Solution to Monthly Problem #11418

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July 6, 2009

Problem 11418 asks to evaluate (for complex $|a| > 1$)

$$J := \int_{-\infty}^{\infty} \frac{t^2 \operatorname{sech}^2(t)}{a - \tanh(t)}\, dt.$$

A variable change of $x = \tanh(t)$ produce for $|a| > 1$ that

$$J = \int_{-1}^{1} \frac{\operatorname{arctanh}^2(x)}{a - x}\, dx = \frac{1}{12} \ln^3\left(\frac{a+1}{a-1}\right) + \frac{\pi^2}{12} \ln\left(\frac{a+1}{a-1}\right). \tag{1}$$

The corresponding *Maple 12* code that obtained this is

```
J1:=a->int(t^2*sech(t)^2/(a-tanh(t)),t=-infinity..infinity): J1(a)
assuming a>1;
     / 2             2
limit|t  ln(a + 1) - t  ln(a + 1 + exp(2 t) a - exp(2 t))
     \

            /     (a - 1) exp(2 t)\   1        /     (a - 1) exp(2 t)\
   - t polylog|2, - ----------------| + - polylog|3, - ----------------|
            \          a + 1      /   2        \          a + 1      /

      2                                                  1
   + t  ln(exp(2 t) + 1) + t polylog(2, -exp(2 t)) - - polylog(3, -exp(2 t)),
                                                         2


                  \
  t = infinity, left|
                  /
J2:=simplify(student[changervar](x=tanh(t),J1(a),x)) assuming
a>1:J2;
```

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```
  1    2             1    2             1                   2
- -- Pi  ln(a - 1) + -- Pi  ln(a + 1) - - ln(a - 1) ln(a + 1)
  12                 12                 4

     1          2             1           3   1           3
   + - ln(a - 1)  ln(a + 1) - -- ln(a - 1)  + -- ln(a + 1)
     4                        12              12
```

The newer syntax of

```
\texttt{[IntegrationTools](Change(J1(a),x=tanh(t)^2)}
```

fails to produce the desired evaluation in (1). It returns the correct but less helpful polylogarithmic limit above.

The evaluation would appear to be valid except when $-1 < a < 1$.

A human proof can be obtained from (1) on using the geometric series and integrating term-by-term carefully—which is much easier once the answer is known.

JMB 12.03.2009