Theory of ROOF walks

Richard E. Crandall *

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Abstract: We define and analyze what we call "running-out-of-fuel" (ROOF) walks, whereby each random step uses progressively less fuel (variance), in such a way that the total fuel expenditure is finite. In spite of this fuel constraint, a ROOF walk might still meander arbitrarily far. Herein we analyze the probability-density functions f_n for the walker's position after n steps, establishing in this way connections with other fields of analysis. One interdisciplinary aspect is a probabilistic view on the "sinc surprises" of Baillie–Borwein–Borwein [5] that arise in their study of certain sums and integrals. Aspects of the asymptotic density function $f := f_\infty$ remain mysterious. Yet, for a certain canonical ROOF walk we are able to prove that the density $f(x)$ decays at least superexponentially, in the sense that for certain positive constants A, B, C and sufficiently large x,

$$0 \ < \ f(x) \ < \ Ae^{-Be^{Cx}}.$$

We also discuss a "primes" walk whose density function f is even more exotic—in fact triply superexponentially decaying—with the relevant analysis combining statistics and number theory.

*Center for Advanced Computation, Reed College, Portland OR, `crandall@reed.edu`

1 Running out of fuel

We define a running-out-of-fuel (ROOF) walk as follows.[1] Let a coordinate after n random jumps be denoted

$$x_n = \Delta_1 + \Delta_2 + \cdots + \Delta_n,$$

where the random jumps Δ_j are pairwise statistically independent, with expectations satisfying

$$\langle \Delta_j \rangle = 0, \quad (1)$$

$$\langle |\Delta_1| \rangle > \langle |\Delta_2| \rangle > \langle |\Delta_3| \rangle > \ldots, \quad (2)$$

$$\sum_j \langle \Delta_j^2 \rangle < \infty. \quad (3)$$

The idea is, we speak of a finite amount of "fuel," being the last sum (3) above, which is, at least heuristically, $\langle x_\infty^2 \rangle$. This fuel store can—as we shall see—be identified in certain physics models as a random walker's total expended energy. The inequality chain (2) tells us that the random jumps (to left or right) have essentially decreasing magnitude. This is all reasonable, since we desire a scenario where the random walker continually expends fuel, but can execute infinitely many jumps in doing so.

It is well known—in fact since the days of Rademacher—that x_n approaches a definite limit, with probability one, merely on the assumption of finite total variance (3) above. Throughout the present work, when we talk of this "almost sure" coordinate $x := x_\infty$ and its density function, we shall assume certain existence proofs; for a more thorough discussion of relevant existence/convergence theorems and their converses, see the delightful monographs [15] [17].

In what follows, we denote by $p_j(\Delta)$ the density function of the j-th jump, and by $f_n(x)$ the density function of x_n. For convenience, we shall identify $f(x) := f_\infty(x)$, which ultimate density function does exist for any of our example ROOF walks. It is also the case that, for all of our ROOF examples below, at least, $f_n(x)$ is, for every $n = 1, 2, 3, \ldots$, monotone decreasing away from $x = 0$, as follows from the fact of monotonicity being preserved under convolution (see [17]).

2 Classes of ROOF walks

Herein we cite some examples of ROOF walks. A relatively trivial, classical ROOF walk has jump density function

$$p_n(\Delta) = \frac{1}{2}\left(\delta(\Delta - 1/2^n) + \delta(\Delta + 1/2^n)\right),$$

[1] The ROOF moniker is the present author's, intended for physicists. Previous names in the literature are "fatigued random walker" [15] and named special cases such as "random harmonic series" [17]. The physical interpretations in the present treatment arose more than a decade ago on the author's idea of a particle model with finite-absorption/random-emission, and subsequent discussions of same with O. Bonfim and S. Wolfram.

where the Dirac delta-function notation means that the n-th coordinate is

$$x_n = \pm\frac{1}{2} \pm \frac{1}{2^2} \pm \frac{1}{2^3} \pm \cdots \pm \frac{1}{2^n},$$

with independently random sign choices. It is well known [15] [17] that the ultimate density for this "binary" ROOF walk is

$$f(x) := f_\infty(x) = \frac{1}{2}\mathbf{1}_{[-1,1]}(x),$$

i.e., an equidistribution pedestal on $x \in [-1, 1]$. This result is intuitively clear, on the realization that balanced -binary representation of reals is unique.

The above example—in giving a pedestal for ultimate density—has all bounded walks, for any choice of signs. Because of wide connections with other mathematical domains, we next define:

Class-D ROOF walks: These are "diverging" ROOF walks in the following sense: Along with the defining relations (1)-(3), we also posit

$$\sum \langle|\Delta_j|\rangle \ = \ \infty.$$

Note that this divergence condition is not satisfied by the "binary" walk above (where the relevant sum over $\langle|\Delta_j|\rangle$ is simply 1). But for a ROOF walk of class D, we see that *the walker can go arbitrarily far away from the origin.* It is generally true, in fact, that a class-D walk has an ultimate density $f := f_\infty$ that is everywhere positive.[2] We next turn to specific subclasses of class-D walks.

$\mathbf{D}_0$ walks: A relatively trivial—albeit interesting—subclass of class-D ROOF walks involves the n-th-step density function

$$p_n(\Delta) = \frac{1}{\sqrt{2\pi v_n}} e^{-\Delta^2/(2v_n)},$$

i.e., ever-thinner, centered Gaussian jumps with real, positive variances $v_1 > v_2 > \ldots$, satisfying $\sum v_j < \infty$. In this instance we reside in class D if

$$\sum \sqrt{v_i} \ = \ \infty.$$

We know that any two Gaussian densities convolve into a Gaussian density of summed variance, so the *final* density function for the ultimate coordinate x_∞ involves the assumed-finite sum $\sum v_n$, as

$$f_\infty(x) = \frac{1}{\sqrt{2\pi \sum v_n}} e^{-x^2/(2\sum v_n)}.$$

[2] We leave to others the issue of proving (or disproving) positivity of $f := f_\infty$ for *every* class-D walk; yet, proofs for our main examples are found in [17].

It is interesting where this ROOF scenario is basically the only kind for which the classical central-limit theorem can be (conceptually, accidentally if you will) applied: Indeed, for none of our other examples does the final density function have Gaussian character. So walks in subclass D_0 are completely solvable in an obvious sense—and yet, we pay a price for said solvability. Namely, the concept of "meandering arbitrarily far" as one runs out of fuel is trivial since on any one step the walker can go arbitrarily far under its instantaneous Gaussian.

D_1 walks: This subclass we define to involve discrete jumps, in that the density function for jump Δ_j is two superimposed Dirac-delta spikes:

$$p_j(\Delta) = \frac{1}{2}\left(\delta(\Delta - b_j) + \delta(\Delta + b_j)\right),$$

where the real, positive b_j satisfy $b_1 > b_2 > \dots$, with $\sum b_j^2 < \infty$, and class-D membership is then assured by

$$\sum b_j = \infty.$$

Note that expectations for class D_1 are given simply by

$$\langle|\Delta_j|\rangle = b_j, \quad \langle\Delta_j^2\rangle = b_j^2.$$

Now we can give the notion of "arbitrary meandering while running out of fuel" some teeth: The density function $f(x) := f_\infty(x)$ is defined and everywhere positive (see, as before, [17]). We should admit that the phenomena attendant on sublass D_1 remain to some extent shrouded in mystery. For example, we *do not* know the exact density f for any explicit set of b_j, and we do not even know an exact $f(0)$, say, for any D_1 walk.

D_2 walks: This subclass we define to involve ever-shrinking jumps again, but with each density function being a pedestal:

$$p_j(\Delta) = \frac{1}{2c_j}\mathbf{1}_{[-c_j, c_j]}(\Delta),$$

where the real, positive c_j satisfy $c_1 > c_2 > \dots$, with $\sum c_j^2 < \infty$, and class-D membership is then assured by

$$\sum c_j = \infty.$$

The expectations for class D_2 are:

$$\langle|\Delta_j|\rangle = \frac{1}{2}c_j, \quad \langle\Delta_j^2\rangle = \frac{1}{3}c_j^2.$$

One of the many delightful surprises that ROOF walks offer is this: It can happen that a class-D_2 walk *has the very same final density function* $f := f_\infty$ as a corresponding walk from class-D_1. This remarkable fact—known to other researchers in one guise or another—will be shown from the probabilistic perspective in what follows.

3 Physics models for ROOF walkers

The present author proposes the following physics model for any ROOF walker of class D_1. We refer to the Figure 1 pictorial to aid our intuition. Ponder a *Gedanken*—"thought experiment"—involving a sled on a frictive track. The law of motion we contemplate is actually reasonable, and used in many physics applications, namely

$$m\frac{d^2x}{dt^2} = -\gamma\frac{dx}{dt}.$$

Here m is sled mass and γ is the coefficient of sliding friction. There is on-board fuel, yet to simplify the analysis we shall neglect the initial fuel mass compared to m and simply assume m is constant for all time.

The equation of motion is not enough, of course; we also need imagine a "demon" on the sled, and the demon has an algorithm for precisely how to expend on-board fuel. To this end, consider first the space-time trajectory under initial condition $x(0) = 0$, $dx/dt(0) = v$. The exact solution to the frictive motion is

$$x(t) = \frac{mv}{\gamma}\left(1 - e^{-\gamma t/m}\right).$$

Note that this motion, once begun with initial velocity v—say by a quick burst of fuel-burn—will never stop as $t \to \infty$; rather, the sled will asymptotically approach a jump distance of mv/γ. So let us assume a fixed, finite jump-time τ, so that in time τ the sled moves $x(\tau)$, at which time τ we use more fuel-burn to *brake* the sled from its velocity $dx/dt(\tau) = ve^{-\gamma\tau/m}$ to a standstill. Thus, the braking action works opposite to the sign of initial velocity v.

Now, to execute a ROOF walk, the control demon knows an infinite set of initial speeds $(v_1, v_2, \dots)$ and randomly selects direction $\pm$ for the initial velociti $v = v_n$ on the n-th jump. Thus, the jump distances that occur every time interval τ are

$$\Delta_n = \pm\frac{mv_n}{\gamma}\left(1 - e^{-\gamma\tau/m}\right).$$

Moreover, each jump operation involves energy outlay

$$E_n = \frac{1}{2}mv_n^2 + \frac{1}{2}mv_n^2\, e^{-2\gamma\tau/m},$$

with the two components here being, respectively, the initial thrust energy to achieve speed $|v_n|$ and the braking energy to remove the speed after time τ.[3]

[3]It could be argued that this energy budget neither takes into account thrusting into the frictive medium, or braking into said medium. However, if the thrusting and braking is done with infinitesimally brief energy pulses (and so, virtually infinite power pulses yet with the finite energy), then the energy needed is effectively that of giving/taking the kinetic energy of the sled.

Now for generality within the class D_1: For the defining sets of incremental distances $b_1, b_2, \ldots$ for a class-D_1 walk, choose physical constants and units such that

$$m = \gamma := 1$$

and the time quantum

$$\tau := \log\left(2 + \sqrt{3}\right),$$

in which case the control demon can assign velocities

$$v_n := \frac{1+\sqrt{3}}{2}\, b_n$$

and so expend successive fuel energies

$$E_n := b_n^2$$

to effect the ROOF walk. This model—which, by thus assigning physical units appropriately—covers all ROOF walks of class D_1, and this is what Figure 1 shows. It is an interesting scaling property of class D_1 that the time quantum $\log(2+\sqrt{3})$ can be fixed, across the entire class.

