeta(5)+365064\zeta(5)^2+103128M(2,8)$$
$$\left.+45120\zeta(2)M(2,6)\right) \quad (1016)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^6}{k^3(k+1)^2}=\frac{1}{24}\left(46368\zeta(7)+10440\zeta(2)\zeta(5)+21384\zeta(3)\zeta(4)-52085\zeta(8)\right.$$
$$+2892\zeta(2)\zeta(3)^2-34704\zeta(3)\zeta(5)-4044M(2,6)+7474\zeta(9)-13122\zeta(3)\zeta(6)$$
$$\left.+6048\zeta(4)\zeta(5)+1953\zeta(2)\zeta(7)-544\zeta(3)^3\right) \quad (1017)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^6}{k^2(k+1)^3}=\frac{1}{24}\left(46368\zeta(7)+10440\zeta(2)\zeta(5)+21384\zeta(3)\zeta(4)-51583\zeta(8)\right.$$
$$+2832\zeta(2)\zeta(3)^2-34344\zeta(3)\zeta(5)-3984M(2,6)+6146\zeta(9)-12582\zeta(3)\zeta(6)$$
$$\left.+5832\zeta(4)\zeta(5)+1953\zeta(2)\zeta(7)-536\zeta(3)^3\right) \quad (1018)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^6}{k(k+1)^4}=\frac{1}{384}\left(247296\zeta(7)+55680\zeta(2)\zeta(5)+114048\zeta(3)\zeta(4)-272432\zeta(8)\right.$$
$$+14784\zeta(2)\zeta(3)^2-181248\zeta(3)\zeta(5)-20928M(2,6)+98336\zeta(9)-201312\zeta(3)\zeta(6)$$
$$+93312\zeta(4)\zeta(5)+31248\zeta(2)\zeta(7)-8576\zeta(3)^3+779835\zeta(10)-490704\zeta(3)\zeta(7)$$
$$+245544\zeta(3)^2\zeta(4)-15600\zeta(2)\zeta(3)\zeta(5)-339864\zeta(5)^2-94728M(2,8)$$
$$\left.-45120\zeta(2)M(2,6)\right) \quad (1019)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^6}{(k+1)^5}=\frac{1}{192}\left(686799\zeta(11)+74512\zeta(2)\zeta(9)-262484\zeta(3)\zeta(8)\right.$$
$$-362208\zeta(4)\zeta(7)-182584\zeta(5)\zeta(6)-18080\zeta(2)\zeta(3)^3+120384\zeta(3)^2\zeta(5)$$
$$\left.-8160\zeta(3)M(2,6)-23360M(3,8)\right) \quad (1020)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^6}{k^4(k+2)}=\frac{1}{768}\left(48+240\zeta(2)+1632\zeta(3)+6852\zeta(4)+13704\zeta(5)+2928\zeta(2)\zeta(3)\right.$$
$$+25164\zeta(6)+3216\zeta(3)^2+30912\zeta(7)+6960\zeta(2)\zeta(5)+14256\zeta(3)\zeta(4)$$
$$-70116\zeta(8)+3936\zeta(2)\zeta(3)^2-46752\zeta(3)\zeta(5)-5472M(2,6)+59792\zeta(9)$$
$$-104976\zeta(3)\zeta(6)+48384\zeta(4)\zeta(5)+15624\zeta(2)\zeta(7)-4352\zeta(3)^3+814101\zeta(10)$$
$$-529680\zeta(3)\zeta(7)+253944\zeta(3)^2\zeta(4)+1200\zeta(2)\zeta(3)\zeta(5)-365064\zeta(5)^2$$
$$\left.-103128M(2,8)-45120\zeta(2)M(2,6)\right) \quad (1021)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^6}{k^3(k+1)(k+2)}=\frac{1}{96}\left(12+60\zeta(2)+408\zeta(3)+1713\zeta(4)+3426\zeta(5)+732\zeta(2)\zeta(3)\right.$$
$$+6291\zeta(6)+804\zeta(3)^2-54096\zeta(7)-12180\zeta(2)\zeta(5)-24948\zeta(3)\zeta(4)$$
$$+52587\zeta(8)-2952\zeta(2)\zeta(3)^2+35064\zeta(3)\zeta(5)+4104M(2,6)-14948\zeta(9)$$
$$\left.+26244\zeta(3)\zeta(6)-12096\zeta(4)\zeta(5)-3906\zeta(2)\zeta(7)+1088\zeta(3)^3\right) \quad (1022)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^6}{k^2(k+1)^2(k+2)} = \frac{1}{48}\big(12+60\zeta(2)+408\zeta(3)+1713\zeta(4)+3426\zeta(5)$$
$$+732\zeta(2)\zeta(3)+6291\zeta(6)+804\zeta(3)^2+38640\zeta(7)+8700\zeta(2)\zeta(5)$$
$$+17820\zeta(3)\zeta(4)-51583\zeta(8)+2832\zeta(2)\zeta(3)^2-34344\zeta(3)\zeta(5)$$
$$-3984M(2,6)\big) \tag{1023}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^6}{k(k+1)^3(k+2)} = \frac{1}{24}\big(12+60\zeta(2)+408\zeta(3)+1713\zeta(4)+3426\zeta(5)+732\zeta(2)\zeta(3)$$
$$+6291\zeta(6)+804\zeta(3)^2-7728\zeta(7)-1740\zeta(2)\zeta(5)-3564\zeta(3)\zeta(4)-6146\zeta(9)$$
$$+12582\zeta(3)\zeta(6)-5832\zeta(4)\zeta(5)-1953\zeta(2)\zeta(7)+536\zeta(3)^3\big) \tag{1024}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^6}{(k+1)^4(k+2)} = \frac{1}{384}\big(384+1920\zeta(2)+13056\zeta(3)+54816\zeta(4)+109632\zeta(5)$$
$$+23424\zeta(2)\zeta(3)+201312\zeta(6)+25728\zeta(3)^2-272432\zeta(8)+14784\zeta(2)\zeta(3)^2$$
$$-181248\zeta(3)\zeta(5)-20928M(2,6)-98336\zeta(9)+201312\zeta(3)\zeta(6)-93312\zeta(4)\zeta(5)$$
$$-31248\zeta(2)\zeta(7)+8576\zeta(3)^3+779835\zeta(10)-490704\zeta(3)\zeta(7)$$
$$+245544\zeta(3)^2\zeta(4)-15600\zeta(2)\zeta(3)\zeta(5)-339864\zeta(5)^2-94728M(2,8)$$
$$-45120\zeta(2)M(2,6)\big) \tag{1025}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^6}{k^3(k+2)^2} = \frac{1}{192}\big(204+756\zeta(2)+4344\zeta(3)+14427\zeta(4)+20382\zeta(5)$$
$$+4644\zeta(2)\zeta(3)+18570\zeta(6)+2940\zeta(3)^2+6489\zeta(7)+1116\zeta(2)\zeta(5)$$
$$+5940\zeta(3)\zeta(4)-52085\zeta(8)+2892\zeta(2)\zeta(3)^2-34704\zeta(3)\zeta(5)-4044M(2,6)$$
$$+14948\zeta(9)-26244\zeta(3)\zeta(6)+12096\zeta(4)\zeta(5)+3906\zeta(2)\zeta(7)$$
$$-1088\zeta(3)^3\big) \tag{1026}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^6}{k^2(k+1)(k+2)^2} = \frac{1}{96}\big(216+816\zeta(2)+4752\zeta(3)+16140\zeta(4)+23808\zeta(5)$$
$$+5376\zeta(2)\zeta(3)+24861\zeta(6)+3744\zeta(3)^2-47607\zeta(7)-11064\zeta(2)\zeta(5)$$
$$-19008\zeta(3)\zeta(4)+502\zeta(8)-60\zeta(2)\zeta(3)^2+360\zeta(3)\zeta(5)+60M(2,6)\big) \tag{1027}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^6}{k(k+1)^2(k+2)^2} = \frac{1}{16}\big(76+292\zeta(2)+1720\zeta(3)+5951\zeta(4)+9078\zeta(5)$$
$$+2036\zeta(2)\zeta(3)+10384\zeta(6)+1516\zeta(3)^2-2989\zeta(7)-788\zeta(2)\zeta(5)$$
$$-396\zeta(3)\zeta(4)-17027\zeta(8)+924\zeta(2)\zeta(3)^2-11328\zeta(3)\zeta(5)-1308M(2,6)\big) \tag{1028}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^6}{(k+1)^3(k+2)^2} = \frac{1}{24}\big(240+936\zeta(2)+5568\zeta(3)+19566\zeta(4)+30660\zeta(5)$$
$$+6840\zeta(2)\zeta(3)+37443\zeta(6)+5352\zeta(3)^2-16695\zeta(7)-4104\zeta(2)\zeta(5)$$
$$-4752\zeta(3)\zeta(4)-51081\zeta(8)+2772\zeta(2)\zeta(3)^2-33984\zeta(3)\zeta(5)-3924M(2,6)$$
$$-6146\zeta(9)+12582\zeta(3)\zeta(6)-5832\zeta(4)\zeta(5)-1953\zeta(2)\zeta(7)+536\zeta(3)^3\big) \tag{1029}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^6}{k^2(k+2)^3} = \frac{1}{192}\big(1716 + 4644\zeta(2) + 22296\zeta(3) + 56835\zeta(4) + 51078\zeta(5)$$
$$+12276\zeta(2)\zeta(3) - 6129\zeta(6) + 444\zeta(3)^2 - 33786\zeta(7) - 7596\zeta(2)\zeta(5)$$
$$-21636\zeta(3)\zeta(4) - 117204\zeta(8) - 14808\zeta(2)\zeta(3)^2 + 38088\zeta(3)\zeta(5) + 11496M(2,6)$$
$$+12292\zeta(9) - 25164\zeta(3)\zeta(6) + 11664\zeta(4)\zeta(5) + 3906\zeta(2)\zeta(7)$$
$$-1072\zeta(3)^3\big) \quad (1030)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^6}{k(k+1)(k+2)^3} = \frac{1}{96}\big(-1932 - 5460\zeta(2) - 27048\zeta(3) - 72975\zeta(4) - 74886\zeta(5)$$
$$-17652\zeta(2)\zeta(3) - 18732\zeta(6) - 4188\zeta(3)^2 + 81393\zeta(7) + 18660\zeta(2)\zeta(5)$$
$$+40644\zeta(3)\zeta(4) + 116702\zeta(8) + 14868\zeta(2)\zeta(3)^2 - 38448\zeta(3)\zeta(5) - 11556M(2,6)$$
$$-12292\zeta(9) + 25164\zeta(3)\zeta(6) - 11664\zeta(4)\zeta(5) - 3906\zeta(2)\zeta(7)$$
$$+1072\zeta(3)^3\big) \quad (1031)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^6}{(k+1)^2(k+2)^3} = \frac{1}{48}\big(2160 + 6336\zeta(2) + 32208\zeta(3) + 90828\zeta(4) + 102120\zeta(5)$$
$$+23760\zeta(2)\zeta(3) + 49884\zeta(6) + 8736\zeta(3)^2 - 90360\zeta(7) - 21024\zeta(2)\zeta(5)$$
$$-41832\zeta(3)\zeta(4) - 167783\zeta(8) - 12096\zeta(2)\zeta(3)^2 + 4464\zeta(3)\zeta(5) + 7632M(2,6)$$
$$+12292\zeta(9) - 25164\zeta(3)\zeta(6) + 11664\zeta(4)\zeta(5) + 3906\zeta(2)\zeta(7)$$
$$-1072\zeta(3)^3\big) \quad (1032)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^6}{k(k+2)^4} = \frac{1}{768}\big(38352 + 74928\zeta(2) + 297696\zeta(3) + 562980\zeta(4) + 259368\zeta(5)$$
$$+60336\zeta(2)\zeta(3) - 452160\zeta(6) - 73200\zeta(3)^2 - 297468\zeta(7) - 51312\zeta(2)\zeta(5)$$
$$-321840\zeta(3)\zeta(4) - 358960\zeta(8) - 75504\zeta(2)\zeta(3)^2 + 270912\zeta(3)\zeta(5) + 59568M(2,6)$$
$$-178672\zeta(9) + 311664\zeta(3)\zeta(6) - 105408\zeta(4)\zeta(5) - 60696\zeta(2)\zeta(7) + 7232\zeta(3)^3$$
$$+779835\zeta(10) - 490704\zeta(3)\zeta(7) + 245544\zeta(3)^2\zeta(4) - 15600\zeta(2)\zeta(3)\zeta(5)$$
$$-339864\zeta(5)^2 - 94728M(2,8) - 45120\zeta(2)M(2,6)\big) \quad (1033)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^6}{(k+1)(k+2)^4} = \frac{1}{384}\big(46080 + 96768\zeta(2) + 405888\zeta(3) + 854880\zeta(4) + 558912\zeta(5)$$
$$+130944\zeta(2)\zeta(3) - 377232\zeta(6) - 56448\zeta(3)^2 - 623040\zeta(7) - 125952\zeta(2)\zeta(5)$$
$$-484416\zeta(3)\zeta(4) - 825768\zeta(8) - 134976\zeta(2)\zeta(3)^2 + 424704\zeta(3)\zeta(5)$$
$$+105792M(2,6) - 129504\zeta(9) + 211008\zeta(3)\zeta(6) - 58752\zeta(4)\zeta(5) - 45072\zeta(2)\zeta(7)$$
$$+2944\zeta(3)^3 + 779835\zeta(10) - 490704\zeta(3)\zeta(7) + 245544\zeta(3)^2\zeta(4)$$
$$-15600\zeta(2)\zeta(3)\zeta(5) - 339864\zeta(5)^2 - 94728M(2,8) - 45120\zeta(2)M(2,6)\big)$$
