A Compendium of BBP-Type Formulas for Mathematical Constants

David H. Bailey*

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Abstract

A 1996 paper by the author, Peter Borwein and Simon Plouffe showed that any mathematical constant given by an infinite series of a certain type has the property that its n-th digit in a particular number base could be calculated directly, without needing to compute any of the first $n-1$ digits, by means of a simple algorithm that does not require multiple-precision arithmetic. Several such formulas were presented in that paper, including formulas for the constants π and $\log 2$. Since then, numerous other formulas of this type have been found. This paper presents a compendium of currently known results of this sort, both proven and conjectured. Experimentally obtained results which are not yet proven have been checked to high precision and are marked with a $\stackrel{?}{=}$. Fully established results are as indicated in the citations and references below.

1 Introduction

This is a collection of formulas for various mathematical constants that are of the form similar to that first noted in the "BBP" paper [13]. That article presented the following formula for π (which was discovered using Ferguson's PSLQ integer relation finding algorithm [21, 14]):

$$\pi = \sum_{k=0}^{\infty} \frac{1}{16^k} \left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6} \right). \tag{1}$$

It was shown in [13] that this formula permits one to calculate the n-th hexadecimal or binary digit of π, without computing any of the first $n-1$ digits, by means of a simple algorithm that does not require multiple-precision arithmetic. A more recent paper by

*Lawrence Berkeley National Laboratory, Berkeley, CA 94720, USA, dhbailey@lbl.gov. Supported in part by the Director, Office of Computational and Technology Research, Division of Mathematical, Information, and Computational Sciences of the U.S. Department of Energy, under contract number DE-AC02-05CH11231.

Borwein, Galway and Borwein showed that there are no other degree-1 BBP-type for π, except those, such as Formula 1, whose "base" is a power of two. However, as we shall see below, in the case of π^2 there is both a base-2 formula (see Formula 26) and a base-3 formula (see Formula 73).

Further, as shown in [13], several other well-known constants also have this individual digit-computation property. One of these is $\log 2$, based on the following centuries-old formula:

$$\log 2 \quad = \quad \sum_{k=1}^{\infty} \frac{1}{k 2^k}. \tag{2}$$

In general, any constant C that can be written in the form

$$C \quad = \quad \sum_{k=0}^{\infty} \frac{p(k)}{q(k) b^k},$$

where p and q are integer polynomials, $\deg(p) < \deg(q)$, and $q(k)$ is nonzero for nonnegative k, possesses this individual digit-computation property. Note that Formula 1 can be written in this form, since the four fractions can be combined into one, yielding

$$\pi \quad = \quad \sum_{k=0}^{\infty} \frac{47 + 151k + 120k^2}{16^k (15 + 194k + 712k^2 + 1024k^3 + 512k^4)}.$$

Since the publication of [13], other papers have presented formulas of this type for various constants, including several constants that arise in quantum field theory [18, 19, 15]. More recently, interest in BBP-type formulas has been heightened by the observation that the question of the statistical randomness of the digit expansions of these constants can be reduced to the following hypothesis regarding the behavior of a particular class of chaotic iterations [15]:

Hypothesis A (from the paper [15]). Denote by $r_n = p(n)/q(n)$ a rational-polynomial function, i.e. $p, q \in Z[X]$. Assume further that $0 \le \deg p < \deg q$, with r_n nonsingular for positive integers n. Choose an integer $b \ge 2$ and initialize $x_0 = 0$. Then the sequence $x = (x_0, x_1, x_2, \ldots)$ determined by the iteration

$$x_n \quad = \quad (b x_{n-1} + r_n) \bmod 1$$

either has a finite attractor or is equidistributed in $[0, 1)$.

Assuming this hypothesis, it is shown in [15] that any BBP-type constant is either normal to base b (i.e., any n-long string digits appears in the base b expansion with limiting frequency b^{-n}), or else it is rational. No proof of Hypothesis A was presented in [15], and indeed it is likely that Hypothesis A is rather difficult to prove. However, it should be emphasized that even particular instances of Hypothesis A, if established, would have interesting consequences. For example, if it could be established that the specific iteration given by $x_0 = 0$, and

$$x_n \quad = \quad (2 x_{n-1} + \frac{1}{n}) \bmod 1$$

is equidistributed in $[0, 1)$, then it would follow that $\log 2$ is normal to base 2. In a similar vein, if it could be established that the iteration given by $x_0 = 0$ and

$$x_n = \left(16x_{n-1} + \frac{120n^2 - 89n + 16}{512n^4 - 1024n^3 + 712n^2 - 206n + 21}\right) \bmod 1$$

is equidistributed in $[0, 1)$, then it would follow that π is normal to base 16 (and thus to base 2 also).