We presume that class-D_0 ROOF walks can be modeled similarly; e.g., for class D_0 one might imagine the demon has Maxwell-distributed gas on-board, and somehow allows Gaussian-distributed velocities for emanating gas particles.[4]

There are other interesting *Gedankens* that generate ROOF walks. For the "convex" friction law

$$m\frac{d^2x}{dt^2} = -\gamma \frac{\frac{dx}{dt}}{\sqrt{\left|\frac{dx}{dt}\right|}}$$

one can show that a sled starting with an initial velocity v_n actually comes to a halt in *finite* time. Thus, the sled demon can employ initial velocity bursts only, needing no braking mechanism. Taking $\gamma = m = 1$, an exact trajectory, given initial speed $v_n > 0$, is

$$x(t) = \frac{2}{3}v_n^{3/2} - \left(\left(\frac{2}{3}\right)^{1/3} v_n^{1/2} - \frac{1}{3}\left(\frac{3}{2}\right)^{2/3} t\right)^3,$$

from which we may casually read off the total distance-to-halt, assuming random choice of direction (left-right), as

$$\Delta_n = \pm\frac{2}{3}v_n^{3/2}.$$

If we choose velocities $v_n := (3/(2n))^{2/3}$, we have divergence of $\sum\langle|\Delta_j|\rangle = 1+1/2+1/3+\ldots$. So this frictive walk is in class D_1: The variance is $\sum\langle\Delta_j^2\rangle = \pi^2/6$, yet the sled can go "anywhere"

[4]Of course, such a scenario requires gas molecules of zero mass—or some other appropriate "law of the infinitesimal"—to avoid just a single emanating molecule dissipating by itself all the finite energy!

on finite fuel. It is of interest that for this convex-friction model, the total energy burn is not the same as the total (finite) variance; rather

$$E = \sum_n \frac{1}{2} m v_n^2 = \frac{3^{4/3}}{2^{7/3}} \zeta(4/3),$$

which is itself finite.

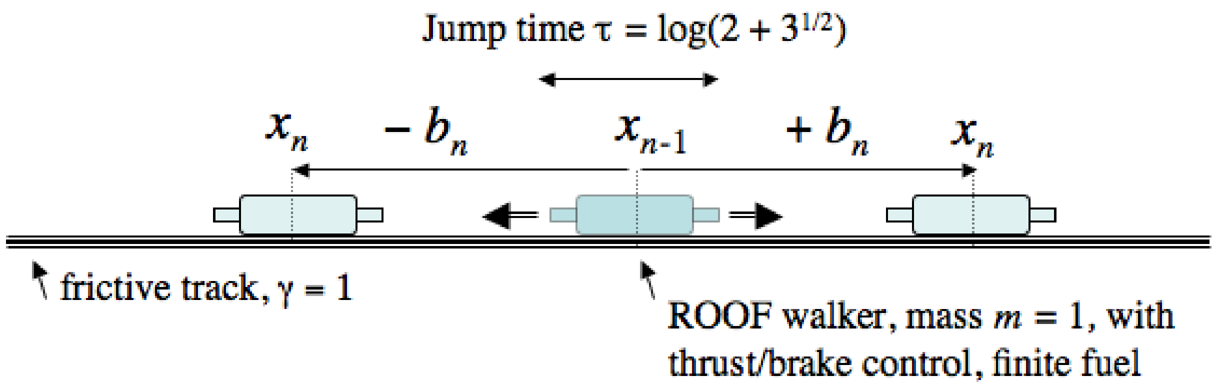

Figure 1: A physics *Gedanken* for class-D_1 ROOF walks. On the n-th jump, a fuel (i.e. energy) parcel b_n^2 is used to move the sled a distance $\pm b_n$, under the friction law $m\, d^2x/dt^2 = -\gamma\, dx/dt$. Remarkably, by burning thus a *finite* total fuel $\sum b_n^2$, and in spite of the friction, the sled still reaches arbitrarily remote positions with nonzero probability, being as $\sum b_n$ diverges. For $b_n := \frac{1}{2n}$, the ultimate position density will be the everywhere-positive $f(x)$ in Figure 2 (see Theorem (1) for equivalence of D_1, D_2 classes).

4 Density-function calculus

From our original prescription for the n-th position,

$$x_n = \Delta_1 + \Delta_2 + \cdots + \Delta_n,$$

we may write the n-th density function, for any integer $m \in [2, n]$, as the convolution

$$f_n(x) = \int_{-\infty}^{\infty} f_{m-1}(x-y) g_{m,n}(y)\, dy, \tag{4}$$

where $g_{m,n}$ here is the density function for the random sum

$$y = \Delta_m + \Delta_{m+1} + \cdots + \Delta_n.$$

For example, for $m = n$ we have the more transparent relation

$$f_n(x) = \int_{-\infty}^{\infty} f_{n-1}(x-y)p_n(y)\ dy,$$

where $p_n(y) = g_{n,n}(y)$ is, as before, the transitional density for the single jump Δ_n. We shall soon be exploiting the freedom of choice in (4) for the intermediate index m. Note that all densities are normalized in this theory; i.e., for all positive integers n and $m \in [2, n]$

$$\int_{x\in R} p_n(x)\ dx = \int_{x\in R} f_n(x)\ dx = \int_{x\in R} g_{m,n}(x)\ dx = 1.$$

In a classical vein, a complementary view on density convolution involves characteristic functions; namely, for the given transition densities $p_n(\Delta)$ we consider the Fourier transform

$$\hat{p}_n(\omega) := \int_{-\infty}^{\infty} p_n(\Delta)e^{-i\omega\Delta}\ d\Delta.$$

Then the familiar classical result is that convolution in configuration space turns into mutliplication in spectral space, that is

$$f_n(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\prod_{j=1}^{n}\hat{p}_j(\omega)e^{i\omega x}\ d\omega. \tag{5}$$

Now we can cite specific integral forms for class-D_1 and class-D_2 density functions, namely:

Density function for class-D_1 walks (discrete-Dirac-delta jumps by $\pm b_n$):

$$\begin{aligned}\hat{p}_n(\omega) &= \cos(b_n\omega),\\ f_n(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x}\ d\omega\ \prod_{j=1}^{n}\cos(b_j\omega).\end{aligned} \tag{6}$$

Density function for class-D_2 walks (pedestal jumps of widths $2c_n$):

$$\begin{aligned}\hat{p}_n(\omega) &= \mathrm{sinc}(c_n\omega),\\ f_n(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x}\ d\omega\ \prod_{j=1}^{n}\mathrm{sinc}(c_j\omega).\end{aligned} \tag{7}$$

We should state right off one of the remarkable results found in previous literature. We shall say that two processes are *statistically equivalent* if their respective density functions ϕ, f satisfy a scaling law $\phi(x) = cf(cx)$ for some positive constant c.

Theorem 1 *The D_1 walk with $b_n := \frac{1}{2n}$ is statistically equivalent to the D_2 walk with $c_n := \frac{1}{2n-1}$, in the sense that both walks have the same asymptotic density function $f := f_\infty$.*

Proof: This follows immediately from (6) and (7), on the observation that for all real ω

$$\prod_{j=1}^{\infty} \cos\left(\frac{\omega}{2j}\right) = \prod_{j=1}^{\infty} \operatorname{sinc}\left(\frac{\omega}{2j-1}\right).$$

This is established in [6] [8]; moreover, [17] has a probabilistic argument for the statistical equivalence of the two walks, which argument exploits the fact that an odd-reciprocal sequence is naturally buried in the harmonic series. **QED**

But there are a great many other properties to be gleaned from Fourier representations, as we shall see.

5 Focus on class-D_2 ROOF walks

Referring to Figure 2, we envision the canonical class-D_2 walk, that is, the figure assumes $c_n := \frac{1}{2n-1}$; yet, we now proceed for general c_n for the class.

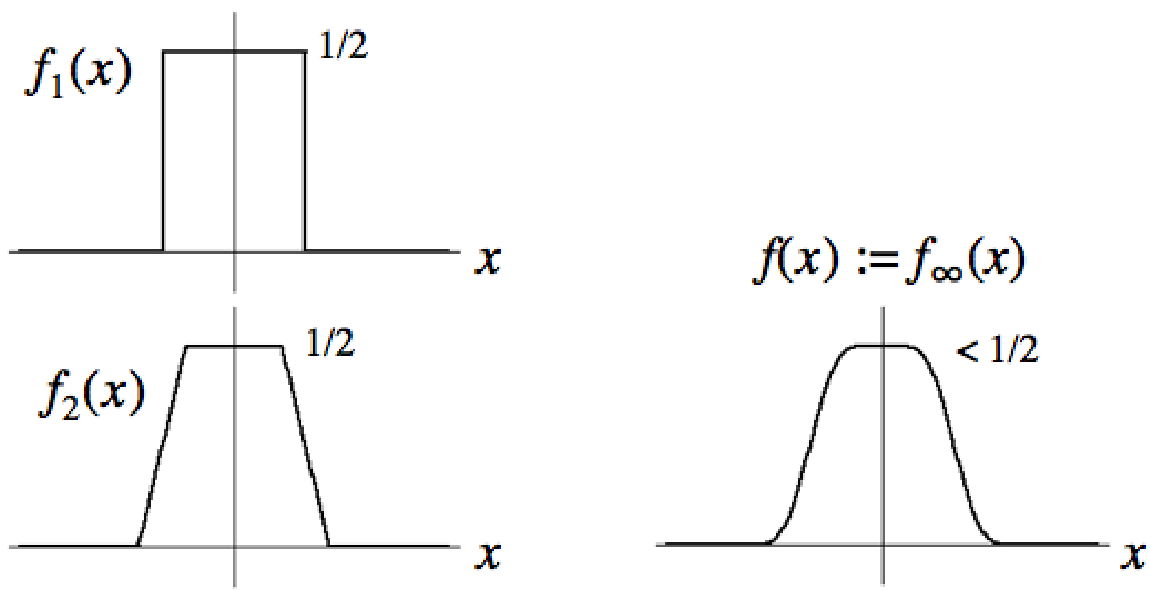

Figure 2: The class-D_2 ROOF walk with canonical assignments $c_n := 1/(2n-1)$. The upper-left plot is for the first random jump: A pedestal density over $[-1, 1]$ with height $1/2$. The lower-left plot is the density after 2 jumps: The top-height is still $1/2$, but straight lines connect $f_2(c_1 - c_2 = 2/3) = 1/2$, $f_2(c_1 + c_2 = 4/3) = 0$. At lower-right is a plot of $f := f_\infty$, which plot remarkably has $f(0) < 1/2$, yet barely so. It is also (barely) true that $f(1) < 1/4$, and that for large x, the decay of (the everywhere-positive) f is at least superexponential; for example, we prove $f(2\pi) < 10^{-13680}$.

For any c_n appropriate to class D_2, a useful representation also following from the convolution theory is

$$f_n(x) = \frac{1}{2^n} \int_{[-1,1]^n} \delta(c_1 z_1 + c_2 z_2 + \cdots + c_n z_n - x) \ \mathcal{D}\vec{z}. \tag{8}$$

Here, $\mathcal{D}\vec{z}$ is the volume element $dz_1 dz_2 \cdots dz_n$, and the integration is over the n-dimensional centered unit cube. This geometrically powerful representation can be derived easily from iteration of (4), or from the subsequent Fourier theory using $\delta(\omega) = 1/(2\pi) \int e^{i\omega z} dz$. We shall later use this delta-representation (8) in connection with the Baillie–Borwein–Borwein theory of sinc integrals. We note that relations such as (8) have been used previously in the study of multidimensional sinc integrals [9].

