Solution to Monthly Problem 11515

David H. Bailey* Jonathan M. Borwein†

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The *Monthly* problem #11515 asks to evaluate

$$S(t) \quad = \quad \sum_{n=1}^{\infty} 4^n \sin^4(t/2^n).$$

Since the sum converges so rapidly, it is a simple matter to numerically evaluate this function for various values of t, say using *Mathematica* or *Maple*. For instance, we quickly found that $S(\pi) = \pi^2$, correct to 100 digits. Further experimentation yielded, within just a minute or two, the conjecture that $S(t) = T(t)$, where

$$T(t) \quad = \quad t^2 - \sin^2 t.$$

This conjecture can be established as follows: First note that both $S(t)$ and $T(t)$ satisfy the same recursion:

$$\begin{aligned} S(2t) - 4S(t) &= 4\sin^4 t \\ T(2t) - 4T(t) &= 4\sin^4 t. \end{aligned}$$

The identity for the S function follows by noting that the RHS is merely the first term of the summation for $S(2t)$ (the remaining terms cancel). The identity for the T function follows by simple trigonometry.

We now reason as follows. First of all, the identity $S(0) = T(0)$ is trivially true. Now given some $\theta \neq 0$, we have, by the above identity,

$$\begin{aligned} S(\theta) - T(\theta) &= S(\theta/2) - T(\theta/2) = S(\theta/4) - T(\theta/4) = S(\theta/8) - T(\theta/8) \cdots \\ &= S(0) - T(0) = 0, \end{aligned}$$

by the continuity of the two functions $S(t)$ and $T(t)$ at zero. The continuity of $S(t)$ at zero follows from the Weierstrass M-test, since on the interval $[-1, 1]$ we have $|4^n \sin^4(t/2^n)| \leq 4^n/16^n = 1/4^n$, which is summable.

*Lawrence Berkeley National Laboratory, Berkeley, CA 94720, dhbailey@lbl.gov. Supported in part by the Director, Office of Computational and Technology Research, Division of Mathematical, Information, and Computational Sciences of the U.S. Department of Energy, under contract number DE-AC02-05CH11231.

†School of Mathematical and Physical Sciences, University of Newcastle, Callaghan, NSW 2308, Australia, jonathan.borwein@newcastle.edu.au. Supported in part by the Australian Research Council.

We note in passing that the proof is greatly facilitated by our "knowing" the result by prior experimentation. Indeed, the proof is very much in the spirit of the Wilf-Zeilberger algorithm, which is discussed at length in [1].

References

[1] Marko Petkovsek, Herbert S. Wilf, Doron Zeilberger, $A = B$, AK Peters, Natick, NH, 1996.