One additional impetus for the study of BPP-type constants comes from a recent paper by Lagarias [24], who demonstrates a connection to G-functions and to a conjecture of Furstenberg from ergodic theory. Lagarias' analysis suggests that there may be a special signficance to constants that have BBP-type formulas in two or more bases — say both a base 2 and a base 3 formula.

This paper is a compendium of the growing set of BBP-type formulas that have been found by various researchers. Part of these formulas are collected here from previously published sources. In other cases, formulas whose existence has been demonstrated in the literature are presented here explicitly for the first time. Still others are new, having been found using the author's PSLQ program [14] in the course of this research.

The PSLQ integer relation algorithm [21] or one of its variants [14] can be used to find formulas such as those listed in this paper as follows. Suppose, for example, that it is conjectured that a given constant α satisfies a BBP-type formula of the form

$$\alpha = \frac{1}{r}\sum_{k=0}^{\infty} \frac{1}{b^k}\left(\frac{a_1}{(kn+1)^s} + \frac{a_2}{(kn+2)^s} + \cdots + \frac{a_n}{(kn+n)^s}\right),$$

where r and a_k are unknown integers, for a specified selection of the parameters b, s and n. To apply PSLQ, first calculate the vector $(\sum_{k\ge 0} 1/(b^k(kn+j)^s),\ 1 \le j \le n)$, as well as α itself, to very high precision, then use this $(n+1)$-long vector (including α at the end) as input to a PSLQ program. If a solution vector (a_j) is found with sufficiently high numerical fidelity, then

$$\alpha = \frac{-1}{a_{n+1}}\sum_{k=0}^{\infty} \frac{1}{b^k}\left(\frac{a_1}{(kn+1)^s} + \frac{a_2}{(kn+2)^s} + \cdots + \frac{a_n}{(kn+n)^s}\right)$$

(at least to the level of numeric precision used).

This compendium is not intended to be a comprehensive listing of all such formulas. In most cases a formula is not listed here if it is merely

1. a telescoping sum.

2. a formal rewriting of another formula on the list.

3. a straightforward formal manipulation starting with another formula in the list.

4. an integer linear combination of two or more formulas already in the list.

Item 1 refers to a summation such as

$$S = \sum_{k=1}^{\infty} \frac{1}{b^k} \left(\frac{b^2}{k} - \frac{1}{k+2} \right),$$

which, if split into two summations, has the property that the terms of the first series cancel with offset terms of the second series, so that S reduces to a rational number (in this example, $S = b + 1/2$). Item 2 refers to the fact that a formula with base b and length n can be rewritten as a formula with base b^r and length rn. Item 4 refers to the fact that the rational linear sum of two BBP series can, in many cases, be written as a single BBP series. This is clear if the two individual series have the same base b. If one has base b^r and the other has base b^s, their sum can be written as a single BBP series with base $b^{\mathrm{lcm}(r,s)}$ [15]. Along this line, many of the formulas listed below possess variants that can be obtained by adding to the listed formula a rational multiple of one of the zero relations listed in Section 11.

The formulas are listed below using a notation introduced in [15]:

$$P(s, b, n, A) = \sum_{k=0}^{\infty} \frac{1}{b^k} \sum_{j=1}^{n} \frac{a_j}{(kn+j)^s}. \qquad (3)$$

where s, b and n are integers, and $A = (a_1, \ a_2, \ \cdots, \ a_n)$ is a vector of integers. For instance, using this notation we can write formulas 1 and 2 more compactly as follows:

$$\pi = P(1, 16, 8, (4, 0, 0, -2, -1, -1, 0, 0)) \qquad (4)$$

$$\log 2 = \frac{1}{2} P(1, 2, 1, (1)). \qquad (5)$$

In most cases below, the representation shown using this notation is a translation from the original source. Also, in some cases the formula listed here is not precisely the one mentioned in the cited reference — an equivalent one is listed here instead — but the original discoverer is given due credit. In cases where the formula has been found experimentally (i.e., by using the PSLQ integer relation finding algorithm), and no formal proof is available, the relation is listed here with the $\stackrel{?}{=}$ notation instead of an equal sign.