$$(1034)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^6}{(k+2)^5} = \frac{1}{384}\big(-80640 - 112896\zeta(2) - 368256\zeta(3) - 494496\zeta(4) - 35136\zeta(5)$$
$$+5760\zeta(2)\zeta(3) + 565536\zeta(6) + 111360\zeta(3)^2 + 197280\zeta(7) + 31104\zeta(2)\zeta(5)$$
$$+202752\zeta(3)\zeta(4) - 471672\zeta(8) - 133440\zeta(2)\zeta(3)^2 + 549504\zeta(3)\zeta(5)$$
$$+116160M(2,6) + 144320\zeta(9) - 364800\zeta(3)\zeta(6) + 258624\zeta(4)\zeta(5) + 36000\zeta(2)\zeta(7)$$
$$-26880\zeta(3)^3 + 449109\zeta(10) - 387360\zeta(3)\zeta(7) - 9720\zeta(3)^2\zeta(4)$$
$$+124560\zeta(2)\zeta(3)\zeta(5) - 122328\zeta(5)^2 - 68040M(2,8) - 11520\zeta(2)M(2,6) + 1373598\zeta(11)$$
$$+149024\zeta(2)\zeta(9) - 524968\zeta(3)\zeta(8) - 724416\zeta(4)\zeta(7) - 365168\zeta(5)\zeta(6)$$
$$-36160\zeta(2)\zeta(3)^3 + 240768\zeta(3)^2\zeta(5) - 16320\zeta(3)M(2,6) - 46720M(3,8)\big) \quad (1035)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^7}{k^4} = \frac{1}{1152}\big(16370805\zeta(11) + 1684144\zeta(2)\zeta(9) + 5889744\zeta(3)\zeta(8)$$
$$-10724760\zeta(4)\zeta(7) - 10480104\zeta(5)\zeta(6) + 844032\zeta(2)\zeta(3)^3$$
$$-2330496\zeta(3)^2\zeta(5) - 1431360\zeta(3)M(2,6) - 630336M(3,8)\big) \quad (1036)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^7}{k^3(k+1)} = \frac{1}{23040}\big(153310720\zeta(8) + 3870720\zeta(2)\zeta(3)^2 + 35078400\zeta(3)\zeta(5)$$
$$-88429120\zeta(9) - 28372800\zeta(3)\zeta(6) - 45812160\zeta(4)\zeta(5) - 18933120\zeta(2)\zeta(7)$$
$$-1290240\zeta(3)^3 - 149534919\zeta(10) + 92839680\zeta(3)\zeta(7) - 52912440\zeta(3)^2\zeta(4)$$
$$+6345360\zeta(2)\zeta(3)\zeta(5) + 69396840\zeta(5)^2 + 18196920M(2,8) + 9072000\zeta(2)M(2,6)\big) \quad (1037)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^7}{k^2(k+1)^2} = \frac{1}{72}\big(-958192\zeta(8) - 24192\zeta(2)\zeta(3)^2 - 219240\zeta(3)\zeta(5) + 545743\zeta(9)$$
$$+177330\zeta(3)\zeta(6) + 284436\zeta(4)\zeta(5) + 118332\zeta(2)\zeta(7) + 8064\zeta(3)^3\big) \quad (1038)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^7}{k(k+1)^3} = \frac{1}{23040}\big(153310720\zeta(8) + 3870720\zeta(2)\zeta(3)^2 + 35078400\zeta(3)\zeta(5)$$
$$-86208640\zeta(9) - 28372800\zeta(3)\zeta(6) - 45207360\zeta(4)\zeta(5) - 18933120\zeta(2)\zeta(7)$$
$$-1290240\zeta(3)^3 - 149375151\zeta(10) + 89769600\zeta(3)\zeta(7) - 52206840\zeta(3)^2\zeta(4)$$
$$+7514640\zeta(2)\zeta(3)\zeta(5) + 67262760\zeta(5)^2 + 17612280M(2,8) + 9072000\zeta(2)M(2,6)\big) \quad (1039)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^7}{(k+1)^4} = \frac{1}{1152}\big(16196565\zeta(11) + 1630384\zeta(2)\zeta(9) + 5721072\zeta(3)\zeta(8)$$
$$-10468728\zeta(4)\zeta(7) - 10317144\zeta(5)\zeta(6) + 837312\zeta(2)\zeta(3)^3$$
$$-2330496\zeta(3)^2\zeta(5) - 1411200\zeta(3)M(2,6) - 616896M(3,8)\big) \quad (1040)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^7}{k^3(k+2)} = \frac{1}{46080}\big(5760 + 34560\zeta(2) + 288000\zeta(3) + 1546560\zeta(4) + 4340160\zeta(5)$$
$$+927360\zeta(2)\zeta(3) + 14115480\zeta(6) + 1834560\zeta(3)^2 + 12782160\zeta(7) + 2923200\zeta(2)\zeta(5)$$
$$+5957280\zeta(3)\zeta(4) + 38327680\zeta(8) + 967680\zeta(2)\zeta(3)^2 + 8769600\zeta(3)\zeta(5)$$
$$-44214560\zeta(9) - 14186400\zeta(3)\zeta(6) - 22906080\zeta(4)\zeta(5) - 9466560\zeta(2)\zeta(7)$$
$$-645120\zeta(3)^3 - 149534919\zeta(10) + 92839680\zeta(3)\zeta(7) - 52912440\zeta(3)^2\zeta(4)$$
$$+6345360\zeta(2)\zeta(3)\zeta(5) + 69396840\zeta(5)^2 + 18196920M(2,8) + 9072000\zeta(2)M(2,6)\big) \quad (1041)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^7}{k^2(k+1)(k+2)} = \frac{1}{576}\big(144+864\zeta(2)+7200\zeta(3)+38664\zeta(4)+108504\zeta(5)$$
$$+23184\zeta(2)\zeta(3)+352887\zeta(6)+45864\zeta(3)^2+319554\zeta(7)+73080\zeta(2)\zeta(5)$$
$$+148932\zeta(3)\zeta(4)-2874576\zeta(8)-72576\zeta(2)\zeta(3)^2-657720\zeta(3)\zeta(5)$$
$$+1105364\zeta(9)+354660\zeta(3)\zeta(6)+572652\zeta(4)\zeta(5)+236664\zeta(2)\zeta(7)$$
$$+16128\zeta(3)^3\big) \tag{1042}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^7}{k(k+1)^2(k+2)} = \frac{1}{288}\big(144+864\zeta(2)+7200\zeta(3)+38664\zeta(4)+108504\zeta(5)$$
$$+23184\zeta(2)\zeta(3)+352887\zeta(6)+45864\zeta(3)^2+319554\zeta(7)+73080\zeta(2)\zeta(5)$$
$$+148932\zeta(3)\zeta(4)+958192\zeta(8)+24192\zeta(2)\zeta(3)^2+219240\zeta(3)\zeta(5)$$
$$-1077608\zeta(9)-354660\zeta(3)\zeta(6)-565092\zeta(4)\zeta(5)-236664\zeta(2)\zeta(7)$$
$$-16128\zeta(3)^3\big) \tag{1043}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^7}{(k+1)^3(k+2)} = \frac{1}{23040}\big(23040+138240\zeta(2)+1152000\zeta(3)+6186240\zeta(4)+17360640\zeta(5)$$
$$+3709440\zeta(2)\zeta(3)+56461920\zeta(6)+7338240\zeta(3)^2+51128640\zeta(7)$$
$$+11692800\zeta(2)\zeta(5)+23829120\zeta(3)\zeta(4)-86208640\zeta(9)-28372800\zeta(3)\zeta(6)$$
$$-45207360\zeta(4)\zeta(5)-18933120\zeta(2)\zeta(7)-1290240\zeta(3)^3+149375151\zeta(10)$$
$$-89769600\zeta(3)\zeta(7)+52206840\zeta(3)^2\zeta(4)-7514640\zeta(2)\zeta(3)\zeta(5)$$
$$-67262760\zeta(5)^2-17612280M(2,8)-9072000\zeta(2)M(2,6)\big) \tag{1044}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^7}{k^2(k+2)^2} = \frac{1}{576}\big(1296+5904\zeta(2)+42192\zeta(3)+185004\zeta(4)+396216\zeta(5)$$
$$+87696\zeta(2)\zeta(3)+878271\zeta(6)+122472\zeta(3)^2+288513\zeta(7)+59976\zeta(2)\zeta(5)$$
$$+198072\zeta(3)\zeta(4)+243058\zeta(8)+63000\zeta(2)\zeta(3)^2-256536\zeta(3)\zeta(5)-54936M(2,6)$$
$$-1091486\zeta(9)-354660\zeta(3)\zeta(6)-568872\zeta(4)\zeta(5)-236664\zeta(2)\zeta(7)$$
$$-16128\zeta(3)^3\big) \tag{1045}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^7}{k(k+1)(k+2)^2} = \frac{1}{576}\big(2736+12672\zeta(2)+91584\zeta(3)+408672\zeta(4)+900936\zeta(5)$$
$$+198576\zeta(2)\zeta(3)+2109429\zeta(6)+290808\zeta(3)^2+896580\zeta(7)+193032\zeta(2)\zeta(5)$$
$$+545076\zeta(3)\zeta(4)-2388460\zeta(8)+53424\zeta(2)\zeta(3)^2-1170792\zeta(3)\zeta(5)$$
$$-109872M(2,6)-1077608\zeta(9)-354660\zeta(3)\zeta(6)-565092\zeta(4)\zeta(5)-236664\zeta(2)\zeta(7)$$
$$-16128\zeta(3)^3\big) \tag{1046}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^7}{(k+1)^2(k+2)^2} = \frac{1}{144}\big(1440+6768\zeta(2)+49392\zeta(3)+223668\zeta(4)+504720\zeta(5)$$
$$+110880\zeta(2)\zeta(3)+1231158\zeta(6)+168336\zeta(3)^2+608067\zeta(7)+133056\zeta(2)\zeta(5)$$
$$+347004\zeta(3)\zeta(4)-715134\zeta(8)+38808\zeta(2)\zeta(3)^2-475776\zeta(3)\zeta(5)-54936M(2,6)$$
$$-1077608\zeta(9)-354660\zeta(3)\zeta(6)-565092\zeta(4)\zeta(5)-236664\zeta(2)\zeta(7)$$
$$-16128\zeta(3)^3\big) \tag{1047}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^7}{k(k+2)^3} = \frac{1}{46080}\big(927360+3179520\zeta(2)+19376640\zeta(3)+68423040\zeta(4)+108054720\zeta(5)$$
$$+24635520\zeta(2)\zeta(3)+140607960\zeta(6)+21107520\zeta(3)^2-39775680\zeta(7)$$
$$-9979200\zeta(2)\zeta(5)-13134240\zeta(3)\zeta(4)-243547760\zeta(8)-19353600\zeta(2)\zeta(3)^2$$
$$+16269120\zeta(3)\zeta(5)+12821760M(2,6)-1803200\zeta(9)-98737440\zeta(3)\zeta(6)$$
$$+16587360\zeta(4)\zeta(5)+3657600\zeta(2)\zeta(7)-4247040\zeta(3)^3-149375151\zeta(10)$$
$$+89769600\zeta(3)\zeta(7)-52206840\zeta(3)^2\zeta(4)+7514640\zeta(2)\zeta(3)\zeta(5)$$
$$+67262760\zeta(5)^2+17612280M(2,8)+9072000\zeta(2)M(2,6)\big) \quad (1048)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^7}{(k+1)(k+2)^3} = \frac{1}{23040}\big(1036800+3686400\zeta(2)+23040000\zeta(3)+84769920\zeta(4)$$
$$+144092160\zeta(5)+32578560\zeta(2)\zeta(3)+224985120\zeta(6)+32739840\zeta(3)^2-3912480\zeta(7)$$
$$-2257920\zeta(2)\zeta(5)+8668800\zeta(3)\zeta(4)-339086160\zeta(8)-17216640\zeta(2)\zeta(3)^2$$
$$-30562560\zeta(3)\zeta(5)+8426880M(2,6)-44907520\zeta(9)-112923840\zeta(3)\zeta(6)$$
$$-6016320\zeta(4)\zeta(5)-5808960\zeta(2)\zeta(7)-4892160\zeta(3)^3-149375151\zeta(10)$$
$$+89769600\zeta(3)\zeta(7)-52206840\zeta(3)^2\zeta(4)+7514640\zeta(2)\zeta(3)\zeta(5)$$
$$+67262760\zeta(5)^2+17612280M(2,8)+9072000\zeta(2)M(2,6)\big) \quad (1049)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^7}{(k+2)^4} = \frac{1}{1152}(138240+354816\zeta(2)+1838592\zeta(3)+5175072\zeta(4)+5821632\zeta(5)$$
$$+1338624\zeta(2)\zeta(3)+2929008\zeta(6)+443520\zeta(3)^2-4509648\zeta(7)-959616\zeta(2)\zeta(5)$$
$$-3108672\zeta(3)\zeta(4)-13269200\zeta(8)-1725696\zeta(2)\zeta(3)^2+4491648\zeta(3)\zeta(5)$$
$$+1314432M(2,6)-327264\zeta(9)+101808\zeta(3)\zeta(6)+362880\zeta(4)\zeta(5)-145152\zeta(2)\zeta(7)$$
$$-59136\zeta(3)^3+16376535\zeta(10)-10304784\zeta(3)\zeta(7)+5156424\zeta(3)^2\zeta(4)$$
$$-327600\zeta(2)\zeta(3)\zeta(5)-7137144\zeta(5)^2-1989288M(2,8)-947520\zeta(2)M(2,6)$$