Net we aim to prove three salient bounds relevant to class-D_2 walks. The asymptotic (final) density function $f := f_\infty$ of the walk has properties:

- $f(0) < \frac{1}{2c_1}$,
- $f(c_1) < \frac{1}{4c_1}$,
- For the canonical case $c_n := 1/(2n-1)$, the decay of $f(x)$ is at least superexponential, meaning that for sufficiently large x, and positive constants A, B, C, we have

$$0 < f(x) < A e^{-B e^{Cx}}.$$

We note right off that various of our literature references have already established the two bounds for $f(0), f(c_1)$ respectively; our intent is to prove these via the present nomenclature and conceptual framework. The third, superexponential-decay result is new as far as the present author is aware.

To begin our bounding analysis , we employ the convolution relation (4) in the specific $(m = 2)$ form

$$f_n(x) = \int_{-\infty}^{\infty} f_1(x - y) g_{2,n}(y) \ dy. \tag{9}$$

In what follows, we denote probabilities by $\mathcal{P}$, and recall that for a class-D_2 walk, the first density function is $f_1(x) = p_1(x) = \frac{1}{2c_1} \mathbf{1}_{[-c_1 . c_1]}(x)$. Also, let us denote generally the tails $y_m = \Delta_m + \cdots + \Delta_n$; e.g., the $g_{2,n}$ function in (9) above is the density for random variable y_2.

Theorem 2 *For a class-D_2 ROOF walk, we have, for all positive integers n and also $n = \infty$,*

$$f_n(0) = \frac{1}{2c_1} \quad \text{if } c_2 + \ldots c_n \leq c_1,$$

$$< \frac{1}{2c_1} \quad \text{otherwise.}$$

Similarly,

$$f_n(c_1) = \frac{1}{4c_1} \qquad \text{if } c_2 + \ldots c_n \le 2c_1,$$
$$< \frac{1}{4c_1} \quad \text{otherwise.}$$

We also have exact sum rules

$$\sum_{k \in Z} f_n(2kc_1) = \sum_{k \in Z} f_n((2k+1)c_1) = \frac{1}{2c_1},$$

and so we have an overall sum rule

$$\sum_{k \in Z} f_n(k) = \frac{1}{c_1}.$$

Finally, for integer $m > 1$ and $x > c_1 + \ldots c_{m-1}$,

$$f_n(x) \;\le\; \frac{1}{2c_1}\mathcal{P}(x < y_m + c_1 + \ldots c_{m-1}.)$$

Remark: Note that the sum rules themselves imply the first two inequalities for $f(0), f(c_1)$—i.e., for $n = \infty$ ultimate densities. The sum rules also yield interesting theoretical and numerical side-connections (see Corollary 2 and Appendix).

Proof: We have immediately from (9) and the stated form for f_1 the relation

$$f_n(x) = \frac{1}{2c_1}\mathcal{P}(y_2 \in [x - c_1, x + c_1]) \tag{10}$$

From this we infer

$$f_n(0) = \frac{1}{2c_1}\mathcal{P}(y_2 \in [-c_1, c_1]) = \frac{1}{2c_1}(1 - 2\mathcal{P}(y_2 > c_1)), \tag{11}$$

and this settles both branches of the $f_n(0)$ claim of the theorem. Similarly,

$$f_n(c_1) = \frac{1}{2c_1}\mathcal{P}(y_2 \in [0, 2c_1]) = \frac{1}{4c_1}(1 - 2\mathcal{P}(y_2 > 2c_1)), \tag{12}$$

thus settling the claims for $f_n(c_1)$.

Now to prove the sum rules. Using (10) again, we have

$$f_n(0) + 2f_n(2c_1) + \cdots = \frac{1}{2c_1}\left(\mathcal{P}(|y_2| \in [0, c_1]) + \mathcal{P}(|y_2| \in [c_1, 3c_1]) + \mathcal{P}(|y_2| \in [3c_1, 5c_1]) + \ldots\right)$$
$$= \frac{1}{2c_1} \cdot 1.$$

A similar argument reveals that

$$f_n(c_1) + f_n(3c_1) + \cdots = \frac{1}{4c_1},$$

and this establishes the sum rules. (Not only do the sum rules hold for all positive integers n, they have to hold for the ultimate density f, i.e. for $n = \infty$, because it is known (see introductory section) the sequence $f_n \to f$ pointwise.)

Finally, going back to (4) for general index m and given that $x > c_1 + \ldots c_{m-1}$, we have

$$f_n(x) \le (\sup_z f_n(z))\ \mathcal{P}(y_m > x - c_1 - c_2 - \cdots - c_m).$$

However, the sup is bounded above by $1/(2c_1)$, as follows trivially from (10), and we obtain the theorem's claimed bound on $f_n(x)$.

QED

In order to render the general bound on $f_n(x)$ in Theorem 2 more useful, we next cite a result concerning expectations of exponential arguments. The following theorem is a generalization of the expectation theory found in [17]:

Theorem 3 *For a class-D_2 ROOF walk, take $n \ge m$, set $v := \sum_{k=m}^n c_k^2$, and denote by $g_{m,n}(y)$ the density for the random variable $y = \Delta_m + \cdots + \Delta_n$. Then for any real t, under the density $g_{m,n}$ we have a bounded expectation*

$$\langle e^{ty} \rangle \ \le\ e^{\frac{1}{6}vt^2}.$$

Moreover, for positive real u we have, under the stated density, a probability bound

$$\mathcal{P}(y > u)\ \le\ e^{-\frac{3u^2}{2v}}.$$

Proof:

$$\langle e^{ty} \rangle = \prod_{j\in[m,n]} \frac{1}{2c_j}\int_{-c_j}^{c_j} e^{ty}\ dy = \prod_{j\in[m,n]} \frac{\sinh(c_j t)}{c_j t} \le e^{\frac{1}{6}vt^2}.$$

This last bound follows immediately from

$$\frac{\sinh z}{z} = 1 + \frac{z^2}{3!} + \frac{z^4}{5!} + \cdots \le 1 + \frac{z^2/6}{1!} + \frac{z^4/6^2}{2!} + \cdots \le e^{z^2/6}.$$

Next we invoke a beautiful theorem of Markov (see [17]), saying that for $u > 0$,

$$\mathcal{P}(y > u) \le \inf_{t>0} \langle e^{ty} \rangle e^{-tu}.$$

This can readily be seen in our present context by writing

$$\mathcal{P}(y > u) = \int_u^\infty g_{m,n}(y)\ dy \le \int_{-\infty}^\infty g_{m,n}(y) e^{t(y-u)}\ dy.$$

But all of this means we can take $t := 3u/v$ in the inf argument and thereby derive the resulting bound for $\mathcal{P}(y > u)$.

QED

Theorem 4 *For the class-D_2 ROOF walk, denote for $m > 1$ the quantities $a_m := \sum_{k=1}^{m-1} c_k$ and $V_m := \sum_{k=m}^\infty c_k^2$. Then the density $f := f_\infty$ satisfies*

$$0 < \frac{1}{2c_1} - f(0) < \frac{1}{c_1} e^{-\frac{3c_1^2}{2V_2}},$$

$$0 < \frac{1}{4c_1} - f(c_1) < \frac{1}{2c_1} e^{-\frac{6c_1^2}{V_2}}.$$

Moreover, for $x > c_1$,

$$0 < f(x) < \frac{1}{2c_1} \inf_{a_m < x} e^{-\frac{3}{2}\frac{(x-a_m)^2}{V_m}}.$$

Proof: Recall that $f(x)$ is everywhere positive [17], so that the first two inequality chains, for $f(0), f(c_1)$ respectively, follow directly from relations (11), (12) and the final statement of Theorem 3. Then for $x > 1$, we can apply the last statements of Theorems 2 and 3 to get the desired inf inequality.

QED

Corollary 1 *For the canonical Class-D_2 ROOF walk (assignments $c_n := 1/(2n-1)$) the asymptotic density $f := f_\infty$ satisfies*

$$0 < \frac{1}{2} - f(0) < e^{-\frac{12}{\pi^2-8}} < 2 \cdot 10^{-3},$$

$$0 < \frac{1}{4} - f(1) < \frac{1}{2} e^{-\frac{48}{\pi^2-8}} < 10^{-11},$$

$$0 < f(x > 1) < \frac{1}{2} e^{-\frac{12(x-1)^2}{\pi^2-8}}.$$

Moreover, for large x the density decays at least superexponentially, in the sense that for $x \ge 3$

$$0 < f(x) < \frac{1}{2} e^{12} e^{-\frac{6}{e^4} e^{2x}}.$$

Remark: The bounds are nonoptimal; in truth, $1/2 - f(0) \approx 10^{-6}$ and $1/4 - f(1) \approx 10^{-43}$. The superexponential-decay bound says $f(3) < 10^{-14}$ while the numerical value is evidently $f(3) \approx 5 \cdot 10^{-43}$. A special case we discuss later is $f(2\pi) < 10^{-13680}$, and for this we do not know an accurate mantissa, even to one significant decimal. Interestingly: It could be that for *general* class-D_2 ROOF walks $f(x)$ decays at least doubly exponentially, i.e. perhaps we always have $f(x) < A\exp(-B\exp(Cx))$ within class D_2; we do not know if this is true, although this corollary for our canonical case is suggestive. (Later we mention a ROOF walk for which f decays *triply* superexponentially, and so still bounded above by a doubly superexponential decay.)

Proof: For this canonical ROOF walk, $V_2 = 1/3^2 + 1/5^2 + \cdots = \pi^2/8 - 1$ and so the first three inequality chains of the corollary result from Theorem 4. As for the last, superexponential inequality, observe that for $m \geq 2$,

$$a_m < 1 + \frac{1}{2}\log m,$$

$$V_m < \frac{1}{4(m-1)}.$$

These can be proved using elementary integral bounds on the function $1/(2z-1)$. Now for $x \geq 3$ set

$$m := \lfloor e^{2(x-2)} \rfloor,$$

so that $m \geq 2$, thus

$$f(x) < \frac{1}{2}e^{-(3/2)(x-1-(1/2)\cdot 2(x-2))^2 \cdot 4(m-1)} \leq \frac{1}{2}e^{-(3/2)\cdot(4(m-1))}$$

$$\leq \frac{1}{2}e^{12}e^{-6e^{2x-2}},$$

which is the desired superexponential bound.

QED

6 Numerical evaluation of density functions

For class D_2 ROOF walks, one might contemplate using the Fourier decomposition (7), performing numerical integration so as to approximate $f_n(x)$ numerically. And, with enough good fortune, perhaps some high-precision values of $f(x)$ for the asymptotic density function f can be obtained—via tight approximations for the infinite sinc-product. Actually there are two related approaches, but ones that differ interestingly in the details.