The P notation formulas listed below have been checked using a computer program that parses the LaTeX source of this document, then computes the left-hand and right-hand sides of these formulas to 2000 decimal digit accuracy.

Additional contributions to this compendium are welcome — please send a note to the author at `dhbailey@lbl.gov`.

2 Logarithm formulas

Clearly $\log n$ can be written with a binary BBP formula (i.e. a formula with $b = 2^m$ for some integer m) provided n factors completely using primes whose logarithms have binary BBP formulas — one merely combines the individual series for the different primes into a single binary BBP formula. We have seen above that $\log 2$ possesses a binary BBP formula, and so does the $\log 3$, by the following reasoning:

$$
\begin{aligned}
\log 3 &= 2\log 2 + \log\left(1 - \frac{1}{4}\right) \;=\; 2\sum_{k=1}^{\infty}\frac{1}{k2^k} - \sum_{k=1}^{\infty}\frac{1}{k4^k} \\
&= \frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{4^k}\left(\frac{2}{2k+1} + \frac{1}{2k+2}\right) - \frac{1}{4}\sum_{k=0}^{\infty}\frac{1}{4^k}\left(\frac{2}{2k+2}\right) \\
&= \sum_{k=0}^{\infty}\frac{1}{4^k}\left(\frac{1}{2k+1}\right) \;=\; P(1,4,2,(1,0)). \qquad (6)
\end{aligned}
$$

In a similar manner, one can show, by examining the factorization of 2^n+1 and 2^n-1, where n is an integer, that numerous other primes have this property. Harley [22] further extended this list of primes by writing

$$
\mathrm{Re}\left(\log\left(1 \pm \frac{1+i}{2^n}\right)\right) \;=\; \left(\frac{1}{2} - n\right)\log 2 + \frac{1}{2}\log(2^{2n-1} \pm 2^n + 1),
$$

where Re denotes the real part. He first noted that the Taylor series of the left-hand side can be written as a binary BBP-type formula. He then applied Aurefeuille's factorization formula

$$
2^{4n-2} + 1 \;=\; (2^{2n-1} + 2^n + 1)(2^{2n-1} - 2^n + 1)
$$

to the right-hand side. More recently, Jonathan Borwein has observed that both of these sets of results can be derived by working with the single expression

$$
\mathrm{Re}\left(\log\left(1 \pm \frac{(1+i)^k}{2^n}\right)\right).
$$

A preliminary list of primes p such that $\log p$ has a binary BBP formula was given in [13]. This list has now been augmented by the author to the following:

$$
\begin{aligned}
&2,\ 3,\ 5,\ 7,\ 11,\ 13,\ 17,\ 19,\ 29,\ 31,\ 37,\ 41,\ 43,\ 61,\ 73,\ 109,\ 113,\ 127,\ 151,\\
&241,\ 257,\ 337,\ 397,\ 683,\ 1321,\ 1613,\ 2113,\ 2731,\ 5419,\ 8191, 43691,\ 61681,\\
&87211,\ 131071,\ 174763,\ 262657,\ 524287,\ 2796203, 15790321,\ 18837001,\\
&22366891,\ 4278255361,\ 4562284561,\ 2932031007403,\ 4363953127297,\\
&4432676798593. \qquad (7)
\end{aligned}
$$

This list is certainly not complete, and it is unknown whether or not all primes have this property, or even whether the list of such primes is finite or infinite. The actual formulas for $\log p$ for the primes above are generally straightforward to derive and are not shown here.

One can also obtain BBP formulas in non-binary bases for the logarithms of certain integers and rational numbers. One example is given by the base ten formula 81 below, which was used in [13] to compute the ten billionth decimal digit of $\log(9/10)$.