$$+16196565\zeta(11)+1630384\zeta(2)\zeta(9)+5721072\zeta(3)\zeta(8)-10468728\zeta(4)\zeta(7)$$
$$-10317144\zeta(5)\zeta(6)+837312\zeta(2)\zeta(3)^3-2330496\zeta(3)^2\zeta(5)-1411200\zeta(3)M(2,6)$$
$$-616896M(3,8)) \quad (1050)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^8}{k^3} = \frac{1}{72}(2824380\zeta(11)+277304\zeta(2)\zeta(9)+1926401\zeta(3)\zeta(8)$$
$$-1998972\zeta(4)\zeta(7)-2270310\zeta(5)\zeta(6)+243648\zeta(2)\zeta(3)^3-803808\zeta(3)^2\zeta(5)$$
$$-341280\zeta(3)M(2,6)-113760M(3,8)) \quad (1051)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^8}{k^2(k+1)} = \frac{1}{480}(13336000\zeta(9)+7093200\zeta(3)\zeta(6)+6432000\zeta(4)\zeta(5)$$
$$+2807280\zeta(2)\zeta(7)+322560\zeta(3)^3-18741581\zeta(10)-6689520\zeta(3)\zeta(7)$$
$$+524640\zeta(3)^2\zeta(4)-1452480\zeta(2)\zeta(3)\zeta(5)-4247040\zeta(5)^2-485280M(2,8)$$
$$-299520\zeta(2)M(2,6)) \quad (1052)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^8}{k(k+1)^2} = \frac{1}{240}\left(6668000\zeta(9)+3546600\zeta(3)\zeta(6)+3216000\zeta(4)\zeta(5)\right.$$
$$+1403640\zeta(2)\zeta(7)+161280\zeta(3)^3-9295879\zeta(10)-3314520\zeta(3)\zeta(7)$$
$$+258540\zeta(3)^2\zeta(4)-733800\zeta(2)\zeta(3)\zeta(5)-2098980\zeta(5)^2-238860M(2,8)$$
$$\left.-149760\zeta(2)M(2,6)\right) \quad (1053)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^8}{(k+1)^3} = \frac{1}{72}\left(2839707\zeta(11)+274424\zeta(2)\zeta(9)+1906367\zeta(3)\zeta(8)\right.$$
$$-1976076\zeta(4)\zeta(7)-2252940\zeta(5)\zeta(6)+242208\zeta(2)\zeta(3)^3-798912\zeta(3)^2\zeta(5)$$
$$\left.-339120\zeta(3)M(2,6)-113040M(3,8)\right) \quad (1054)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^8}{k^2(k+2)} = \frac{1}{2880}\left(720+5040\zeta(2)+49680\zeta(3)+324000\zeta(4)+1162800\zeta(5)\right.$$
$$+247680\zeta(2)\zeta(3)+5303460\zeta(6)+692640\zeta(3)^2+8496540\zeta(7)+1931040\zeta(2)\zeta(5)$$
$$+4042800\zeta(3)\zeta(4)+19063670\zeta(8)+483840\zeta(2)\zeta(3)^2+4368960\zeta(3)\zeta(5)$$
$$+20004000\zeta(9)+10639800\zeta(3)\zeta(6)+9648000\zeta(4)\zeta(5)+4210920\zeta(2)\zeta(7)$$
$$+483840\zeta(3)^3-56224743\zeta(10)-20068560\zeta(3)\zeta(7)+1573920\zeta(3)^2\zeta(4)$$
$$\left.-4357440\zeta(2)\zeta(3)\zeta(5)-12741120\zeta(5)^2-1455840M(2,8)-898560\zeta(2)M(2,6)\right)$$
$$(1055)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^8}{k(k+1)(k+2)} = \frac{1}{144}\left(-72-504\zeta(2)-4968\zeta(3)-32400\zeta(4)-116280\zeta(5)\right.$$
$$-24768\zeta(2)\zeta(3)-530346\zeta(6)-69264\zeta(3)^2-849654\zeta(7)-193104\zeta(2)\zeta(5)$$
$$-404280\zeta(3)\zeta(4)-1906367\zeta(8)-48384\zeta(2)\zeta(3)^2-436896\zeta(3)\zeta(5)$$
$$+2000400\zeta(9)+1063980\zeta(3)\zeta(6)+964800\zeta(4)\zeta(5)+421092\zeta(2)\zeta(7)$$
$$\left.+48384\zeta(3)^3\right) \quad (1056)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^8}{(k+1)^2(k+2)} = \frac{1}{720}\left(720+5040\zeta(2)+49680\zeta(3)+324000\zeta(4)+1162800\zeta(5)\right.$$
$$+247680\zeta(2)\zeta(3)+5303460\zeta(6)+692640\zeta(3)^2+8496540\zeta(7)+1931040\zeta(2)\zeta(5)$$
$$+4042800\zeta(3)\zeta(4)+19063670\zeta(8)+483840\zeta(2)\zeta(3)^2+4368960\zeta(3)\zeta(5)$$
$$-27887637\zeta(10)-9943560\zeta(3)\zeta(7)+775620\zeta(3)^2\zeta(4)-2201400\zeta(2)\zeta(3)\zeta(5)$$
$$\left.-6296940\zeta(5)^2-716580M(2,8)-449280\zeta(2)M(2,6)\right) \quad (1057)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^8}{k(k+2)^2} = \frac{1}{1440}\left(6840+37080\zeta(2)+317160\zeta(3)+1726560\zeta(4)+4930200\zeta(5)\right.$$
$$+1074240\zeta(2)\zeta(3)+16758210\zeta(6)+2273040\zeta(3)^2+16566750\zeta(7)$$
$$+3678480\zeta(2)\zeta(5)+8776440\zeta(3)\zeta(4)+14292825\zeta(8)+1501920\zeta(2)\zeta(3)^2$$
$$-2962080\zeta(3)\zeta(5)-1098720M(2,6)-11550160\zeta(9)-1773300\zeta(3)\zeta(6)$$
$$-6477840\zeta(4)\zeta(5)-2627820\zeta(2)\zeta(7)-80640\zeta(3)^3-27887637\zeta(10)$$
$$-9943560\zeta(3)\zeta(7)+775620\zeta(3)^2\zeta(4)-2201400\zeta(2)\zeta(3)\zeta(5)-6296940\zeta(5)^2$$
$$\left.-716580M(2,8)-449280\zeta(2)M(2,6)\right) \quad (1058)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^8}{(k+1)(k+2)^2}=\frac{1}{720}\big(7200+39600\zeta(2)+342000\zeta(3)+1888560\zeta(4)+5511600\zeta(5)$$
$$+1198080\zeta(2)\zeta(3)+19409940\zeta(6)+2619360\zeta(3)^2+20815020\zeta(7)$$
$$+4644000\zeta(2)\zeta(5)+10797840\zeta(3)\zeta(4)+23824660\zeta(8)+1743840\zeta(2)\zeta(3)^2$$
$$-777600\zeta(3)\zeta(5)-1098720M(2,6)-21552160\zeta(9)-7093200\zeta(3)\zeta(6)$$
$$-11301840\zeta(4)\zeta(5)-4733280\zeta(2)\zeta(7)-322560\zeta(3)^3-27887637\zeta(10)$$
$$-9943560\zeta(3)\zeta(7)+775620\zeta(3)^2\zeta(4)-2201400\zeta(2)\zeta(3)\zeta(5)-6296940\zeta(5)^2$$
$$-716580M(2,8)-449280\zeta(2)M(2,6)\big) \quad (1059)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^8}{(k+2)^3}=\frac{1}{2880}\big(129600+541440\zeta(2)+4008960\zeta(3)+18103680\zeta(4)+40584960\zeta(5)$$
$$+9020160\zeta(2)\zeta(3)+98753280\zeta(6)+13881600\zeta(3)^2+48610080\zeta(7)$$
$$+10391040\zeta(2)\zeta(5)+28817280\zeta(3)\zeta(4)-74914520\zeta(8)-1733760\zeta(2)\zeta(3)^2$$
$$-18086400\zeta(3)\zeta(5)-120960M(2,6)-65558080\zeta(9)-70648320\zeta(3)\zeta(6)$$
$$-25611840\zeta(4)\zeta(5)-12371040\zeta(2)\zeta(7)-3091200\zeta(3)^3-149375151\zeta(10)$$
$$+89769600\zeta(3)\zeta(7)-52206840\zeta(3)^2\zeta(4)+7514640\zeta(2)\zeta(3)\zeta(5)$$
$$+67262760\zeta(5)^2+17612280M(2,8)+9072000\zeta(2)M(2,6)-113588280\zeta(11)$$
$$-10976960\zeta(2)\zeta(9)-76254680\zeta(3)\zeta(8)+79043040\zeta(4)\zeta(7)+90117600\zeta(5)\zeta(6)$$
$$-9688320\zeta(2)\zeta(3)^3+31956480\zeta(3)^2\zeta(5)+13564800\zeta(3)M(2,6)+4521600M(3,8)\big)$$
$$(1060)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^9}{k^2}=\frac{1}{64}\big(7739347\zeta(11)+2048432\zeta(2)\zeta(9)+5357920\zeta(3)\zeta(8)$$
$$+8811792\zeta(4)\zeta(7)+10526056\zeta(5)\zeta(6)-294208\zeta(2)\zeta(3)^3+2064192\zeta(3)^2\zeta(5)$$
$$+540096\zeta(3)M(2,6)+199936M(3,8)\big) \quad (1061)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^9}{k(k+1)}=\frac{1}{40}\big(17039209\zeta(10)+3158190\zeta(3)\zeta(7)+704820\zeta(3)^2\zeta(4)$$
$$+928080\zeta(2)\zeta(3)\zeta(5)+1767000\zeta(5)^2+37320\zeta(2)M(2,6)\big) \quad (1062)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^9}{(k+1)^2}=\frac{1}{64}\big(7676163\zeta(11)+2050992\zeta(2)\zeta(9)+5357920\zeta(3)\zeta(8)$$
$$+8776416\zeta(4)\zeta(7)+10489496\zeta(5)\zeta(6)-292928\zeta(2)\zeta(3)^3+2058432\zeta(3)^2\zeta(5)$$
$$+538176\zeta(3)M(2,6)+199296M(3,8)\big) \quad (1063)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^9}{k(k+2)}=\frac{1}{160}\big(80+640\zeta(2)+7280\zeta(3)+55780\zeta(4)+243200\zeta(5)+51600\zeta(2)\zeta(3)$$
$$+1416700\zeta(6)+185160\zeta(3)^2+3186110\zeta(7)+721920\zeta(2)\zeta(5)+1525680\zeta(3)\zeta(4)$$
$$+12880535\zeta(8)+365400\zeta(2)\zeta(3)^2+2739360\zeta(3)\zeta(5)-37320M(2,6)+9964440\zeta(9)$$
$$+5312700\zeta(3)\zeta(6)+4812120\zeta(4)\zeta(5)+2105460\zeta(2)\zeta(7)+241760\zeta(3)^3$$
$$+34078418\zeta(10)+6316380\zeta(3)\zeta(7)+1409640\zeta(3)^2\zeta(4)+1856160\zeta(2)\zeta(3)\zeta(5)$$
$$+3534000\zeta(5)^2+74640\zeta(2)M(2,6)\big) \quad (1064)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^9}{(k+1)(k+2)} = \frac{1}{16}\big(16+128\zeta(2)+1456\zeta(3)+11156\zeta(4)+48640\zeta(5)+10320\zeta(2)\zeta(3)$$
$$+283340\zeta(6)+37032\zeta(3)^2+637222\zeta(7)+144384\zeta(2)\zeta(5)+305136\zeta(3)\zeta(4)$$
$$+2576107\zeta(8)+73080\zeta(2)\zeta(3)^2+547872\zeta(3)\zeta(5)-7464M(2,6)+1992888\zeta(9)$$
$$+1062540\zeta(3)\zeta(6)+962424\zeta(4)\zeta(5)+421092\zeta(2)\zeta(7)+48352\zeta(3)^3\big) \quad (1065)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^9}{(k+2)^2} = \frac{1}{320}\big(-3200-20160\zeta(2)-200960\zeta(3)-1307040\zeta(4)-4662400\zeta(5)$$
$$-1005120\zeta(2)\zeta(3)-21224080\zeta(6)-2841600\zeta(3)^2-33558720\zeta(7)$$
$$-7513920\zeta(2)\zeta(5)-16977600\zeta(3)\zeta(4)-83974040\zeta(8)-3958080\zeta(2)\zeta(3)^2$$
$$-9227520\zeta(3)\zeta(5)+1763520M(2,6)+3246560\zeta(9)-7064400\zeta(3)\zeta(6)$$
$$+3355200\zeta(4)\zeta(5)+1044720\zeta(2)\zeta(7)-321920\zeta(3)^3+111550548\zeta(10)$$
$$+39774240\zeta(3)\zeta(7)-3102480\zeta(3)^2\zeta(4)+8805600\zeta(2)\zeta(3)\zeta(5)$$
$$+25187760\zeta(5)^2+2866320M(2,8)+1797120\zeta(2)M(2,6)+38380815\zeta(11)$$
$$+10254960\zeta(2)\zeta(9)+26789600\zeta(3)\zeta(8)+43882080\zeta(4)\zeta(7)+52447480\zeta(5)\zeta(6)$$
$$-1464640\zeta(2)\zeta(3)^3+10292160\zeta(3)^2\zeta(5)+2690880\zeta(3)M(2,6)+996480M(3,8)\big)$$
$$(1066)$$

