Solution to Monthly Problem 11410

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The *Monthly* problem #11410 [1] asks to find

$$\lim_{x\to 0}\left(1/2\ln(\cos\phi)+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\sin^2(nx)\sin^2(n\phi)}{n^3x^2}\right)x^{-2}.$$

Let $F(x,\phi)$ denote the value of the function whose limit is to be evaluated, and let $G(\phi)=\lim_{x\to 0}F(x,\phi)$.

We first experimented computationally, using Mathematica, with the expression on the right-hand side and found that for $\phi = k\pi/24$, where $1 \le k \le 11$, the numerical values (to 60 significant digits) of $G(\phi)$ are

0.000722182505041636292457460071443741585418266365564313420145961
0.00299153207185378441209227641568793884286579103160322865736550
0.00714886980224207926652593964919182678586067705192099390194336
0.01388888888888888888888888888888888888888888888888888888888889
0.0245329461033693146089005025756105624316079933061006223635078
0.0416666666666666666666666666666666666666666666666666666666667
0.0707665155173791563728019364204745262731885041536456255992876
0.1250000000000000000000000000000000000000000000000000000000000
0.242851130197757920733474060350808173214139322948079006098057
0.580341801261479548921241056917645394490467542301730104675968
2.40397835587420989272584010093247116970978523617468943861706

These numerical values immediately suggested that $G(\pi/6) = 1/72$, that $G(\pi/4) = 1/24$, and that $G(\pi/3) = 1/8$. With some more advanced computing, using the PSLQ integer relation algorithm, we found that the irrational values

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are roots of the quadratic polynomial $-1+336t-576t^2$, the cubic polynomial $-4+575t-2160t^2-576t^3$, or the quartic polynomial $-1+14405-77184t^2+829440t^3-331776t^4$.

Examining the problem as stated, we then found, using the Fourier series for $\log(2\cos(\phi/2))$, that the function F can be written as

$$\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{\sin^2(nx)}{n^2x^2}-1\right)\frac{\sin^2(n\phi)}{n^2x^2}.$$

Maple 12 was unable to evaluate this as written, but after applying the substitution $\sin^2\theta=(1-\cos(2\theta))/2$, it was able to evaluate the sum in terms of logarithms and trilogarithms and reduce the entire limit expression to

$$G(\phi) \quad = \quad -1/12+\frac{1}{6\cos(2\phi)+6},$$

which then can be further simplified to

$$G(\phi) \quad = \quad \frac{\tan^2\phi}{24}.$$

As a check, we computed the values of this expression numerically for $\phi=k\pi/24$, where $1\le k\le 11$, and found that they agreed with the values given above to 60-digit accuracy.

References

[1] Omran Kouba, "Problem 11410," *American Mathematical Monthly*, vol. 116, no. 1 (Jan 2009), pg. 83.