3 Arctangent formulas

Shortly after the original BBP paper appeared in 1996, Adamchik and Wagon observed that [11]

$$\arctan 2 = \frac{1}{8} P(1, 16, 8, (8, 0, 4, 0, -2, 0, -1, 0)). \tag{8}$$

More recently, binary BBP formulas have been found for $\arctan q$ for a large set of rational numbers q. These experimental results, which were obtained by the author using the PSLQ program, coincide exactly in the cases studied so far with the set of rationals given by $q = |\mathrm{Im}(T)/\mathrm{Re}(T)|$ or $|\mathrm{Re}(T)/\mathrm{Im}(T)|$, where

$$T = \prod_{k=1}^{m} \left(1 \pm \frac{(1+i)^{u_k}}{2^{v_k}}\right)^{w_k}. \tag{9}$$

The arctangents of these q clearly possess binary BBP formulas, because $\mathrm{Im}(\log T)$ decomposes into a linear sum of terms, the Taylor series of which are binary BBP formulas. The author is indebited to Jonathan Borwein for this observation. See also [16, pg. 344]. Alternatively, one can write Formula 9 as

$$T = \prod_{k=1}^{m} \left(1 \pm \frac{i}{2^{t_k}}\right)^{u_k} \left(1 \pm \frac{1+i}{2^{v_k}}\right)^{w_k} \tag{10}$$

for various m-long nonnegative integer vectors $t,\ u,\ v,\ w$ and choices of signs as shown. For example, setting $t = (1,1),\ u = (1,1),\ v = (1,3),\ w = (1,1)$, with signs $(1,-1,-1,1)$, gives the result $T = 25/32 - 5i/8$, which yields $q = 4/5$. Indeed, one can obtain the formula

$$\begin{aligned}\arctan\left(\frac{4}{5}\right) &= \frac{1}{2^{17}} P(1, 2^{20}, 40, (0, 2^{19}, 0, -3 \cdot 2^{17}, -15 \cdot 2^{15}, 0, 0, 5 \cdot 2^{15}, 0, 2^{15}, 0, \\ &\quad -3 \cdot 2^{13}, 0, 0, 5 \cdot 2^{10}, 5 \cdot 2^{11}, 0, 2^{11}, 0, 2^{10}, 0, 0, 0, 5 \cdot 2^{7}, 15 \cdot 2^{5}, 128, 0, \\ &\quad -96, 0, 0, 0, 40, 0, 8, -5, -6, 0, 0, 0, 0)).\end{aligned} \tag{11}$$

In this manner, it can be seen that binary BBP formulas exist for the arctangents of the following rational numbers. Only those rationals with numerators < denominators ≤

50 are listed here:

$$1/2,\ 1/3,\ 2/3,\ 1/4,\ 3/4,\ 1/5,\ 2/5,\ 3/5,\ 4/5,\ 1/7,\ 3/7,\ 4/7,\ 6/7,$$
$$1/8,\ 7/8,\ 1/9,\ 2/9,\ 7/9,\ 8/9,\ 3/10,\ 2/11,\ 3/11,\ 7/11,\ 8/11,\ 10/11,$$
$$1/12,\ 5/12,\ 1/13,\ 6/13,\ 7/13,\ 9/13,\ 11/13,\ 3/14,\ 5/14,\ 1/15,\ 4/15,$$
$$8/15,\ 1/16,\ 11/16,\ 13/16,\ 15/16,\ 1/17,\ 6/17,\ 7/17,\ 11/17,\ 15/17,$$
$$16/17,\ 1/18,\ 13/18,\ 4/19,\ 6/19,\ 7/19,\ 8/19,\ 9/19,\ 11/19,\ 17/19,$$
$$1/21,\ 16/21,\ 3/22,\ 7/22,\ 9/22,\ 19/22,\ 2/23,\ 4/23,\ 6/23,\ 7/23,$$
$$11/23,\ 14/23,\ 15/23,\ 7/24,\ 11/24,\ 23/24,\ 13/25,\ 19/25,\ 21/25,$$
$$7/26,\ 23/26,\ 5/27,\ 11/27,\ 2/29,\ 3/29,\ 15/29,\ 17/29,\ 24/29,\ 28/29,$$
$$17/30,\ 1/31,\ 5/31,\ 8/31,\ 12/31,\ 13/31,\ 17/31,\ 18/31,\ 22/31,\ 27/31,$$
$$1/32,\ 9/32,\ 31/32,\ 1/33,\ 4/33,\ 10/33,\ 14/33,\ 19/33,\ 31/33,\ 32/33,$$
$$7/34,\ 27/34,\ 13/35,\ 25/36,\ 5/37,\ 9/37,\ 10/37,\ 16/37,\ 29/37,\ 36/37,$$
$$1/38,\ 5/38,\ 13/38,\ 21/38,\ 20/39,\ 23/39,\ 37/39,\ 9/40,\ 3/41,\ 23/41,$$
$$27/41,\ 28/41,\ 38/41,\ 11/42,\ 19/42,\ 37/42,\ 6/43,\ 19/43,\ 23/43,$$
$$32/43,\ 33/43,\ 7/44,\ 23/44,\ 27/44,\ 3/46,\ 9/46,\ 17/46,\ 35/46,\ 37/46,$$
$$1/47,\ 13/47,\ 14/47,\ 16/47,\ 19/47,\ 27/47,\ 19/48,\ 3/49,\ 8/49,\ 13/49,$$
$$18/49,\ 31/49,\ 37/49,\ 43/49,\ 29/50,\ 49/50. \qquad (12)$$