Formulas for order $r = m + n = 12$:

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^{11}} = \frac{1}{4} (13\zeta(12) - 4\zeta(3)\zeta(9) - 4\zeta(5)\zeta(7)) \quad (1067)$$

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{H(k)}{k^{10}(k+1)} = \frac{1}{4} \big(&-4\zeta(2) + 8\zeta(3) - 5\zeta(4) + 12\zeta(5) - 4\zeta(2)\zeta(3) - 7\zeta(6) + 2\zeta(3)^2 \\ &+16\zeta(7) - 4\zeta(2)\zeta(5) - 4\zeta(3)\zeta(4) - 9\zeta(8) + 4\zeta(3)\zeta(5) + 20\zeta(9) \\ &-4\zeta(3)\zeta(6) - 4\zeta(4)\zeta(5) - 4\zeta(2)\zeta(7) - 11\zeta(10) + 4\zeta(3)\zeta(7) + 2\zeta(5)^2 \\ &+24\zeta(11) - 4\zeta(2)\zeta(9) - 4\zeta(3)\zeta(8) - 4\zeta(4)\zeta(7) - 4\zeta(5)\zeta(6)) \end{aligned} \quad (1068)$$

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{H(k)}{k^{9}(k+1)^2} = \frac{1}{4} (&36\zeta(2) - 68\zeta(3) + 35\zeta(4) - 72\zeta(5) + 24\zeta(2)\zeta(3) + 35\zeta(6) \\ &-10\zeta(3)^2 - 64\zeta(7) + 16\zeta(2)\zeta(5) + 16\zeta(3)\zeta(4) + 27\zeta(8) - 12\zeta(3)\zeta(5) \\ &-40\zeta(9) + 8\zeta(3)\zeta(6) + 8\zeta(4)\zeta(5) + 8\zeta(2)\zeta(7) + 11\zeta(10) - 4\zeta(3)\zeta(7) \\ &-2\zeta(5)^2\big) \end{aligned} \quad (1069)$$

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{H(k)}{k^{8}(k+1)^3} = \frac{1}{4} (&144\zeta(2) - 256\zeta(3) + 104\zeta(4) - 180\zeta(5) + 60\zeta(2)\zeta(3) + 70\zeta(6) \\ &-20\zeta(3)^2 - 96\zeta(7) + 24\zeta(2)\zeta(5) + 24\zeta(3)\zeta(4) + 27\zeta(8) - 12\zeta(3)\zeta(5) \\ &-20\zeta(9) + 4\zeta(3)\zeta(6) + 4\zeta(4)\zeta(5) + 4\zeta(2)\zeta(7)) \end{aligned} \quad (1070)$$

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{H(k)}{k^{7}(k+1)^4} = \frac{1}{4} (&336\zeta(2) - 560\zeta(3) + 168\zeta(4) - 248\zeta(5) + 84\zeta(2)\zeta(3) + 70\zeta(6) \\ &-20\zeta(3)^2 - 64\zeta(7) + 16\zeta(2)\zeta(5) + 16\zeta(3)\zeta(4) + 9\zeta(8) - 4\zeta(3)\zeta(5)\big) \end{aligned} \quad (1071)$$

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{H(k)}{k^{6}(k+1)^5} = \frac{1}{2} (&252\zeta(2) - 392\zeta(3) + 77\zeta(4) - 114\zeta(5) + 42\zeta(2)\zeta(3) + 16\zeta(6) \\ &-4\zeta(3)^2 - 8\zeta(7) + 2\zeta(2)\zeta(5) + 2\zeta(3)\zeta(4)\big) \end{aligned} \quad (1072)$$

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{H(k)}{k^{5}(k+1)^6} = \frac{1}{2} (&252\zeta(2) - 364\zeta(3) + 35\zeta(4) - 96\zeta(5) + 42\zeta(2)\zeta(3) - 4\zeta(6) \\ &+4\zeta(3)^2 - 6\zeta(7) + 2\zeta(2)\zeta(5) + 2\zeta(3)\zeta(4)\big) \end{aligned} \quad (1073)$$

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{H(k)}{k^{4}(k+1)^7} = \frac{1}{4} \big(&-336\zeta(2) + 448\zeta(3) + 172\zeta(5) - 84\zeta(2)\zeta(3) + 30\zeta(6) - 20\zeta(3)^2 \\ &+48\zeta(7) - 16\zeta(2)\zeta(5) - 16\zeta(3)\zeta(4) + 5\zeta(8) - 4\zeta(3)\zeta(5)) \end{aligned} \quad (1074)$$

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{H(k)}{k^{3}(k+1)^8} = \frac{1}{4} (&144\zeta(2) - 176\zeta(3) - 16\zeta(4) - 120\zeta(5) + 60\zeta(2)\zeta(3) - 30\zeta(6) \\ &+20\zeta(3)^2 - 72\zeta(7) + 24\zeta(2)\zeta(5) + 24\zeta(3)\zeta(4) - 15\zeta(8) + 12\zeta(3)\zeta(5) \\ &-16\zeta(9) + 4\zeta(3)\zeta(6) + 4\zeta(4)\zeta(5) + 4\zeta(2)\zeta(7)) \end{aligned} \quad (1075)$$

$$\sum_{k=1}^{\infty}\frac{H(k)}{k^2(k+1)^9} = \frac{1}{4}\left(-36\zeta(2)+40\zeta(3)+7\zeta(4)+48\zeta(5)-24\zeta(2)\zeta(3)+15\zeta(6)\right.$$
$$-10\zeta(3)^2+48\zeta(7)-16\zeta(2)\zeta(5)-16\zeta(3)\zeta(4)+15\zeta(8)-12\zeta(3)\zeta(5)$$
$$+32\zeta(9)-8\zeta(3)\zeta(6)-8\zeta(4)\zeta(5)-8\zeta(2)\zeta(7)+7\zeta(10)-4\zeta(3)\zeta(7)$$
$$\left.-2\zeta(5)^2\right) \tag{1076}$$

$$\sum_{k=1}^{\infty}\frac{H(k)}{k(k+1)^{10}} = \frac{1}{4}\left(4\zeta(2)-4\zeta(3)-\zeta(4)-8\zeta(5)+4\zeta(2)\zeta(3)-3\zeta(6)+2\zeta(3)^2\right.$$
$$-12\zeta(7)+4\zeta(2)\zeta(5)+4\zeta(3)\zeta(4)-5\zeta(8)+4\zeta(3)\zeta(5)-16\zeta(9)$$
$$+4\zeta(3)\zeta(6)+4\zeta(4)\zeta(5)+4\zeta(2)\zeta(7)-7\zeta(10)+4\zeta(3)\zeta(7)+2\zeta(5)^2$$
$$\left.-20\zeta(11)+4\zeta(2)\zeta(9)+4\zeta(3)\zeta(8)+4\zeta(4)\zeta(7)+4\zeta(5)\zeta(6)\right) \tag{1077}$$