One approach is used in the work [3] [4] to evaluate integrals of the type that appear in our characteristic relation (6). Those authors were working in a scenario statistically equivalent to a ROOF model having parameters $b_n := 1/(2n)$. (We have hereby scaled the jumps b_n with respect to [3] for our own purposes, yet statistical equivalence prevails.) The two referenced works employ the expansion (valid for $|\omega| < \pi/2$)

$$\log\cos\omega = \sum_{j=1}^{\infty} \frac{(-1)^j \left(2^{2j}-1\right) B_{2j}}{(2j)!j}\omega^{2j}$$

with a partitioned integral

$$f_\infty(x) \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x}\; d\omega\; e^{\sum_{k=m(\omega)}^{\infty}\log\cos(\omega/(2k))} \prod_{j=1}^{m(\omega)-1}\cos(\omega/(2j)). \tag{13}$$

Here, $m(\omega)$ is an integer-valued function that bounds $|\omega|/\pi$ from above—so that the log-cos series is always valid during numerical integration. They were able to obtain in this way what is equivalent to an extreme-precision value for $f(0) = f_\infty(0)$ for this canonical class-D_1 walk (see our Appendix for listed values).

A second approach is to exploit the equivalence Theorem 1 by considering instead the canonical class D_2 walk with $c_n := 1/(2n-1)$, for which

$$f_\infty(x) \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x}\; d\omega\; e^{\sum_{k=m(\omega)}^{\infty}\log(\mathrm{sinc}(\omega/(2k-1))} \prod_{j=1}^{m(\omega)-1}\mathrm{sinc}(\omega/(2j-1)). \tag{14}$$

In this alternative formulation we would use

$$\log\mathrm{sinc}\;\omega = \sum_{j=1}^{\infty} \frac{(-1)^j 2^{2j-1} B_{2j}}{(2j)!j}\omega^{2j},$$

valid this time for the range $|\omega| < \pi$.

Note that we are saying the f_∞ in (14) is the *same* function as the f_∞ in (13).

In [17], where the two-ROOF-walk equivalence we are exploiting here is argued probabilistically, appears the germ of a fine idea, as follows: That the characteristic kernel for either of the equivalent asymptotic walks can be partitioned as a product of trigonometric (cos or sinc) terms—corresponding to the random variable $\Delta_1 + \cdots + \Delta_{m-1}$—times a product of characteristics for the "tail variable" $y_m := \Delta_m + \Delta_{m+1} + \ldots$. However, it is argued in [17] that *the distribution of y_m for large m is essentially Gaussian*. But this would mean a reasonable approximation might be the relatively simple construction

$$f_\infty(x) \approx \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x} e^{-v_m\omega^2/2}\; d\omega \prod_{j=1}^{m(\omega)-1}\mathrm{sinc}(\omega/(2j-1)),$$

wherein one witnesses the term $e^{-v_m\omega^2/2}$—the characteristic function for a Gaussian of variance $v_m = \langle y_m^2\rangle$. A beautiful aspect of this argument is that it is actually addressing the *first* term of the log-sinc series. Indeed, take for example the log-sinc series through a few terms:[5]

$$\log \operatorname{sinc} \omega = -\frac{\omega^2}{6} - \frac{\omega^4}{180} - \frac{\omega^6}{2835} - \frac{\omega^8}{37800} - \frac{\omega^{10}}{467775} - \ldots$$

and note that the tail y_m claimed to be near-Gaussian—has density even closer to

$$g_m(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega y} e^{-a_2\omega^2 - a_4\omega^4 - \ldots} \, d\omega,$$

where

$$a_2 := \frac{1}{6} \sum_{k \geq m} \frac{1}{(2k-1)^2},$$

$$a_4 := \frac{1}{180} \sum_{k \geq m} \frac{1}{(2k-1)^4},$$

and so on. The $g_m(y)$ integral here is problematic. All the present author knows along such lines is that by keeping only a_2, a_4, the result $g_m(0)$ is a Bessel-$K_{1/4}$ evaluation.

It is interesting that even though the log-cos and log-sinc series have finite radii of convergence, we can in these instances ignore that and simply choose optimal cutoff m for a desired final precision on the desired $f_\infty(x)$. This freedom is due to the happy fact of all series elements in both of these log-series being negative, even outside the convergence domain. The downside to this way of thinking is that we do not yet know an analytical relation giving the optimal parameters such as cutoff m and series cutoff degree for a desired ultimate precision for $f_\infty(x)$. Still, it appears possible in principle to obtain arbitrary precision—albeit at steeply ramping cost—via empirical tuning of the relevnt parameters, as we have done for our Appendix.

7 Experimental and theoretical surprises

7.1 The "sinc surprises" of Baillie–Borwein–Borwein

A recent work by R. Baillie, D. Borwein and J. Borwein [5] on certain sums and integrals has the word "surprising" in its very title, for indeed, those researchers considered various product functions—in our probabilistic language these are characteristic functions of class-D_2 ROOF walks—of a form we denote

$$\Pi^{(n)}(\omega) := \prod_{j=1}^{n} \operatorname{sinc}(c_j \omega)$$

[5] The notion that all terms here are ω-powers divided by integers is specious: It may well be that the term shown with ω^{10} is the last such "pure" term.

and managed to build identities that sometimes fail with—shall we say—surprisingly minuscule errors.[6]

We remind ourselves that there are ROOF walks for which the c_j for j up through any finite n can be "anything monotonic," i.e. subject only to the constraint $c_j > c_k$ for $j > k$, and still represent the initial pedestal-half-widths of a finite-total-variance walk, as in the chain (2). That is, only when we contemplate infinite products, $n \to \infty$, denoting these Π_∞ when they exist, are we concerned about convergence issues for the c_j and the ensuing products $\Pi^{(n)}$. For finite n, then, any pairwise distinct real constants c_j in the definition of $\Pi^{(n)}$ are effectively allowed, since they can always be sorted into decreasing order.[7] In what follows, therefore, we shall assume that in the construction of a sinc-product $\Pi^{(n)}$ the multiplicands adopt the class-D_2 ROOF walk's natural sort, with the real, positive c_j strictly decreasing.

Now, two interesting constructs residing at the core of the previous research [5] are what we shall call a sinc-sum and a sinc-integral, respectively

$$S_n := \sum_{k \in Z} \Pi^{(n)}(k), \tag{15}$$

$$I_n := \int_{\omega \in R} \Pi^{(n)}(\omega)\ d\omega. \tag{16}$$

We reserve the freedom to formally allow $n = \infty$, when, of course, Π_∞ converges as an infinite product. This notion is tantamount, in our probabilistic view, to discussing the density function limit $f_n \to f_\infty =: f$.

Of great interest are sim-integral error terms addressed in [5] and which we equivalently denote

$$\rho_n := S_n - I_n.$$

In regard to such error terms, the aforementioned authors of [5] made surprising discoveries of the following type: It can happen that the sum-integral error here is $\rho_n = 0$ for some very large initial set of n indices, but suddenly some ρ_{n_0} does *not* vanish. Similarly, there can be long stretches of n values for which not only $\rho_n = 0$ but both S_n, I_n maintain a steady value, said value being invariant with increasing n until we reach some sudden index n_1 where both S_{n_1}, I_{n_1} depart(s) from the steady value (and so the error ρ_{n_1} at that sudden juncture may or may not vanish).

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[6]The philosopher Wittgenstein said "There can never be surprises in logic," yet when mathematicians speak of surprise we all know what is really meant: something "unexpected." The present author maintains that believing in "unexpectedness" does not violate Wittgenstein's thesis, in that the sheer unpredictability of where analysis might lead pervades all of mathematics. After all, the eminent Carl Ludwig Siegel did say "One cannot guess the real difficulties of a problem before having solved it"—in such a sense, surprise perhaps really does have a place in mathematics, logic notwithstanding.

[7]Evidently nothing prevents us from using.more generally, nonincreasing c_j-sequences; i.e., the pairwise-distinctness condition is not fundamentally required. However, much of our analysis goes through more efficiently when no c_j pairs coincide.

In the ensuing subsections we cover some observations and results in regard to these "sinc surprises" and related analyses.

7.2 Exact finite forms for sinc-sums S_n

It turns out to be possible, for arbitrary c_j, to evaluate any S_n (for finite n) in finite form. We shall derive the general formula and indicate later how this sometimes leads to immediate results for the integrals I_n. Note that this manner of sum evaluation is foreshadowed in [5]; here we attempt to generalize such ideas to cover arbitrary sinc-sums (for finite n).

First there is a general polylogarithmic identity, starting from

$$\Pi^{(n)}(\omega) = \frac{1}{(2i\omega)^n \prod_{j=1}^n c_j} \prod_{j=1}^n \left(e^{ic_j\omega} - e^{-ic_j\omega}\right)$$

with a sinc-sum emerging as a combinatorial entity

$$S_n = 1 + 2\frac{1}{(2i)^n T(\vec{c})} \sum_{\vec{\beta}\in\{-1,1\}^n} T(\vec{\beta})\ \mathrm{Li}_n\left(e^{i\vec{\beta}\cdot\vec{c}}\right), \tag{17}$$

where for any vector $\vec{r}$, we denote $T(\vec{r}) := \prod_{j=1}^n r_j$, while the vector index $\vec{\beta}$ runs over all 2^n balanced-binary n-vectors, $\vec{c} := (c_1, \ldots, c_n)$, and $\mathrm{Li}_n(z) := \sum_{k\geq 1} z^k/k^n$ is the standard polylogarithm. Happily, $\Re(\mathrm{Li}_n), \Im(\mathrm{Li}_n)$ here can be evaluated in finite form as n is even, odd respectively. In fact, for real z one has polynomial evaluations (see [5], [11] and references therein)

$$\Re\left(\mathrm{Li}_n\left(e^{iz}\right)\right) = \sum_{k=1}^{\infty} \frac{\cos kz}{k^n} = Q_n B_n\left(\left\{\frac{z}{2\pi}\right\}\right); \quad n \text{ even}, \tag{18}$$

$$\Im\left(\mathrm{Li}_n\left(e^{iz}\right)\right) = \sum_{k=1}^{\infty} \frac{\sin kz}{k^n} = Q_n B_n\left(\left\{\frac{z}{2\pi}\right\}\right); \quad n \text{ odd}, \tag{19}$$

where B_n is the standard Bernoulli polynomial, $\{\ \}$ indicates (mod 1) fractional part, and

$$Q_n := (-1)^{\lfloor n/2\rfloor - 1} 2^{n-1} \frac{\pi^n}{n!}.$$

In spite of the fact of (17) having 2^n summands if naively summed, this manner of polylogarithmic evaluation does indeed give a finite form for S_n. We do not yet know a method that will significantly accelerate this summation, so down to polynomial complexity in n rather than 2^n, although something should be possible because there is so much redundancy in the combinatorics. (Of course, the manner in which the alternating signs of the terms in $\vec{\beta}\cdot\vec{c}$ enter tells us that the basic complexity of a ROOF walk is somehow embedded in S_n itself.)

Using the prescription (17), (18), (19) for evaluation of S_n one can establish the following: For the class-D_2 ROOF-walk assignment $c_j := 1/(2j-1)$ we have sinc-sum evaluations

$$S_1, S_2, \ldots, S_7 = \pi,$$

whereas

$$S_8 = \pi \frac{467807924713440738696537864469}{467807924720320453655260875000}, \tag{20}$$

a striking result from earlier research; see [5], and note that the corresponding sinc-integrals I_n will "track" the above S_n values—in fact the matching goes on up to surprisingly large n, as we shall see. It will turn out for example that the origin density of the canonical ROOF walk in question, after 8 jumps, has $f_8(0) = S_8/(2\pi)$, i.e. a value just a touch below $1/2$.