It was recently noted in a paper by Borwein, Galway and Borwein [17], which includes a significant structure theory for BBP-type arctangent formulas, that $\arctan(1/6) = \arctan(1/5) - \arctan(1/31)$, so that $\arctan(1/6)$ also has a binary BBP formula. Similarly, $\arctan(5/6) = \arctan(1) - \arctan(1/5) + \arctan(1/9) - \arctan(1/255)$, and $\arctan(1/11) = \arctan(1) - \arctan(5/6)$. By performing PSLQ over the set of arctangents of the above list, augmented by 1/6 and 5/6, one finds that most are linearly dependent on the others. Indeed, by eliminating redundant elements, one can reduce the list to the following set whose arctangents appear to be linearly independent:

$$1/2,\ 1/3,\ 2/3,\ 1/4,\ 2/5,\ 4/5,\ 1/6,\ 5/6,\ 7/8,\ 3/10,\ 4/15,\ 1/16,\ 6/19,$$
$$32/33,\ 13/38. \qquad (13)$$

From the basis set (13), by performing PSLQ, one can find relations for all of the rationals in the list (12), and, in addition, relations for the following additional rationals:

$$5/7,\ 1/11,\ 4/13,\ 7/16,\ 4/17,\ 9/17,\ 3/19,\ 9/20,\ 13/21,\ 15/21,$$
$$20/21,\ 21/22,\ 9/23,\ 10/23,\ 2/25,\ 8/26,\ 19/26,\ 8/27,\ 14/27,\ 19/27,$$
$$23/27,\ 3/28,\ 17/28,\ 11/29,\ 14/29,\ 26/29,\ 25/31,\ 7/32,\ 13/33,$$
$$19/34,\ 12/35,\ 19/35,\ 11/36,\ 31/36,\ 20/37,\ 22/37,\ 9/38,\ 31/38,$$
$$2/39,\ 25/39,\ 13/40,\ 1/41,\ 13/41,\ 24/41,\ 27/41,\ 31/42,\ 1/43,\ 2/43,$$
$$15/43,\ 36/43,\ 35/44,\ 7/45,\ 11/45,\ 41/45,\ 43/46,\ 23/47,\ 25/47,$$
$$29/47,\ 29/48,\ 2/49,\ 10/49,\ 1/50,\ 41/50. \qquad (14)$$

Even after merging our list (12) by this new list (14), note that not all "small" rationals appear. For instance, it is not known whether arctan(2/7) possesses a binary BBP formula. On the other hand, Kunle Adegoke, Jaume Oliver Lafont and Olawanle Layeni have shown that arctan(5/36) has a binary BBP-type formula [10].

One can obtain BBP formulas in non-binary bases for the arctangents of certain rational numbers by employing appropriate variants of Formulas 9 and 10.

4 Other degree 1 binary formulas

We present here some additional degree 1 binary BBP-type formulas (in other words, in the P notation defined in equation 3 above, $s = 1$, and $b = 2^m$ for some integer $m > 0$). Here $\phi = (1 + \sqrt{5})/2$ is the golden mean.

$$\pi = \frac{1}{4} P(1, 16, 8, (8, 8, 4, 0, -2, -2, -1, 0)) \tag{15}$$

$$\pi = P(1, -4, 4, (2, 2, 1, 0)) \tag{16}$$

$$\pi\sqrt{2} = \frac{1}{8} P(1, 64, 12, (32, 0, 8, 0, 8, 0, -4, 0, -1, 0, -1, 0)) \tag{17}$$

