A Non-Normality Result

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September 12, 2007

For an integer $b \geq 2$ we say that a real number α is *b-normal* (or *normal base b*) if, for all $m > 0$, every m-long string of digits in the base-b expansion of α appears, in the limit, with frequency b^{-m}.

Consider the constant

$$\alpha = \sum_{n=1}^{\infty} \frac{1}{3^n 2^{3^n}}.$$

This constant was originally proven 2-normal by Stoneham in 1973 [3]. More recently (in 2002) it was proved 2-normal using a different approach by Bailey and Crandall [1], and (in 2006) using a third approach by Bailey and Misieurwicz [2]. Here is a related result:

Theorem 1 *α is* not *6-normal.*

Proof: Let the notation $\{\cdot\}$ denote fractional part. Note that the base-6 digits immediately following position n in the base-6 expansion of α can be obtained by computing $\{6^n \alpha\}$, which can be written as follows:

$$\{6^n \alpha\} = \left\{ \sum_{m=1}^{\lfloor \log_3 n \rfloor} 3^{n-m} 2^{n-3^m} \right\} + \left\{ \sum_{m=\lfloor \log_3 n \rfloor + 1}^{\infty} 3^{n-m} 2^{n-3^m} \right\}.$$

Now note that the first portion of this expression is *zero*, since all terms of the summation are integers. That leaves the second expression.

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Supported in part by the Director, Office of Computational and Technology Research, Division of Mathematical, Information, and Computational Sciences of the U.S. Department of Energy, under contract number DE-AC02-05CH11231.

Consider the case when $n = 3^M$, where $M \geq 1$ is an integer, and examine just the first term of the second summation. We see that this expression is

$$3^{3^M-(M+1)} 2^{3^M-3^{M+1}} \;=\; 3^{3^M-M-1} 2^{-2\cdot 3^M} \;=\; (3/4)^{3^M}/3^{M+1}. \qquad (1)$$

We can generously bound the sum of all terms of the second summation by 1.00001 times this amount, for all $M \geq 1$, and by many times closer to unity for all $M \geq 2$, etc. Thus we have

$$\{6^{3^m}\alpha\} \;\approx\; \frac{\left(\frac{3}{4}\right)^{3^m}}{3^{m+1}},$$

and this approximation is as accurate as one wishes (in ratio) for all sufficiently large m.

Given the very small size of the expression $(3/4)^{3^m}/3^{m+1}$ for even moderate-sized m, it is clear the base-6 expansion will have very long stretches of zeroes beginning at positions $3^m + 1$. For example, by explicitly computing α to high precision, one can produce the following counts of consecutive zeroes Z_m that immediately follow position 3^m in the base-6 expansion of α:

m	3^m	Z_m
1	3	1
2	9	3
3	27	6
4	81	16
5	243	42
6	729	121
7	2187	356
8	6561	1058
9	19683	3166
10	59049	9487

In total, there 14,276 zeroes in these ten segments, which, including the last segment, span the first $59049 + 14276 = 68536$ base-6 digits of α. In this tabulation we have of course ignored the many zeroes in the large "random" segments of the expansion. In any event, the fraction of the first 68,536 digits that are zero is at least $14276/68536 = 0.20839$, which is significantly more than the expected value $1/6 = 0.166666\ldots$.

This reckoning can be made more rigorous by noting that for any $\epsilon > 0$, there is some M such that for all $m > M$ we have

$$3^m[\log_6(3/4) - \epsilon] \;<\; Z_m \;<\; 3^m[\log_6(3/4) + \epsilon]$$

Thus, the fraction of zeroes in the first n base-6 digits of α must, infinitely often, exceed

$$\frac{4}{3} \cdot \frac{\log_6(3/4)}{1 + \log_6(3/4)} \;\approx\; 0.18446111\ldots,$$

which is clearly greater than 1/6. This means that α is not 6-normal.

This result is easily extended to the following:

Theorem 2 *The constant α is* not *b-normal for any base b of the form $2^m 3^n$, where $m \geq n \geq 1$.*

References

[1] David H. Bailey and Richard E. Crandall, "Random Generators and Normal Numbers," *Experimental Mathematics*, vol. 11 (2002), no. 4, pg. 527–546.

[2] David H. Bailey and Michal Misiurewicz, "A Strong Hot Spot Theorem," *Proceedings of the American Mathematical Society*, vol. 134 (2006), no. 9, pg. 2495–2501.

[3] R. Stoneham, "On Absolute (j, epsilon)-Normality in the Rational Fractions with Applications to Normal Numbers," *Acta Arithmetica*, vol. 22 (1973), pg. 277–286.