$$\sum_{k=1}^{\infty}\frac{H(k)}{(k+1)^{11}} = \frac{1}{4}\left(9\zeta(12)-4\zeta(3)\zeta(9)-4\zeta(5)\zeta(7)\right) \tag{1078}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^2}{k^{10}} = M(2,10) \tag{1079}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^2}{k^9(k+1)} = \frac{1}{24}\left(72\zeta(3)-102\zeta(4)+84\zeta(5)-24\zeta(2)\zeta(3)-97\zeta(6)+48\zeta(3)^2\right.$$
$$+144\zeta(7)-24\zeta(2)\zeta(5)-60\zeta(3)\zeta(4)-24M(2,6)+220\zeta(9)-84\zeta(3)\zeta(6)$$
$$-60\zeta(4)\zeta(5)-24\zeta(2)\zeta(7)+8\zeta(3)^3-24M(2,8)+312\zeta(11)-24\zeta(2)\zeta(9)$$
$$\left.-108\zeta(3)\zeta(8)-60\zeta(4)\zeta(7)-84\zeta(5)\zeta(6)+24\zeta(3)^2\zeta(5)\right) \tag{1080}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^2}{k^8(k+1)^2} = \frac{1}{24}\left(576\zeta(3)-780\zeta(4)+504\zeta(5)-144\zeta(2)\zeta(3)-485\zeta(6)\right.$$
$$+240\zeta(3)^2+576\zeta(7)-96\zeta(2)\zeta(5)-240\zeta(3)\zeta(4)-72M(2,6)+440\zeta(9)$$
$$\left.-168\zeta(3)\zeta(6)-120\zeta(4)\zeta(5)-48\zeta(2)\zeta(7)+16\zeta(3)^3-24M(2,8)\right) \tag{1081}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^2}{k^7(k+1)^3} = \frac{1}{12}\left(1008\zeta(3)-1302\zeta(4)+648\zeta(5)-192\zeta(2)\zeta(3)-485\zeta(6)\right.$$
$$+240\zeta(3)^2+432\zeta(7)-72\zeta(2)\zeta(5)-180\zeta(3)\zeta(4)-36M(2,6)+110\zeta(9)$$
$$\left.-42\zeta(3)\zeta(6)-30\zeta(4)\zeta(5)-12\zeta(2)\zeta(7)+4\zeta(3)^3\right) \tag{1082}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^2}{k^6(k+1)^4} = \frac{1}{24}\left(4032\zeta(3)-4956\zeta(4)+1896\zeta(5)-624\zeta(2)\zeta(3)-1007\zeta(6)\right.$$
$$\left.+504\zeta(3)^2+576\zeta(7)-96\zeta(2)\zeta(5)-240\zeta(3)\zeta(4)-24M(2,6)\right) \tag{1083}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^2}{k^5(k+1)^5} = \frac{1}{12}\left(2520\zeta(3)-2940\zeta(4)+900\zeta(5)-360\zeta(2)\zeta(3)-335\zeta(6)\right.$$
$$\left.+180\zeta(3)^2+84\zeta(7)-24\zeta(2)\zeta(5)-24\zeta(3)\zeta(4)\right) \tag{1084}$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k^4(k+1)^6} = \frac{1}{24}\left(4032\zeta(3) - 4452\zeta(4) + 1224\zeta(5) - 624\zeta(2)\zeta(3) - 467\zeta(6) \right.$$
$$+288\zeta(3)^2 + 96\zeta(7) - 96\zeta(2)\zeta(5) + 48\zeta(3)\zeta(4) + 84\zeta(8) - 48\zeta(3)\zeta(5)$$
$$\left. -24M(2,6)\right) \quad (1085)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k^3(k+1)^7} = \frac{1}{12}\left(1008\zeta(3) - 1050\zeta(4) + 312\zeta(5) - 192\zeta(2)\zeta(3) - 185\zeta(6) \right.$$
$$+120\zeta(3)^2 + 72\zeta(7) - 72\zeta(2)\zeta(5) + 36\zeta(3)\zeta(4) + 126\zeta(8) - 72\zeta(3)\zeta(5)$$
$$\left. -36M(2,6) - 2\zeta(9) + 18\zeta(3)\zeta(6) + 6\zeta(4)\zeta(5) - 12\zeta(2)\zeta(7) - 4\zeta(3)^3\right) \quad (1086)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k^2(k+1)^8} = \frac{1}{24}\left(576\zeta(3) - 564\zeta(4) + 216\zeta(5) - 144\zeta(2)\zeta(3) - 185\zeta(6) \right.$$
$$+120\zeta(3)^2 + 96\zeta(7) - 96\zeta(2)\zeta(5) + 48\zeta(3)\zeta(4) + 252\zeta(8) - 144\zeta(3)\zeta(5)$$
$$-72M(2,6) - 8\zeta(9) + 72\zeta(3)\zeta(6) + 24\zeta(4)\zeta(5) - 48\zeta(2)\zeta(7) - 16\zeta(3)^3$$
$$\left. +108\zeta(10) - 48\zeta(3)\zeta(7) - 24\zeta(5)^2 - 24M(2,8)\right) \quad (1087)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k(k+1)^9} = \frac{1}{24}\left(72\zeta(3) - 66\zeta(4) + 36\zeta(5) - 24\zeta(2)\zeta(3) - 37\zeta(6) + 24\zeta(3)^2 \right.$$
$$+24\zeta(7) - 24\zeta(2)\zeta(5) + 12\zeta(3)\zeta(4) + 84\zeta(8) - 48\zeta(3)\zeta(5) - 24M(2,6)$$
$$-4\zeta(9) + 36\zeta(3)\zeta(6) + 12\zeta(4)\zeta(5) - 24\zeta(2)\zeta(7) - 8\zeta(3)^3 + 108\zeta(10)$$
$$-48\zeta(3)\zeta(7) - 24\zeta(5)^2 - 24M(2,8) - 48\zeta(11) - 24\zeta(2)\zeta(9) + 60\zeta(3)\zeta(8)$$
$$\left. +12\zeta(4)\zeta(7) + 36\zeta(5)\zeta(6) - 24\zeta(3)^2\zeta(5)\right) \quad (1088)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{(k+1)^{10}} = \frac{1}{2}\left(-11\zeta(12) + 4\zeta(3)\zeta(9) + 4\zeta(5)\zeta(7) + 2M(2,10)\right) \quad (1089)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^3}{k^9} = \frac{1}{22112}\left(-355355\zeta(12) + 221120\zeta(3)\zeta(9) + 265344\zeta(5)\zeta(7) \right.$$
$$+33168\zeta(3)^2\zeta(6) - 5528\zeta(3)^4 - 49752\zeta(2)\zeta(5)^2 - 99504\zeta(2)\zeta(3)\zeta(7)$$
$$\left. +82920M(2,10)\right) \quad (1090)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^3}{k^8(k+1)} = \frac{1}{480}\left(-4800\zeta(4) + 4800\zeta(5) + 480\zeta(2)\zeta(3) - 2790\zeta(6) + 1200\zeta(3)^2 \right.$$
$$+6930\zeta(7) + 960\zeta(2)\zeta(5) - 6120\zeta(3)\zeta(4) + 2975\zeta(8) + 600\zeta(2)\zeta(3)^2$$
$$-2880\zeta(3)\zeta(5) - 1320M(2,6) + 10420\zeta(9) - 5820\zeta(3)\zeta(6) - 6120\zeta(4)\zeta(5)$$
$$+1440\zeta(2)\zeta(7) + 960\zeta(3)^3 + 4983\zeta(10) - 3840\zeta(3)\zeta(7) - 240\zeta(3)^2\zeta(4)$$
$$\left. +1680\zeta(2)\zeta(3)\zeta(5) - 2160\zeta(5)^2 - 1560M(2,8) + 480M(3,8)\right) \quad (1091)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^3}{k^7(k+1)^2} = \frac{1}{480}\left(33600\zeta(4) - 32400\zeta(5) - 3360\zeta(2)\zeta(3) + 13950\zeta(6) \right.$$
$$-6000\zeta(3)^2 - 27720\zeta(7) - 3840\zeta(2)\zeta(5) + 24480\zeta(3)\zeta(4) - 8925\zeta(8)$$
$$-1800\zeta(2)\zeta(3)^2 + 8640\zeta(3)\zeta(5) + 3960M(2,6) - 20840\zeta(9) + 11640\zeta(3)\zeta(6)$$
$$+12240\zeta(4)\zeta(5) - 2880\zeta(2)\zeta(7) - 1920\zeta(3)^3 - 4983\zeta(10) + 3840\zeta(3)\zeta(7)$$
$$\left. +240\zeta(3)^2\zeta(4) - 1680\zeta(2)\zeta(3)\zeta(5) + 2160\zeta(5)^2 + 1560M(2,8)\right) \quad (1092)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^3}{k^6(k+1)^3} = \frac{1}{96}\big(20160\zeta(4) - 18720\zeta(5) - 2016\zeta(2)\zeta(3) + 5778\zeta(6) - 2592\zeta(3)^2$$
$$-8316\zeta(7) - 1152\zeta(2)\zeta(5) + 7344\zeta(3)\zeta(4) - 1785\zeta(8) - 360\zeta(2)\zeta(3)^2$$
$$+1728\zeta(3)\zeta(5) + 792M(2,6) - 2084\zeta(9) + 1164\zeta(3)\zeta(6) + 1224\zeta(4)\zeta(5)$$
$$-288\zeta(2)\zeta(7) - 192\zeta(3)^3\big) \tag{1093}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^3}{k^5(k+1)^4} = \frac{1}{96}\big(33600\zeta(4) - 30000\zeta(5) - 3360\zeta(2)\zeta(3) + 6570\zeta(6)$$
$$-3360\zeta(3)^2 - 6258\zeta(7) - 960\zeta(2)\zeta(5) + 5688\zeta(3)\zeta(4) - 595\zeta(8)$$
$$-120\zeta(2)\zeta(3)^2 + 576\zeta(3)\zeta(5) + 264M(2,6)\big) \tag{1094}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^3}{k^4(k+1)^5} = \frac{1}{96}\big(33600\zeta(4) - 28800\zeta(5) - 3360\zeta(2)\zeta(3) + 4770\zeta(6) - 3120\zeta(3)^2$$
$$-4242\zeta(7) - 960\zeta(2)\zeta(5) + 4392\zeta(3)\zeta(4) + 43\zeta(8) + 120\zeta(2)\zeta(3)^2$$
$$-288\zeta(3)\zeta(5) + 24M(2,6)\big) \tag{1095}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^3}{k^3(k+1)^6} = \frac{1}{96}\big(20160\zeta(4) - 16560\zeta(5) - 2016\zeta(2)\zeta(3) + 2538\zeta(6) - 2160\zeta(3)^2$$
$$-4284\zeta(7) - 1152\zeta(2)\zeta(5) + 4752\zeta(3)\zeta(4) + 129\zeta(8) + 360\zeta(2)\zeta(3)^2$$
$$-864\zeta(3)\zeta(5) + 72M(2,6) - 788\zeta(9) + 444\zeta(3)\zeta(6) + 792\zeta(4)\zeta(5)$$
$$-288\zeta(2)\zeta(7) - 96\zeta(3)^3\big) \tag{1096}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^3}{k^2(k+1)^7} = \frac{1}{480}\big(33600\zeta(4) - 26400\zeta(5) - 3360\zeta(2)\zeta(3) + 4950\zeta(6)$$
$$-4800\zeta(3)^2 - 14280\zeta(7) - 3840\zeta(2)\zeta(5) + 15840\zeta(3)\zeta(4) + 645\zeta(8)$$
$$+1800\zeta(2)\zeta(3)^2 - 4320\zeta(3)\zeta(5) + 360M(2,6) - 7880\zeta(9) + 4440\zeta(3)\zeta(6)$$
$$+7920\zeta(4)\zeta(5) - 2880\zeta(2)\zeta(7) - 960\zeta(3)^3 + 1503\zeta(10) - 2400\zeta(3)\zeta(7)$$
$$-240\zeta(3)^2\zeta(4) + 1680\zeta(2)\zeta(3)\zeta(5) - 1440\zeta(5)^2 - 120M(2,8)\big) \tag{1097}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^3}{k(k+1)^8} = \frac{1}{480}\big(4800\zeta(4) - 3600\zeta(5) - 480\zeta(2)\zeta(3) + 990\zeta(6) - 960\zeta(3)^2$$
$$-3570\zeta(7) - 960\zeta(2)\zeta(5) + 3960\zeta(3)\zeta(4) + 215\zeta(8) + 600\zeta(2)\zeta(3)^2$$
$$-1440\zeta(3)\zeta(5) + 120M(2,6) - 3940\zeta(9) + 2220\zeta(3)\zeta(6) + 3960\zeta(4)\zeta(5)$$
$$-1440\zeta(2)\zeta(7) - 480\zeta(3)^3 + 1503\zeta(10) - 2400\zeta(3)\zeta(7) - 240\zeta(3)^2\zeta(4)$$
$$+1680\zeta(2)\zeta(3)\zeta(5) - 1440\zeta(5)^2 - 120M(2,8) + 10560\zeta(11) - 5040\zeta(3)\zeta(8)$$
$$-2160\zeta(4)\zeta(7) - 3600\zeta(5)\zeta(6) + 1440\zeta(3)^2\zeta(5) - 480M(3,8)\big) \tag{1098}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^3}{(k+1)^9} = \frac{1}{22112}\big(161875\zeta(12) - 154784\zeta(3)\zeta(9) - 199008\zeta(5)\zeta(7)$$
$$-33168\zeta(3)^2\zeta(6) + 5528\zeta(3)^4 + 49752\zeta(2)\zeta(5)^2 + 99504\zeta(2)\zeta(3)\zeta(7)$$