More closed forms based on the polylogarithm decomposition appear in our Appendix, where also are found stultifying factorizations for some of the nontrivial multiples of π. For example, the prime factorization of the "surprising" S_8 above is

$$S_8 = \pi \frac{4322433877 \cdot 108227896140339439297}{2^3 \cdot 3^{12} \cdot 5^6 \cdot 7^7 \cdot 11^6 \cdot 13^6}.$$

7.3 Probabilistic approach for sinc-integrals

In regard to the connection between ROOF walks and sinc-integrals, there is an immediate interdisciplinary observation of great utility:

Theorem 5 *The sinc-integrals defined by (16) can be identified as*

$$I_n = 2\pi f_n(0),$$

where f_n is the density function for n steps of the class-D_2 ROOF walk having pedestal widths $2c_j$.

Proof: This is immediate from the characteristic relation (7). **QED**

This theorem connects the probabilistic notions with the theory of sinc-integrals, as exemplified in the following interpretation of the Mares integrals, named after B. Mares [3] [8]:

Corollary 2 *Starting with the function*

$$C(\omega) := \prod_{j=1}^{\infty} \cos\left(\frac{\omega}{j}\right),$$

the Mares integrals we hereby define

$$M_q := \int_0^{\infty} C(\omega) \cos(2q\omega)\ d\omega$$

have the probabilistic interpretation

$$M_q = \frac{\pi}{2} f(q),$$

where f is the ultimate density of the canonical ($c_j := 1/(2j-1)$) class-D_2 ROOF walk. Thus every $M_q, q = 0, 1, 2, 3, \ldots$ is positive, with particular bounds

$$M_0 := \int_0^\infty C(\omega)\ d\omega < \frac{\pi}{4},$$

$$M_2 := \int_0^\infty C(\omega)\cos(2\omega)\ d\omega < \frac{\pi}{8}.$$

In addition, we have sum rules

$$M_0 + 2M_2 + 2M_4 + \cdots = \frac{\pi}{4},$$

$$M_1 + M_3 + M_5 + \cdots = \frac{\pi}{8},$$

and the overall sum rule

$$\sum_{q \in Z} M_q = \frac{\pi}{2}.$$

Proof: Reminding ourselves of the equivalence Theorem 1, we see from the very proof of said theorem and the characteristic relation (6) that $M_q = \frac{\pi}{2} f(q)$. But by bounding Theorem 2, we have $f(0) < 1/2$ and the result follows. For the second integral, the same argument goes through involving Theorem 2 again, with $M_1 = \frac{\pi}{2} f(1) < \frac{\pi}{8}$. Finally, the sum rules for the Mares integrals M_q follow immediately from the probabilistic sum rules in Theorem 2. **QED**

Note that if desired, one may establish rigorous bounds for the Mares integrals, along the lines of our Corollary 1. As for numerical estimates, our Appendix automatically has high-precision evaluations for the Mares integrals, being as they are the given multiples of f values in Corollary 2.

The next observation is a powerful one appearing in various references such as [3] [5]. Our present intent is to give this previously established result a probabilistic connection:

Theorem 6 *For rational, monotonic decreasing c_j, the sinc-integral defined in (16) is a rational multiple of π—alternatively, the origin probability density $f_n(0)$ ($= \frac{1}{2\pi} I_n$) is rational—that is*

$$I_n := \pi \frac{1}{2^{n-1} c_1} \mathrm{Vol}(\mathcal{W}),$$

where $\mathcal{W}$ is the $(n-1)$-dimensional polyhedron defined by the constraints

$$|c_2 z_2 + \ldots c_n z_n| \le c_1, \quad |z_k| \le 1.$$

Proof: The theorem follows immediately from Theorem 5 and the convolution (8). **QED**

Evidently, then, $I_n = \pi/c_1$ if the polyhedron under its defining constraints does not—to use language from [8]—"bite into" the centered hypercube; however, when the constraint is "active" we have a sudden reduction $I_n < \pi/c_1$ and this explains some of the surprises in the realm of sinc-integrals [3]. Note that this geometrical picture involving intersecting regions is consistent with the constraints involved in our Theorem 2.

There is also a finite form available for sinc-integrals at a certain threshold, namely [8, Ex. 27, p. 123], this form being reminiscent of the finite form for sinc-sums, our prescription (17), (18), (19). We cite this result in our probabilistic context:

Theorem 7 *For a class-D_2 ROOF walk, consider an index N such that*

$$c_2 + \cdots + c_N = c_1 + \delta$$

with $0 < \delta \le c_N$. Then the probability density at the origin after N steps is

$$f_N(0) = \frac{1}{2c_1}\left(1 - \frac{\delta^{N-1}}{2^{N-2}(N-1)!\prod_{j=2}^{N} c_j}\right).$$

Proof: We assume the validity of the published exercise, which starts with Bernoulli identities of the type we used above for sinc-sums. Alternatively, one can prove the theorem directly, via our relation (8) for class-D_2 walks, on the notion that the "bite" of the polyhedron is relatively trivial under the given δ-constraints. **QED**

An example of Theorem 7 in action is as follows. Note that for the canonical class-D_2 ROOF walk, it happens that

$$\frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{15} \approx 1.0218,$$

so that we have

$$\begin{aligned} f_8(0) &= \frac{1}{2}\left(1 - \frac{6879714958723010531}{467807924720320453655260875000}\right) \\ &= \frac{1}{2}\frac{467807924713440738696537864469}{467807924720320453655260875000}, \end{aligned}$$

which sure enough involves the same rational as did S_8 in (20).

We next begin an analysis intended to rigorously compare sinc-sums and sinc-integrals.

7.4 Theory of sum-integral error terms ρ_n

An analysis of the sinc-sum-integral error terms starts with Poisson transformation. As in [10] we can Poisson-transform the sum over $k \in Z$, as

$$S_n := \sum_{k\in Z} \Pi^{(n)}(k) = \sum_{\mu\in Z} \int_{\omega\in R} \Pi^{(n)}(\omega) e^{2\pi i \mu\omega}\, d\omega$$

$$= I_n + \sum_{\mu \neq 0} \int_{\omega \in R} \Pi^{(n)}(\omega) e^{2\pi i \mu \omega} \, d\omega,$$

and so the desired error term is

$$\rho_n = 2 \sum_{\mu=1}^{\infty} \int_{-\infty}^{\infty} \Pi^{(n)}(\omega) e^{2\pi i \mu \omega} \, d\omega. \tag{21}$$

This transformation algebra amounts to a proof of the following

Theorem 8 *For a sinc-product $\Pi^{(n)}(\omega) := \prod_{j=1}^{n} \mathrm{sinc}(c_j \omega)$ where the c_j are associated with a class-D_2 ROOF walk, the sinc-sum-integral error is given by*

$$\rho_n := S_n - I_n = 4\pi \sum_{\mu \geq 1} f_n(2\pi\mu),$$

where f_n is the density function for n steps of the ROOF walk.

This exact series representation for the error ρ_n leads, then, to a result known essentially to previous authors; again we endeavor to cast a result in the language of our present framework:

Corollary 3 *For the ROOF-walk c_j of Theorem 8, the error $\rho_n := S_n - I_n$ is never negative, and is positive if and only if $c_1 + c_2 + \cdots + c_n > 2\pi$.*

Proof: In Theorem 8, the constraint $\cdots > 2\pi$ determine whether there be any contributions from $f(2\pi\mu)$ for positive integer μ. **QED**

When the converse constraint $c_1 + \cdots + c_n < 2\pi$ succeeds, ρ_n vanishes and I_n can be calculated in finite form as an instance of S_n from (17); or conversely, S_n can be evaluated via an instance of I_n per Theorem 6. In any case, Corollary 3 determines the precise threshold at which S_n, I_n differ.

As an example, we finally have derived that for the canonical assignment $c_j := 1/(2j-1)$, the errors ρ_n; $n = 1, \ldots 8$ all valish, which means I_8—remarkably enough—has the same peculiar value as S_8, in (20). It will turn out below that the index n must be taken *much* farther to yield a nonzero ρ_n.

But in a different direction for the moment, we find that Theorem 8 has an interesting implication regarding sum-integral identities that are *always* true:

Corollary 4 *For a class-D_2 ROOF walk with all rational c_j, there is a sinc-sum/sinc-integral relation that is true for all finite $n = 1, 2, \ldots,$ namely*

$$S_n := \sum_{k \in Z} \Pi^{(n)}(k) = \int_{\omega \in R} \Pi^{(n)}(\omega) \, \frac{\sin(\pi(K + 1/2)\omega)}{\sin(\pi\omega/2)} \, d\omega,$$

where we define the integer

$$K := \left\lfloor \frac{c_1 + \cdots + c_n}{2\pi} \right\rfloor.$$

Remark: Note once again: If the c_j sum is less than 2π, then $K = 0$ and we recover the equality $S_n = I_n$, hence $\rho_n = 0$. Also, here and elsewhere we use rational c_j simply to avoid the sometimes delicate phenomenon that a sum of c_j values actually equals a multiple of π. Still, appropriate modifications of the analysis should be able to handle irrational c_j.

Proof: This corollary follows from Theorem 8 and a trigonometric identity valid for any nonnegative integer K:

$$\sum_{\mu=-K}^{K} e^{2\pi i\mu\omega} = \frac{\sin(\pi(K+1/2)\omega)}{\sin(\pi\omega/2)}.$$

QED

One of the attractive results from [5] is that for the choice $c_k := 1/(2k-1)$, which assignment we have associated with the canonical class-D_2 ROOF walk, the error $\rho_n := S_n - I_n$ has the surprising property

$$\rho_1, \rho_2, \ldots, \rho_{40249} = 0,$$

whereas

$$\rho_{40250} > 0.$$

(Note that in their work, the index n is offset by 1 with respect to our probability models.) This phenomenon can be interpreted, via Corollary 3, as that of the sum

$$\frac{1}{1} + \frac{1}{3} + \ldots \frac{1}{2N-1} = 2\pi + \delta$$

having

$$0 < \delta < 1/(2N-1)$$

precisely when $N := 40250$. That is, the sum through c_{n-1} does not quite reach the threshold 2π. To be clear with respect to our previous discussions about S_n and I_n, we remind ourselves that even though ρ_n thus vanishes for thousands of low-lying n, we know that both S_n, I_n fall below the value π at $n = 8$, because we have already given enough finite forms for some low-lying S_n (and see Appendix for specifics).

To quantify the error ρ_n in such threshold cases, we establish

Theorem 9 *Assume rational c_j elements for a class-D_2 ROOF walk, with $c_1 < 2\pi$. Then there always exists a unique threshold index N such that*

$$c_1 + c_2 + \cdots + c_N = 2\pi + \delta,$$

with $0 < \delta < c_N$. Moreover, the exact sum-integral error at this threshold is

$$\rho_N := S_N - I_N = \frac{\pi\delta^{N-1}}{2^{N-2}(N-1)!\prod_{j=1}^{N} c_j}.$$

Remark: Minuscule as this "threshold error" ρ_N can be experimentally, it is nevertheless a rational polynomial in π.