$$\pi\sqrt{3} = \frac{9}{32} P(1, 64, 6, (16, 8, 0, -2, -1, 0)) \tag{18}$$

$$\begin{aligned}\pi\sqrt{5} = {} & \frac{5}{2^{19}} P(1, 2^{20}, 40, (2^{19}, 2^{19}, -2^{18}, 0, 0, 2^{17}, 2^{16}, 0, 2^{15}, 0, \\ & 2^{14}, 0, 2^{13}, 2^{13}, 0, 0, -2^{11}, 2^{11}, 2^{10}, 0, -2^9, -2^9, 2^8, 0, 0, \\ & -2^7, -2^6, 0, -2^5, 0, -2^4, 0, -2^3, -2^3, 0, 0, 2, -2, -1, 0))\end{aligned} \tag{19}$$

$$\sqrt{2}\log(1 + \sqrt{2}) = \frac{1}{8} P(1, 16, 8, (8, 0, 4, 0, 2, 0, 1, 0)) \tag{20}$$

$$\sqrt{2}\tan^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{8} P(1, 16, 8, (8, 0, -4, 0, 2, 0, -1, 0)) \tag{21}$$

$$\tan^{-1}\phi = \frac{1}{16} P(1, 16, 8, (8, 16, 4, 0, -2, -4, -1, 0)) \tag{22}$$

$$\tan^{-1}\phi^3 = \frac{1}{8} P(1, 16, 8, (8, 4, 4, 0, -2, -1, -1, 0)) \tag{23}$$

$$\tan^{-1}\phi^5 = \frac{1}{16} P(1, 16, 8, (8, 32, 4, 0, -2, -8, -1, 0)) \tag{24}$$

$$\begin{aligned}\tan^{-1}\phi^7 = {} & \frac{3}{2^{12}} P(1, 2^{12}, 24, (2^{11}, 0, 2^{10}, 0, -2^9, -2^{10}, -2^8, 0, 2^7, 0, \\ & 2^6, 0, -2^5, 0, -2^4, 0, 2^3, 2^4, 2^2, 0, -2, 0, -1, 0))\end{aligned} \tag{25}$$

Formula 15 was first found by Ferguson [21], while 16, which is the alternating sign equivalent of 15, was found independently by Hales and by Adamchik and Wagon [11]. Technically speaking, these formulas can be obtained from the original BBP formula for π (formula 1) by adding 1/4 times relation 108 of Section 12, but they are included here for historical interest, since their discovery predated the discovery of relation 108. Formula

17 appeared in [13]. Formulas 18, 20 and 21 are due to Knuth [23, pg. 628]. Formulas 19 and 22–25 were found by Kunle Adegoke [8, 9].

5 Degree 2 binary formulas

Here are some degree 2 binary formulas (i.e., $s = 2$, and $b = 2^m$ for some integer $m > 0$). The constant G here is Catalan's constant, namely $G = 1 - 1/3^2 + 1/5^2 - 1/7^2 + \cdots = 0.9159655941\ldots$:

$$\pi^2 = P(2, 16, 8, (16, -16, -8, -16, -4, -4, 2, 0)) \tag{26}$$

$$\pi^2 = \frac{9}{8} P(2, 64, 6, (16, -24, -8, -6, 1, 0)) \tag{27}$$

$$\log^2 2 = \frac{1}{6} P(2, 16, 8, (16, -40, -8, -28, -4, -10, 2, -3)) \tag{28}$$

$$\log^2 2 = \frac{1}{32} P(2, 64, 6, (64, -160, -56, -40, 4, -1)) \tag{29}$$

$$G - \frac{1}{8}\pi \log 2 = \frac{1}{16} P(2, 16, 8, (8, 8, 4, 0, -2, -2, -1, 0)) \tag{30}$$

$$\begin{aligned} \pi \log 2 = {} & \frac{1}{256} P(2, 2^{12}, 24, (2^{12}, -2^{13}, -51 \cdot 2^9, 15 \cdot 2^{10}, -2^{10}, 39 \cdot 2^8, 0, \\ & 45 \cdot 2^8, 37 \cdot 2^6, -2^9, 0, 3 \cdot 2^8, -64, 0, 51 \cdot 2^3, 45 \cdot 2^4, 16, 196, 0, \\ & 60, -37, 0, 0, 0)) \end{aligned} \tag{31}$$

$$\begin{aligned} \pi\sqrt{3} \log 2 = {} & \frac{1}{128} P(2, 2^{12}, 24, (9 \cdot 2^9, -27 \cdot 2^9, -9 \cdot 2^{11}, 27 \cdot 2^9, 0, 81 \cdot 2^7, \\ & 9 \cdot 2^6, 45 \cdot 2^8, 9 \cdot 2^8, 0, 0, 9 \cdot 2^6, -72, -216, 9 \cdot 2^5, 9 \cdot 2^6, 0, 162, \\ & -9, 72, -36, 0, 0, 0)) \end{aligned} \tag{32}$$