$$-16584M(2,10)\big) \tag{1099}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^4}{k^8} = M(4,8) \tag{1100}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^4}{k^7(k+1)} = \frac{1}{5760}\big(172800\zeta(5)+34560\zeta(2)\zeta(3)-234960\zeta(6)-17280\zeta(3)^2$$
$$+133200\zeta(7)+28800\zeta(2)\zeta(5)-123840\zeta(3)\zeta(4)+593320\zeta(8)$$
$$+161280\zeta(2)\zeta(3)^2-668160\zeta(3)\zeta(5)-149760M(2,6)+209280\zeta(9)$$
$$-133920\zeta(3)\zeta(6)-123840\zeta(4)\zeta(5)+40320\zeta(2)\zeta(7)+19200\zeta(3)^3$$
$$+619407\zeta(10)-540000\zeta(3)\zeta(7)-9000\zeta(3)^2\zeta(4)+195120\zeta(2)\zeta(3)\zeta(5)$$
$$-212040\zeta(5)^2-109080M(2,8)-11520\zeta(2)M(2,6)-345240\zeta(11)-32640\zeta(2)\zeta(9)$$
$$+142800\zeta(3)\zeta(8)+145440\zeta(4)\zeta(7)+122160\zeta(5)\zeta(6)+9600\zeta(2)\zeta(3)^3$$
$$-69120\zeta(3)^2\zeta(5)+21120M(3,8)\big) \tag{1101}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^4}{k^6(k+1)^2} = \frac{1}{1920}\big(-345600\zeta(5)-69120\zeta(2)\zeta(3)+460320\zeta(6)+34560\zeta(3)^2$$
$$-177600\zeta(7)-38400\zeta(2)\zeta(5)+165120\zeta(3)\zeta(4)-593320\zeta(8)$$
$$-161280\zeta(2)\zeta(3)^2+668160\zeta(3)\zeta(5)+149760M(2,6)-139520\zeta(9)+89280\zeta(3)\zeta(6)$$
$$+82560\zeta(4)\zeta(5)-26880\zeta(2)\zeta(7)-12800\zeta(3)^3-206469\zeta(10)+180000\zeta(3)\zeta(7)$$
$$+3000\zeta(3)^2\zeta(4)-65040\zeta(2)\zeta(3)\zeta(5)+70680\zeta(5)^2+36360M(2,8)$$
$$+3840\zeta(2)M(2,6)) \tag{1102}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^4}{k^5(k+1)^3} = \frac{1}{48}\big(21600\zeta(5)+4320\zeta(2)\zeta(3)-28170\zeta(6)-2160\zeta(3)^2$$
$$+7314\zeta(7)+1680\zeta(2)\zeta(5)-7080\zeta(3)\zeta(4)+14833\zeta(8)+4032\zeta(2)\zeta(3)^2$$
$$-16704\zeta(3)\zeta(5)-3744M(2,6)+1744\zeta(9)-1116\zeta(3)\zeta(6)-1032\zeta(4)\zeta(5)$$
$$+336\zeta(2)\zeta(7)+160\zeta(3)^3\big) \tag{1103}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^4}{k^4(k+1)^4} = \frac{1}{18}\big(10800\zeta(5)+2160\zeta(2)\zeta(3)-13785\zeta(6)-1080\zeta(3)^2+2646\zeta(7)$$
$$+720\zeta(2)\zeta(5)-2880\zeta(3)\zeta(4)+3406\zeta(8)+918\zeta(2)\zeta(3)^2-3816\zeta(3)\zeta(5)$$
$$-846M(2,6)) \tag{1104}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^4}{k^3(k+1)^5} = \frac{1}{48}\big(21600\zeta(5)+4320\zeta(2)\zeta(3)-26970\zeta(6)-2160\zeta(3)^2+5034\zeta(7)$$
$$+1680\zeta(2)\zeta(5)-6360\zeta(3)\zeta(4)+12415\zeta(8)+3312\zeta(2)\zeta(3)^2$$
$$-13824\zeta(3)\zeta(5)-3024M(2,6)+696\zeta(9)-396\zeta(3)\zeta(6)-888\zeta(4)\zeta(5)$$
$$+336\zeta(2)\zeta(7)+128\zeta(3)^3\big) \tag{1105}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^4}{k^2(k+1)^6}=\frac{1}{1920}\big(345600\zeta(5)+69120\zeta(2)\zeta(3)-421920\zeta(6)-34560\zeta(3)^2$$
$$+104640\zeta(7)+38400\zeta(2)\zeta(5)-142080\zeta(3)\zeta(4)+496600\zeta(8)$$
$$+132480\zeta(2)\zeta(3)^2-552960\zeta(3)\zeta(5)-120960M(2,6)+55680\zeta(9)-31680\zeta(3)\zeta(6)$$
$$-71040\zeta(4)\zeta(5)+26880\zeta(2)\zeta(7)+10240\zeta(3)^3+145941\zeta(10)-126240\zeta(3)\zeta(7)$$
$$+840\zeta(3)^2\zeta(4)+38160\zeta(2)\zeta(3)\zeta(5)-39960\zeta(5)^2-22920M(2,8)$$
$$-3840\zeta(2)M(2,6)\big) \tag{1106}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^4}{k(k+1)^7}=\frac{1}{5760}\big(172800\zeta(5)+34560\zeta(2)\zeta(3)-206160\zeta(6)-17280\zeta(3)^2$$
$$+78480\zeta(7)+28800\zeta(2)\zeta(5)-106560\zeta(3)\zeta(4)+496600\zeta(8)+132480\zeta(2)\zeta(3)^2$$
$$-552960\zeta(3)\zeta(5)-120960M(2,6)+83520\zeta(9)-47520\zeta(3)\zeta(6)-106560\zeta(4)\zeta(5)$$
$$+40320\zeta(2)\zeta(7)+15360\zeta(3)^3+437823\zeta(10)-378720\zeta(3)\zeta(7)$$
$$+2520\zeta(3)^2\zeta(4)+114480\zeta(2)\zeta(3)\zeta(5)-119880\zeta(5)^2-68760M(2,8)$$
$$-11520\zeta(2)M(2,6)+28440\zeta(11)+44160\zeta(2)\zeta(9)-10320\zeta(3)\zeta(8)$$
$$-82080\zeta(4)\zeta(7)-24240\zeta(5)\zeta(6)-9600\zeta(2)\zeta(3)^3+34560\zeta(3)^2\zeta(5)$$
$$+1920M(3,8)\big) \tag{1107}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^4}{(k+1)^8}=\frac{1}{5528}(289019\zeta(12)-199008\zeta(3)\zeta(9)-243232\zeta(5)\zeta(7)$$
$$-33168\zeta(3)^2\zeta(6)+5528\zeta(3)^4+49752\zeta(2)\zeta(5)^2+99504\zeta(2)\zeta(3)\zeta(7)$$
$$-49752M(2,10)+5528M(4,8)) \tag{1108}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^5}{k^7}=\frac{1}{265344}(3612841\zeta(12)-884480\zeta(3)\zeta(9)-597024\zeta(5)\zeta(7)$$
$$+364848\zeta(3)^2\zeta(6)+221120\zeta(3)^4+364848\zeta(2)\zeta(5)^2+729696\zeta(2)\zeta(3)\zeta(7)$$
$$-3250464\zeta(3)\zeta(4)\zeta(5)-1028208M(2,10)+663360M(4,8)) \tag{1109}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^5}{k^6(k+1)}=\frac{1}{2304}\big(-411264\zeta(6)-51840\zeta(3)^2+295344\zeta(7)+65664\zeta(2)\zeta(5)$$
$$+76032\zeta(3)\zeta(4)+542488\zeta(8)+152640\zeta(2)\zeta(3)^2-630144\zeta(3)\zeta(5)-135360M(2,6)$$
$$+302144\zeta(9)-469920\zeta(3)\zeta(6)+152064\zeta(4)\zeta(5)+76320\zeta(2)\zeta(7)-11520\zeta(3)^3$$
$$+579897\zeta(10)-519840\zeta(3)\zeta(7)+3240\zeta(3)^2\zeta(4)+185040\zeta(2)\zeta(3)\zeta(5)$$
$$-203832\zeta(5)^2-98280M(2,8)-11520\zeta(2)M(2,6)-3126684\zeta(11)-352064\zeta(2)\zeta(9)$$
$$+1186640\zeta(3)\zeta(8)+1647936\zeta(4)\zeta(7)+880320\zeta(5)\zeta(6)+84480\zeta(2)\zeta(3)^3$$
$$-564480\zeta(3)^2\zeta(5)+34560\zeta(3)M(2,6)+111360M(3,8)\big) \tag{1110}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^5}{k^5(k+1)^2}=\frac{1}{2304}\big(2056320\zeta(6)+259200\zeta(3)^2-1448496\zeta(7)-328320\zeta(2)\zeta(5)$$
$$-380160\zeta(3)\zeta(4)-1627464\zeta(8)-457920\zeta(2)\zeta(3)^2+1890432\zeta(3)\zeta(5)$$
$$+406080M(2,6)-604288\zeta(9)+939840\zeta(3)\zeta(6)-304128\zeta(4)\zeta(5)-152640\zeta(2)\zeta(7)$$
$$+23040\zeta(3)^3-579897\zeta(10)+519840\zeta(3)\zeta(7)-3240\zeta(3)^2\zeta(4)$$
$$-185040\zeta(2)\zeta(3)\zeta(5)+203832\zeta(5)^2+98280M(2,8)+11520\zeta(2)M(2,6)\big) \quad (1111)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^5}{k^4(k+1)^3}=\frac{1}{144}\big(257040\zeta(6)+32400\zeta(3)^2-177534\zeta(7)-41040\zeta(2)\zeta(5)$$
$$-47520\zeta(3)\zeta(4)-134527\zeta(8)-37440\zeta(2)\zeta(3)^2+154368\zeta(3)\zeta(5)+33120M(2,6)$$
$$-18884\zeta(9)+29370\zeta(3)\zeta(6)-9504\zeta(4)\zeta(5)-4770\zeta(2)\zeta(7)$$
$$+720\zeta(3)^3\big) \quad (1112)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^5}{k^3(k+1)^4}=\frac{1}{144}\big(257040\zeta(6)+32400\zeta(3)^2-174006\zeta(7)-41040\zeta(2)\zeta(5)$$
$$-47520\zeta(3)\zeta(4)-132337\zeta(8)-36000\zeta(2)\zeta(3)^2+148032\zeta(3)\zeta(5)+31680M(2,6)$$
$$-14240\zeta(9)+25770\zeta(3)\zeta(6)-9504\zeta(4)\zeta(5)-4770\zeta(2)\zeta(7)$$
$$+720\zeta(3)^3\big) \quad (1113)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^5}{k^2(k+1)^5}=\frac{1}{2304}\big(2056320\zeta(6)+259200\zeta(3)^2-1363824\zeta(7)-328320\zeta(2)\zeta(5)$$
$$-380160\zeta(3)\zeta(4)-1574904\zeta(8)-423360\zeta(2)\zeta(3)^2+1738368\zeta(3)\zeta(5)$$
$$+371520M(2,6)-455680\zeta(9)+824640\zeta(3)\zeta(6)-304128\zeta(4)\zeta(5)-152640\zeta(2)\zeta(7)$$
$$+23040\zeta(3)^3-449109\zeta(10)+387360\zeta(3)\zeta(7)+9720\zeta(3)^2\zeta(4)$$
$$-124560\zeta(2)\zeta(3)\zeta(5)+122328\zeta(5)^2+68040M(2,8)+11520\zeta(2)M(2,6)\big) \quad (1114)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^5}{k(k+1)^6}=\frac{1}{2304}\big(411264\zeta(6)+51840\zeta(3)^2-267120\zeta(7)-65664\zeta(2)\zeta(5)$$
$$-76032\zeta(3)\zeta(4)-524968\zeta(8)-141120\zeta(2)\zeta(3)^2+579456\zeta(3)\zeta(5)+123840M(2,6)$$
$$-227840\zeta(9)+412320\zeta(3)\zeta(6)-152064\zeta(4)\zeta(5)-76320\zeta(2)\zeta(7)+11520\zeta(3)^3$$
$$-449109\zeta(10)+387360\zeta(3)\zeta(7)+9720\zeta(3)^2\zeta(4)-124560\zeta(2)\zeta(3)\zeta(5)$$
$$+122328\zeta(5)^2+68040M(2,8)+11520\zeta(2)M(2,6)+2668908\zeta(11)+275264\zeta(2)\zeta(9)$$
$$-993200\zeta(3)\zeta(8)-1403136\zeta(4)\zeta(7)-705120\zeta(5)\zeta(6)-65280\zeta(2)\zeta(3)^3$$
$$+449280\zeta(3)^2\zeta(5)-34560\zeta(3)M(2,6)-92160M(3,8)\big) \quad (1115)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^6}{k^5(k+1)}=\frac{1}{384}(247296\zeta(7)+55680\zeta(2)\zeta(5)+114048\zeta(3)\zeta(4)-280464\zeta(8)$$
$$+15744\zeta(2)\zeta(3)^2-187008\zeta(3)\zeta(5)-21888M(2,6)+119584\zeta(9)-209952\zeta(3)\zeta(6)$$
$$+96768\zeta(4)\zeta(5)+31248\zeta(2)\zeta(7)-8704\zeta(3)^3+814101\zeta(10)-529680\zeta(3)\zeta(7)$$
$$+253944\zeta(3)^2\zeta(4)+1200\zeta(2)\zeta(3)\zeta(5)-365064\zeta(5)^2-103128M(2,8)$$
$$-45120\zeta(2)M(2,6)-1469286\zeta(11)-166944\zeta(2)\zeta(9)+542488\zeta(3)\zeta(8)$$
$$+790176\zeta(4)\zeta(7)+410848\zeta(5)\zeta(6)+38720\zeta(2)\zeta(3)^3-260352\zeta(3)^2\zeta(5)$$
$$+18240\zeta(3)M(2,6)+51200M(3,8)) \quad (1116)$$