Proof: The sum of the c_j diverges for the ROOF walk, and monotonicity of the c_j settles the existence of the claimed δ and threshold N. From Theorem 8 and convolution (8), we therefore have

$$\rho_N = \frac{\pi}{2^{N-2}} \int_{[-1,1]^N} \delta(c_1 z_1 + c_2 z_2 + \cdots + c_N z_N - 2\pi) \, \mathcal{D}\vec{z},$$

that is, only the density terms $f_N(2\pi), f_N(-2\pi)$ can possibly contribute to the error ρ_N. However, also because of the constraint $\delta < c_N$, the delta-function argument $c_1 z_1 + c_2 z_2 + \cdots + c_N z_N - 2\pi$ cannot vanish if any z_k be negative. Therefore we can change variables $z_k \to 1 - u_k$ and write

$$\rho_N = \frac{\pi}{2^{N-2}} \int_{[0,1]^n} \delta(c_1 u_1 + c_2 u_2 + \cdots + c_N u_N - \delta) \, \mathcal{D}\vec{u},$$

where the new domain is, importantly, $[0,1]^N$. Integrating over just u_1 gives

$$\rho_N = \frac{\pi}{2^{N-2} c_1} \int_{u_j \geq 0; \; c_2 u_2 + \cdots + c_N u_N < \delta} du_2 \cdots \, du_N.$$

This polyhedral integral is easily doable, yielding the desired formula of the theorem. **QED**

Now, in the canonical case ($c_j := 1/(2j-1)$) we obtain the surprising scenario of Baillie et a;. [5], namely, with the aforementioned threshold $N = 40250$ we calculate

$$\sum_{j=1}^{40250} \frac{1}{2j-1} = 2\pi + (\delta = 0.000000234727\ldots),$$

with $\delta < 1/(2 \cdot 40251 - 1)$. We already know that all errors $\rho_1, \ldots, \rho_{40249}$ vanish. But Theorem 9 gives the sum-integral error at threshold, as

$$\rho_{40250} = S_{40250} - I_{40250} = \frac{\pi \delta^{40249}}{2^{40248}} \frac{80499 \, !!}{40249 \, !}$$

$$\approx 8.42 \cdot 10^{-226577}.$$

Here, !! means odd-factorial, as in 5!! := $5 \cdot 3 \cdot 1$. This phenomenon tells us that sheer experimental computation even in the region of 100000 decimal digits would not catch this kind of "surprise!"

What can be said about an arbitrary error ρ_n in the canonical case? A summary result follows immediately from the effective superexponential bound of Corollary 1, together with Theorem 8, as

Theorem 10 (Summary of the Baillie et al. error terms) *For the canonical case ($c_j := 1/(2j-1)$), the error $\rho_n := S_n - I_n$ satisfies:*

$$\rho_1, \ldots, \rho_{40249} = 0,$$

$$0 < \rho_{40250} < 10^{-226576},$$

and for all other n (including $n = \infty$)

$$0 < \rho_n < 10^{-13679}.$$

Remark: In spite of the rigor of these bounds, we do not know over what regions ρ_n might be monotonic in n; neither do we know ρ_∞ nor S_∞ nor I_∞.

7.5 A "primes" ROOF walk

For certain exotic ROOF walks, such as the class-D_2 walk having $c_j := 1/p_j$ where p_j is the j-th prime, the "surprise" effect is yet more striking. Note first that $\sum 1/p_j, \sum 1/p_j^2$ diverges, converges respectively, as is classically known, so class-D_2 membership is assured. We may also assign $b_j = 1/p_j$ to get a class-D_1 "primes" walk. For class D_2 we can obtain a threshold error that is remarkably small, as follows. With a view to Theorem 9, we require a threshold index N—determining the prime p_N—such that

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{p_N} = 2\pi + \delta, \tag{22}$$

with $0 < \delta < 1/p_N$. An immediate question is, if we know N, p_N, what is the threshold error ρ_N from Theorem 9? Happily, there are known rigorous bounds on the n-th prime number; said bounds having been developed in the classic work of Rosser and Schoenfeld [16], and more recently by Dusart [13]. We know that for $n \geq 6$,

$$n \log n \ < \ p_n \ < \ n \log n + n \log\log n, \tag{23}$$

and also some bounds on the prime-counting function: For real $x \geq 599$,

$$\frac{x}{\log x}\left(1 + \frac{0.992}{\log x}\right) \ \leq \ \pi(x) \ \leq \ \frac{x}{\log x}\left(1 + \frac{1.2762}{\log x}\right). \tag{24}$$

The lower bound of (23) leads to

Theorem 11 *For the "primes" ROOF walk threshold N described by (22), the Baillie–Borwein–Borwein error satisfies*

$$0 < \rho_N < \frac{1}{N^N}.$$

Proof: Clearly by direct computation, $N \geq 28$. Since $\delta < 1/p_N$ we have, from Theorem 9 and the bounds (23),

$$\rho_N < \frac{\pi \cdot 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11}{(N \log N)^{N-1}} \frac{1}{2^{N-2}(N-1)!} \prod_{j=6}^{N} j(\log j + \log\log j)$$

$$< \frac{\pi \cdot 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11}{(N \log N)^{N-1}} \frac{1}{2^{N-2}(N-1)!} \frac{N!}{5!} (\log N + \log\log N)^{N-5}.$$

The right-hand side is easily seen to be $< 1/N^N$ for $N \geq 28$. **QED**

The problem now is to estimate N. Baillie et al. [5] estimated, on the basis of the Mertens theorem

$$\sum_{p \leq x} \frac{1}{p} = \log\log x + B + o(1),$$

with B being the Mertens constant, that the threshold N and the prime p_N are *very* large—in fact, each turns out to well exceed a googol (10^{100}), as we shall eventually prove. Incidentally, B is resolvable to high precision, for example

$$B = 0.2614972128476427837554268386086958590515666482611992061920642139 2\ldots$$

accrues easily from an algorithm of E. Bach that, curiously enough, does not employ primes directly, rather using Riemann-zeta evaluations at integers [12, Ex. 1.90].

Let us first approach this threshold problem on the assumption of the Riemann hypothesis (RH), under which it is known that for $x \geq 13.5$,

$$\left| \sum_{p \leq x} \frac{1}{p} - \log\log x - B \right| \; < \; \frac{3x + 4}{8\pi\sqrt{x}},$$

and that the prime count $\pi(x)$ is, for $x \geq 2.01$, well approximated by the logarithmic integral li, in the sense

$$|\pi(x) - \mathrm{li}(x)| \; < \; \sqrt{x} \log x.$$

These facts are found in [1, Equ. 7.1] and [12, Ex. 1.37], and all date back to the celebrated work of Schoenfeld and Rosser. From these (RH-conditional) bounds, one can derive

$$2\pi + \delta - \log\log p_N - B \in [-\epsilon, \epsilon],$$

and so

$$e^{e^{2\pi - B + \delta - \epsilon}} \; < \; p_N \; < \; e^{e^{2\pi - B + \delta + \epsilon}},$$

where ϵ is certainly less then 10^{-50}. It follows—again, under the RH—that at threshold, the prime and index are bounded in the forms

$$p_N \approx 10^{179.05 \pm 0.01},$$

$$N \approx 10^{176.44 \pm 0.01}.$$

These offsets ± 0.01 are quite conservative, yet it follows from Theorem 11 that the Baillie–Borwein–Borwein error has, again on the RH,

$$\rho_N < 10^{-10^{178}}.$$

Let us endeavor, then, to make such arguments unconditional (independent of the RH), to rigorously analyze this fascinating "primes" ROOF walk. We start with

Theorem 12 (Unconditional, no RH) *There exist nonnegative, monotone-nonincreasing sequences* $(c_1, c_2, \dots), (d_1, d_2, \dots)$, *both converging to zero, such that for a fixed* m *the reciprocal-prime sum is constrained for all* $n > m$ *by*

$$\sum_{j=1}^{n} \frac{1}{p_j} - \log\log n \;\in\; [B - c_m,\; B + d_m].$$

In particular, for $n > 3 \cdot 10^8$, *the constraint interval can be taken to be* $[B - 0.047, B + 0.147]$, *while for* $n > 4.41 \cdot 10^{16}$ *we may use* $[B - 0.025, B + 0.093]$.

Remark: Even though the Mertens theorem standardly says $\sum_{p \le x} 1/p - \log\log x - B = o(1)$, i.e. the $\log\log$ argument is the upper limit on primes and not their upper index, the notion that both $c_m, d_m \to 0$ is equivalent to said theorem, since $\log\log(x := p_n) - \log\log n = o(1)$. Also, the present author conjectures that the c_m can all be taken to be 0; equivalenetly, for $n > 1$ the difference $\sum_{j \le n} 1/p_j - \log\log n$ is *always* $\ge B$.

Proof: It is elementary that for $h(x)$ positive, continuous, and monotone decreasing on $[k, n]$,

$$\int_{k+1}^{n+1} h(x)\, dx \;\le\; \sum_{j=k+1}^{n} h(j) \;\le\; \int_{k}^{n} h(x)\, dx.$$

(This can be shown quickly by replacing $h(j) \to h(\lfloor x \rfloor)$ in the summation, then converting to an integral over $h(\lfloor x \rfloor)$.) Take an integer $k \in [6, n-1]$. The integral bounds can be applied along with $p_n > n \log n$ from (23), taking $h(x) := 1/(x \log x)$, to obtain

$$\sum_{j=1}^{n} \frac{1}{p_j} \le \log\log n + B + \inf_{k \in [6, n-1]} \left(\sum_{j=1}^{k} \frac{1}{p_j} - \log\log(k) - B \right).$$

But we also have, using the other bound $p_n > n(\log n + \log\log n)$ from (23),

$$\sum_{j=1}^{n} \frac{1}{p_j} \ge \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{p_{k-1}} + \int_{k}^{n+1} \frac{dx}{x(\log x + \log\log x)}.$$

Now we observe

$$\int_a^b \frac{dx}{x(\log x + \log\log x)} = \int_{\log a}^{\log b} \frac{du(1+1/u)}{u+\log u} - \int_{\log a}^{\log b} \frac{du}{u(u+\log u)}.$$

It follows that

$$\sum_{j=1}^{n} \frac{1}{p_j} \geq \log\log n + B + \sup_{k\in[6,n-1]} \left(\sum_{j=1}^{k-1} \frac{1}{p_j} - \log(\log k + \log\log k) - \frac{1}{\log k} - B \right).$$

Now, c_m, d_m for $m \geq 6$ can simply be defined respectively as the $\inf_{k\in[6,m]}(\ ),\ \sup_{k\in[6,m]}(\ )$ above, so the existence of properly monotonic c_m, d_m is established; the classical Mertens theorem says $c_m, d_m \to 0$. Taking $k = m = 3\cdot 10^8$ in the argments of the sup, inf terms, we can settle by direct summation of $1/p$ the bounding interval $[-0.047, 0.147]$. Taking $k = m$ corresponding to the Bach–Sorenson prime $p_m = 1801241230056600523$ (see text), we have from (24) the constraint $m = \pi(p_m) \in [4.39 \cdot 10^{16}, 4.41 \cdot 10^{16}]$ and this, used in the sup, inf terms, is enough to achieve the final explicit constraint interval of the theorem.