$$\sum_{k=1}^{\infty}\frac{H(k)^6}{k^4(k+1)^2} = \frac{1}{384}\left(989184\zeta(7) + 222720\zeta(2)\zeta(5) + 456192\zeta(3)\zeta(4) - 1113824\zeta(8)\right.$$
$$+62016\zeta(2)\zeta(3)^2 - 742272\zeta(3)\zeta(5) - 86592M(2,6) + 239168\zeta(9) - 419904\zeta(3)\zeta(6)$$
$$+193536\zeta(4)\zeta(5) + 62496\zeta(2)\zeta(7) - 17408\zeta(3)^3 + 814101\zeta(10)$$
$$-529680\zeta(3)\zeta(7) + 253944\zeta(3)^2\zeta(4) + 1200\zeta(2)\zeta(3)\zeta(5) - 365064\zeta(5)^2$$
$$\left.-103128M(2,8) - 45120\zeta(2)M(2,6)\right) \tag{1117}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^6}{k^3(k+1)^3} = \frac{1}{4}\left(15456\zeta(7) + 3480\zeta(2)\zeta(5) + 7128\zeta(3)\zeta(4) - 17278\zeta(8)\right.$$
$$+954\zeta(2)\zeta(3)^2 - 11508\zeta(3)\zeta(5) - 1338M(2,6) + 2270\zeta(9) - 4284\zeta(3)\zeta(6)$$
$$\left.+1980\zeta(4)\zeta(5) + 651\zeta(2)\zeta(7) - 180\zeta(3)^3\right) \tag{1118}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^6}{k^2(k+1)^4} = \frac{1}{384}\left(989184\zeta(7) + 222720\zeta(2)\zeta(5) + 456192\zeta(3)\zeta(4) - 1097760\zeta(8)\right.$$
$$+60096\zeta(2)\zeta(3)^2 - 730752\zeta(3)\zeta(5) - 84672M(2,6) + 196672\zeta(9) - 402624\zeta(3)\zeta(6)$$
$$+186624\zeta(4)\zeta(5) + 62496\zeta(2)\zeta(7) - 17152\zeta(3)^3 + 779835\zeta(10)$$
$$-490704\zeta(3)\zeta(7) + 245544\zeta(3)^2\zeta(4) - 15600\zeta(2)\zeta(3)\zeta(5) - 339864\zeta(5)^2$$
$$\left.-94728M(2,8) - 45120\zeta(2)M(2,6)\right) \tag{1119}$$