QED

These bounds on reciprocal-primes sums leads to a rigorous resolution of the threshold problem, in the form

Corollary 5 (Unconditional) *The threshold index N from (22) satisfies*

$$10^{163} < N < 10^{184},$$

and hence, the Baillie–Borwein–Borwein error satisfies

$$\rho_N < 10^{-10^{165}}.$$

Moreover, the "primes" ROOF walk has ultimate probability density decaying at least triply superexponentially; indeed, for $x \geq 2$,

$$0 \ < \ f(x) \ < e^{-e^{2(x-1.36)}e^{e^{x-1.36}}}.$$

In particular,

$$0 \ < \ f(2\pi) \ < \ 10^{-10^{63}}.$$

Thus, for any n exceeding the threshold N, we have

$$0 \ < \ \rho_n \ < \ 10^{-10^{62}}.$$

Remark: These results mean, unconditionally, that the threshold N exceeds a googol, and the threshold error ρ_N is less than 1/googolplex.

Proof: The bound on N is an easy computation from Theorem 12 with its specific interval $[B - 0.025, B + 0.093]$; we have

$$e^{e^{2\pi - B + 0.0251}} > N > e^{e^{2\pi - B - 0.093}}.$$

Then the bound on ρ_N follows from Theorem 11. The bound on $f(x)$ is a consequence of Corollary 1, with the assignment

$$m := \left\lfloor e^{e^{x-1-B-0.093}} \right\rfloor,$$

the only extra bound needed being

$$\sum_{j=m}^{\infty} \frac{1}{p_j^2} \;<\; \frac{1}{(m-1)\log^2 m},$$

as follows from the lower bound on p_j from (23). The last bound on $\rho_{(n>N)}$ follows from Theorem 8.

QED

Incidentally, the situation in regard to *exact* knowledge of reciprocal-prime sums is not infinitely hopeless. Recently, E. Bach and J. Sorenson [1] devised a clever scheme for assessing the threshold index m for such as

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{p_m} = 4 + \delta,$$

with $\delta < 1/p_m$, where one notes here a change of our previous threshold $2\pi \to$ threshold 4. They found the precise threshold prime as

$$p_m = 1801241230056600523,$$

for which those authors derived

$$\sum_{j=1}^{m} \frac{1}{p_j} \;\approx\; 4.0000000000000000000021,$$

as we used in the computations for the proof of our Theorem 12 above. The authors of [1] did not need to determine the exact m for their purposes, although they could have. This is why we used for our Theorem 12 the unconditional bounds (24) on $\pi(x)$ to sufficiently closely estimate the index m.

Can the Baillie–Borwein–Borwein threshold be assessed by sieving, then? The authors of [1] do say:

> "*We note that [the threshold prime for reciprocal-prime sum > 5] is about* $4.2 \cdot 10^{49}$, *so its precise value may remain unknown for all eternity.*"

Those authors are, of course, referring to the shortcomings of prevailing sieve methods and machinery in 50-digit regions. This having been said, the notion of locating the explicit threshold for $\sum 1/p > 2\pi$, as is fundamental to the theory of the "primes" ROOF walk, looms even more stultifying.

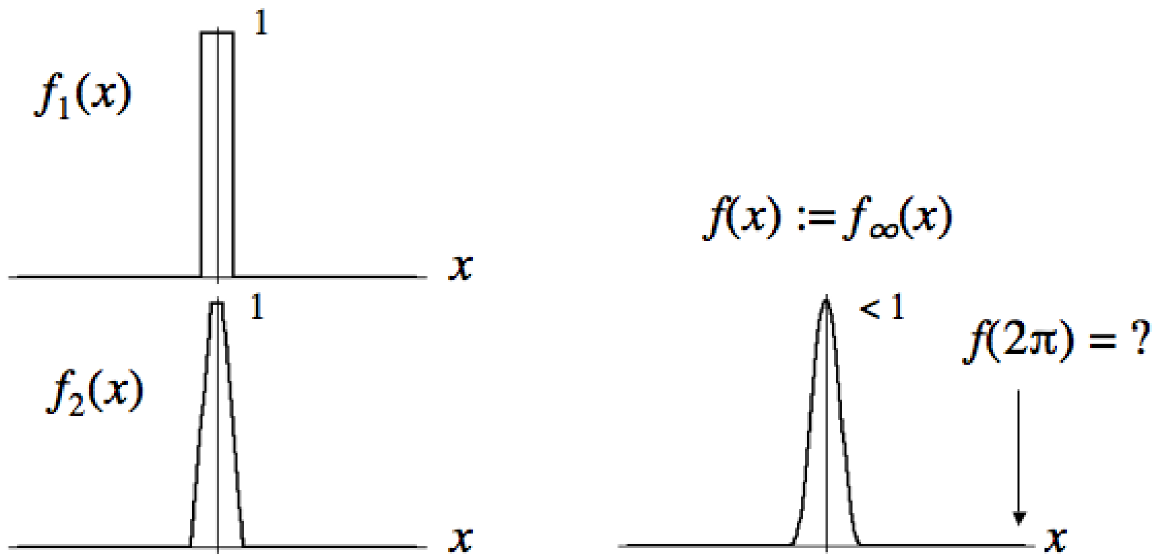

Figure 3: Probability density for a "primes" ROOF walk. On its n-th step the walker jumps uniformly an increment $[-1/p_n, 1/p_n]$ where p_n is the n-th prime number. The upper-left plot is the density f_1, the lower-left is f_2. (The x-axis is compressed; the initial density f_1 is positive on $x \in [-1/2, 1/2]$.) The lower-right plot is for $f := f_\infty$, the everywhere-positive, ultimate density for this walk. The decay of f is *triply superexponential*; for example, we prove that $f(2\pi) < 1/\exp(\exp(\exp(4.99)))$ (Corollary 5).

8 Open problems

- What is an exact evaluation of the discrepancy $1/2 - f(0)$ for the canonical (or for that matter, any!) class-D_2 walk? (In the canonical case, an equivalent question is, what is the Mares integral M_0 in closed form?) Is the situation—that we do not know any closed forms for a single $f(x)$–for class-D_1 walks—just as dificult?

- What can be said about the class-D_2 walk with $c_n = 1/n$? In the present paper this particular "fuel assignment" was not touched upon at all. There could—or could not—be a trivial scaling to analyze this walk in previous terms.

- It is possible to calculate higher moments of ROOF walks. For example, the canonical class-D_1 walk ($b_n := 1/(2n)$ is the jump length) has, as we know, $\langle x_\infty^2 \rangle = \frac{1}{4}\zeta(2)$. It can be derived [15] that $\langle x_\infty^4 \rangle = \frac{3}{4}\zeta(2)^2 - \frac{1}{8}\zeta(4)$, and so on. The question is, how can knowledge of the structure of $f(x)$ accrue from, say, the knowledge of many such moments?

- We advise that ROOF walks outside class D, e.g. walks with *converging* sum $\sum\langle|\Delta_j|\rangle$, are difficult to analyze and many mysteries abound. Though our example walk with $\Delta_n = \pm 1/2^n$ was easy, intuitive, a walk with $\Delta_n = \pm 1/3^n$ is already intricate, conjuring up such entities as the Cantor set of fractional dimension and evidently intractable characteristic. The references [17] [15] discuss some open problems for such ROOF classes.

- When is a class-D_1 walk equivalent to a class-D_2 walk, in the sense of the two walks' ultimate density functions coinciding at all x? Equivalence *does* hold if for every positive integer n we have
$$\sum_{j\geq 1} c_j^{2n} = (4^n - 1)\sum_{j\geq 1} b_j^{2n},$$
as is certainly true for our canonical cases $c_j := 1/(2j-1), b_j := 1/(2j)$.

- Presumably there is a ROOF walk of class D having *quadruply superexponential* decay—so how does one develop a theory of ROOF walks that have arbitrary, cascaded superexponential bounds?

- One aspect of density-function dynamics we have not addressed at all is diffusion theory. Heuristically, for the canonical class-D_1 ROOF walk, having $b_n := 1/(2n)$, we might approximate
$$f_n(x) \approx \frac{1}{2} f_{n-1}(x - 1/(2n)) + \frac{1}{2} f_{n-1}(x + 1/(2n)).$$
This leads to the heuristic differential equation (setting $n := t$, a "time"):
$$\frac{\partial f}{\partial t} = \frac{1}{8t^2}\frac{\partial^2 f}{\partial x^2}.$$

An exact solution is

$$f(x,t) = \frac{1}{\sqrt{a - 1/t}} e^{-2x^2/(a-1/t)},$$

for constant a. But this suggests a limiting Gaussian distribution as $t \to \infty$, which we know to be incorrect (the true $f(x,\infty)$ decays superexponentially in x, after all). Perhaps this exact solution should be used as a propagation kernel, connecting successive finite t values.

- What is the "natural" quantum potential $V(x)$ such that the canonical class-D_2 density $f(x)$ is the Schrödinger ground state of V? We demand that the Schrödinger equation holds, i.e.

$$-\frac{d^2 f}{dx^2} + Vf = E_0 f,$$

where the energy scale is set by $V(0) := 0$. This means that the potential V is given explicitly by

$$V(x) = E_0 + \frac{f''(x)}{f(x)},$$

with eigenvalue $E_0 = -f''(0)/f(0)$. This appears to be a difficult problem. What we can say is that this "natural" energy E_0 is given by

$$E_0 = \frac{4}{M_0} \int_0^\infty \omega^2 C(\omega)\, d\omega,$$

where C is the Mares kernel as in Corollary 2. D. Bailey [2] has calculated this fundamental quantum energy as

$$E_0 \approx 0.013990359117147734511346575283227759543878661010906341350035.$$

(Actually, Bailey can obtain hundreds of digits beyond this.) Also of interest: The graph of $V(x)$ apparently itself has zero-crossings.

- How would one develop a physics theory of a fundamental ROOF particle—say in a full, 3-dimensional setting—that, upon photon-energy absorption, dissipates the absorbed energy in the style of a ROOF walk, thus ending up at a probabilistic position?

9 Appendix

For the numerical and symbolic results herein, we adopt the canonical prescription

$$\pi^{(n)}(\omega) := \prod_{j=1}^{n} \text{sinc}\left(\frac{\omega}{2j-1}\right),$$

and define the sinc-sums and sinc-integrals, respectively:

$$S_n := \sum_{k \in Z} \pi^{(n)}(k), \quad I_n := \int_{\omega \in R} \pi^{(n)}(\omega)\ d\omega.$$

9.1 Exact sinc-sums

We recall that for the canonical assignments $c_j := 1/(2j-1)$ the sinc sums are

$$S_1, S_2, ..., S_7 = \pi.$$

Use of relations (17), (18), (19) give the following, for $n \in [8, 16]$ in the form

$$\frac{S_n}{\pi} = \frac{\text{num}}{\text{den}},$$

where num, den are integers—and the prime factorization of num is given when known. We observe that separately interesting is the factorization of the numerator of $1 - \text{num}/\text{den}$; we do not report these herein, although we do caution researchers that the first few such factorizations are trivial—and that triviality is misleading. (We do give one instance of factorization of $1 - \text{num}/\text{den}$, for the last example $n = 16$ below.)

$$S_8/\pi = \frac{4678079247134407386965378644 69}{46780792472032045365526087500 0},$$

$$\text{num} = 4322433877 \cdot 10822789614033943929 7.$$

$$S_9/\pi = \frac{177086951830561906424973155306284222955698 65119}{177086953941505976474491764937637554675200 00000},$$

num is prime.

$$S_{10}/\pi = \frac{80967996219408975678286868543125354863110611145506053675116 53}{80968003779706499608759190328576347168200750760623815750000 00},$$

$$\text{num} = 109 \cdot 307 \cdot 24196275354692937177864168945738682982132687190480845613 1,$$

$$S_{11}/\pi = \frac{2051563935160591194337436768610392837217226815379395891838337765936509}{2051564503724359411435325207087513361930253427318374450656960000000000},$$

$$\text{num} = 5167 \cdot 397051274464987651313612689880083769540783204060266284466486891027.$$

$$
\begin{gathered}
S_{12}/\pi = 37193167701690492344448194533283488902041049236760438 \\
30296516790118732385138484006728786 3 \\
/ \\
37193188390019359679267753038304609065247968318560293442237 \\
45376099348262566239562500000 0, \\
\text{num} = 457 \cdot 704477 \cdot 423614372509 \cdot 4481160013802705926237 \cdot \\
608581870911745961111277550067787903251962756 99.
\end{gathered}
$$

$$
\begin{gathered}
S_{13}/\pi = 54311089646116984630768274650449120156130445308519 1 \\
93751467179053597576266313204943450734539904399598201240 79 \\
/ \\
54311139113760643460138983791926804752980129908880153 59 \\
173701456675706968166358939600745000000000000000000000, \\
\text{num is composite.}
\end{gathered}
$$

$$
\begin{gathered}
S_{14}/\pi = 93668579368254770022901538277679262941014368030255491 11 \\
4199067590221323144401222248080717527867670545886934135274294680 69 \\
/ \\
93668702988747470522353716916858309344196014509267648553881576518 \\
6138415909651342556693677990179313694763183593750000000 0, \\
\text{num is composite.}
\end{gathered}
$$

$$
\begin{gathered}
S_{15}/\pi = 11065827516567126907053963240111542592075264383241815 87 \\
11753605989327076464755451468091430698696898117300223589 93 \\
09568164969698643947875339545875104788753 7 \\
/ \\
11065847045404214525396328376964652190542839117451331136 15 \\
55507340676877044145563077750327648920069167541590183060 8
\end{gathered}
$$

$$
\begin{gathered}
316356778613513375000000000000000000000000, \\
\mathrm{num} = 313 \cdot 1031 \cdot 2903 \cdot 5569 \cdot 482011381136627 \cdot \\
490221412985821942 03 \cdot 15265526308002685361 \cdot \\
24835459108155475602858136 4947 \cdot \\
2367680487134139644000178219983941204010506257395269626981 1.
\end{gathered}
$$

$$
\begin{gathered}
S_{16}/\pi = 10288574241455161975736690594382871962046253992402535 19 \\
781651784876399870694224539789974108272189041614892539 8988 \\
870835066166007744052418638591660237591063020045382226 8852 \\
643263934963 \\
/ \\
102885971988537707660519059741757265548279300360887901 72 \\
468690003333625836111565713233727527852994542095913709 45 \\
435907838981606429475270406888624733780606156179428100 \\
58593750000000000, \\
\text{num is composite.}
\end{gathered}
$$

For S_{16}/π above, the factorization of the numerator of $1 - \mathrm{num}/\mathrm{den}$ is

$$449 \cdot 20310271 \cdot 47617241426617180429 \cdot P,$$

where P is prime. It does appear that factoring $1 - \mathrm{num}/\mathrm{den}$ is generally easier than factoring $\mathrm{num}/\mathrm{den}$ (see Theorem 7 for hints in such a direction), but this ease may well vanish for large n.

9.2 D. Bailey's calculations and probability densities

D. Bailey's computed values [2] [8, p. 101] [3, p. 220] for the Mares integrals M_q (see our Corollary 2) are as follows:

```
M0 = 0.78538055729863287349258301146733252476165283080347360800414
6917699335235479052147603662085365602506509045709197834959317463160036034276655
4655793070478317411219735302291101275852304766809624821269866748407448374677899
1192997656890603879924119300411734744121340251835043120675312327518451159416622
6328051500397979692746867677549649008771512050047411751537426086086434141273466
2340203432904856702365298120786854984823645457071868728389420781473030254405188
515375759834490183474894245260192233482609683185 ...

M1 = 0.39269908169872415480783042290993786052464543418723159592681
2285162093247139938546179016512747455366777507395573123898403582027154856887600
9365093419344489869213452474548876333219399274808697766142856120560186166274833
4092186922776226736872376792130721133122701989040626907262320706556348432831522
8467835732338980594711041571395979629698980697925450710450859796765509555398866
6516988216867140817746408841650537676651111993388422594453673944761274375324977
1676215036520551311534938104429251335903210314 22...,

M2 = 8.80304940771806153891717627159814381975952015142361979461511
8880943304625005102667331048496338024324565621524280867215515055817668445163777
7339077690143407141786169219167721204257217305620873136437134987718461929608422
6723101072728338583600869437127114540073951502201596913466726744862119921797866
3346537438597941601454053667022413448709161813738699979796065156129977631401477
7488486316619583061414315655242489782286950388733690741641893986318106350023566
61487794848001152223921556433151143616 6575... * 10^(-6),

M3 = 7.40734656631695055788876383803645837578649487840420309266722
0261115241036389927304703421336935169600576506671570270467833038288865981918799
5680769170853223441624993415205683407588579136679802486475518866769517510083366
8876391916669044470124470325768076002674823483234395231078406950464210485421377
2958836013710095126777029491860917452085570098305877551246280109528594482144600
7955605713928810998796815199762814846833092760974792384384047614661360067589066
25622...* 10^(-43),

M4 = 1.42548695386600110540119621157777475570100223045903273417688
1713857423821979425747629152241486074350954314822162545931900095829873725696766
751052804477739229057123321708493560780305 52...* 10^(-319).
```

Numerical values for $f(q) = (2/\pi)M_q$ follow. To get the data below, we used our relation (14) with the (naive—see text) choice $m(\omega) := 100$ and $L = 128$, the latter counting the degree to which the log-sin series is taken. (Note these next values are radically less precise than D. Bailey's equivalent values above; we cite the values below just to indicate what happens with given m, L parameters in the alternative, log-sin method of the text.)

```
f(0) = 0.49998879160937983591179672655936274390974036542331027985
1216510375952777913076163275012186591681410271349074209673928791 5...

f(1) = 0.2499999999999999999999999999999999999999995284336715103517
6566288447952278896539736624087359644335652108526603810367036228395181...

f(2) =  5.604195310082044101636720318628045129817288344860074391 74481
2023611043461918362493906704159294864325462895163035604211685... * 10^(-6)

f(3) =  4.715663284896482343371155204772110346026337591264035566434
7891473396189632963771 60... * 10^(-43).
```

As for the integrity of all the above data,

- All the approximations ($f(0), f(1), f(2), f(3)$) above are consistent with D. Bailey's values for the corresponding Mares integrals, to the displayed precision(s). We remind ourselves that D. Bailey's method is somewhat different—he used the original, log-cos scheme, with more powerful routines and machinery to get his extreme-precision values.
- D. Bailey's values above actually imply (recall Mares-integral/probability-density equivalence)

$$0 < \frac{1}{2} - f(0) - 2f(2) - 2f(4) < 10^{-500},$$

$$0 < \frac{1}{4} - f(1) - f(3) < 10^{-500},$$

  all consistent with the sum rules in Theorem 2 and Corollary 2.
- Values for $f(5)$ and beyond have been quite difficult to obtain, and in view of the superexponential decay of f that we have proven, this difficulty makes sense. After all, we know as in-text that $f(2\pi) < 10^{-13680}$. Even with $f(5)$, it is not so much the computational complexity as the severe dynamic range of terms that hampers us. So we are still without a reasonably precise value for $f(5)$ and beyond. We do know from Corollary 1 that $f(5) < 10^{-1000}$, and D. Bailey has already inferred $f(5) < 10^{-500}$ numerically.

- It is fascinating that modern experimental mathematics still has not achieved a closed form for the canonical probability density $f(0) = (2/\pi)M_0$. The continued fraction displayed below shows at least that $f(0)$ is quite close to 1/2, and is calculated from D. Bailey's extreme-precision value above for M_0:

```
{0, 2, 22304, 4, 1, 1, 2, 1, 25, 6, 9, 1, 1, 1, 2, 1, 2, 34, 4, 10, 29, 9, 1,
33, 1, 2, 1, 1, 6, 1, 2, 9, 21, 1, 12, 1, 6, 1, 9, 14, 9, 8, 5, 7, 1, 2, 27,
2, 1, 14, 5, 2, 1, 3, 1, 1, 1, 1, 5, 30, 2, 1, 1, 2, 2, 96, 1, 1, 12, 5, 1,
2, 3, 2, 2, 2, 1, 18, 7, 4, 1, 2, 1, 15, 1, 1, 2, 41, 1, 1, 17, 1, 35, 1,
307, 18, 14, 1, 134, 1, 13, 12, 1, 3, 33, 2, 1, 11, 1, 7, 5, 2, 1, 1, 19, 2,
27, 88, 11, 1, 7, 1, 1, 4, 11, 1, 2, 8, 1, 1, 2, 1, 1, 5, 1, 1, 1, 1, 17, 55,
1, 4, 2, 4, 3, 1, 1, 1, 1, 3, 5, 2, 88, 44, 53, 2, 7, 1, 2, 7, 4, 1, 1, 1,
33, 2, 1, 31, 6, 1, 1, 1, 4, 348, 3, 1, 1, 33, 1, 8, 1, 1, 5, 1, 79, 2, 7, 2,
1, 1, 2, 31, 1, 1, 9, 1, 3, 3, 2, 3, 5, 1, 1, 2, 2, 1, 2, 8, 2, 1, 1, 1, 1,
474, 29, 3, 1, 1, 4, 15, 1, 12, 1, 3, 1, 6, 1, 19, 1, 1, 1, 1, 1, 114, 1, 1,
2, 5, 2, 2, 2, 1, 9, 1, 12, 2, 1, 2, 3, 4, 1, 1, 2, 4, 1, 1, 2, 3, 1, 3, 3,
50, 1, 3, 9, 9, 1, 1, 1, 1, 2, 1, 5, 1, 1, 3, 1, 3, 1, 1, 1, 1, 1, 6, 2, 1,
1, 6, 1, 1, 1, 5, 1, 1, 1, 3, 1, 10, 7, 1, 1, 1, 1, 21, 1, 3, 4, 3, 1, 1, 5,
1, 18, 1, 3, 3, 1, 1, 30, 4, 1, 1, 8, 1, 1, 4, 1, 2, 1, 2, 6, 1, 1, 88, 5, 1,
1, 2, 3, 3, 4, 3, 1, 22, 6, 2, 1, 45, 1, 1, 27, 2, 1, 8, 3, 2, 1, 6, 3, 1, 1,
2, 1, 6, 1, 1, 3, 1, 1, 1, 3, 3, 4, 1, 2, 1, 21, 1, 1, 2, 1, 4, 1, 1, 511, 1,
1, 15, 4, 12, 1, 235, 4, 2, 10, 4, 1, 4, 12, 16, 7, 1, 1, 1, 3, 1, 1, 17, 1,
1, 1, 2, 6, 1, 4, 1, 6, 1, 3, 3, 2, 1, 2, 13, 1, 2, 1, 1, 1, 4, 5, 1, 1, 3,
1, 1, 4, 1, 2, 20, 897, 4, 4, 2, 13, 3, 13, 7, 52, 1, 1, 1, 1, 1, 1, 5, 2, 3,
1, ...}
```

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