$$\sum_{k=1}^{\infty}\frac{H(k)^6}{k(k+1)^5} = \frac{1}{384}\left(247296\zeta(7) + 55680\zeta(2)\zeta(5) + 114048\zeta(3)\zeta(4) - 272432\zeta(8)\right.$$
$$+14784\zeta(2)\zeta(3)^2 - 181248\zeta(3)\zeta(5) - 20928M(2,6) + 98336\zeta(9) - 201312\zeta(3)\zeta(6)$$
$$+93312\zeta(4)\zeta(5) + 31248\zeta(2)\zeta(7) - 8576\zeta(3)^3 + 779835\zeta(10) - 490704\zeta(3)\zeta(7)$$
$$+245544\zeta(3)^2\zeta(4) - 15600\zeta(2)\zeta(3)\zeta(5) - 339864\zeta(5)^2 - 94728M(2,8)$$
$$-45120\zeta(2)M(2,6) - 1373598\zeta(11) - 149024\zeta(2)\zeta(9) + 524968\zeta(3)\zeta(8)$$
$$+724416\zeta(4)\zeta(7) + 365168\zeta(5)\zeta(6) + 36160\zeta(2)\zeta(3)^3 - 240768\zeta(3)^2\zeta(5)$$
$$\left.+16320\zeta(3)M(2,6) + 46720M(3,8)\right) \tag{1120}$$