$$\begin{aligned} \pi\sqrt{3} \log 2 = {} & \frac{9}{2^{10}} P(2, 2^{12}, 24, (2^{11}, -3 \cdot 2^{11}, 0, -2^{11}, 2^9, 0, 2^8, 2^{10}, 0, 3 \cdot 2^7, 2^6, 0, \\ & -2^5, -3 \cdot 2^5, 0, -2^6, -2^3, 0, -2^2, 2^3, 0, 6, -1, 0)) \end{aligned} \tag{33}$$

$$\begin{aligned} \sqrt{3}\, \mathrm{Cl}_2\left(\frac{\pi}{3}\right) = {} & \frac{9}{5 \cdot 2^{10}} P(2, 2^{12}, 24, (2^{11}, -2^{12}, 0, -2^{10}, 2^9, 0, 2^8, 3 \cdot 2^8, 0, 2^8, 2^6, \\ & 0, -2^5, -2^6, 0, -3 \cdot 2^4, -2^3, 0, -2^2, 2^2, 0, 2^2, -1, 0)) \end{aligned} \tag{34}$$

$$\begin{aligned} G = {} & \frac{1}{2^{10}} P(2, 2^{12}, 24, (2^{10}, 2^{10}, -2^9, -3 \cdot 2^{10}, -256, -2^{11}, -256, \\ & -9 \cdot 2^7, -5 \cdot 2^6, 64, 64, 0, -16, 64, 8, -72, 4, -8, 4, -12, 5, \\ & 4, -1, 0)) \end{aligned} \tag{35}$$

$$\begin{aligned} G = {} & \frac{3}{2^{12}} P(2, 2^{12}, 24, (2^{11}, -2^{11}, -2^{11}, 0, -2^9, -2^{10}, -2^8, 0, -2^8, -2^7, \\ & 2^6, 0, -2^5, 2^5, 2^5, 0, 2^3, 2^4, 2^2, 0, 2^2, 2, -1, 0)) \end{aligned} \tag{36}$$

Formulas 26, 27, 29 and 30 were presented in [13] (although 30 appeared in a 1909 book by Nielsen [25, pg. 105]). Formulas 28 and 32 were found experimentally by the author, using the PSLQ program. Formula 28 was subsequently proved by Kunle Adegoke [4]. Formulas for $\pi \log 2$ and G were first derived by Broadhurst, although the specific explicit formulas given here (31 and 35) were found experimentally by the author using PSLQ. Formula 36 was found by Gery Huvent. Formulas 33 and 34 appeared in [10]. Here Cl_2 is a Clausen function (see section 9). Formula 35 = Formula 36 $+5/2^{11}\times$ Relation 120 $+1/2^{12}\times$ Relation 121. Formula 32 = Formula 33 $+27/2^9\times$ Relation 120 $-9/2^{10}\times$ Relation 121. Formula 31 = $8\times$ (Formula 36 $-$ Formula 30) $-1/2^6\times$ Relation 120 $-1/2^8\times$ Relation 121.

6 Degree 3 binary formulas

$$
\begin{aligned}
\zeta(3) &= \frac{1}{7\cdot 2^8}P(3, 2^{12}, 24, (3\cdot 2^{11}, -21\cdot 2^{11}, 3\cdot 2^{13}, 15\cdot 2^{11}, -3\cdot 2^9, 3\cdot 2^{10}, \\
&\quad 3\cdot 2^8, 0, -3\cdot 2^{10}, -21\cdot 2^7, -192, -3\cdot 2^9, -96, -21\cdot 2^5, -3\cdot 2^7, 0, \\
&\quad 24, 48, -12, 120, 48, -42, 3, 0)) \qquad (37) \\
\log^3 2 &= \frac{1}{256}P(3, 2^{12}, 24, (0, 3\cdot 2^{13}, -27\cdot 2^{12}, 3\cdot 2^{14}, 0, 93\cdot 2^9, 0, 3\cdot 2^{14}, 27\cdot 2^9, \\
&\quad 3\cdot 2^9, 0, 75\cdot 2^6, 0, 3\cdot 2^7, 27\cdot 2^6, 3\cdot 2^{10}, 0, 93\cdot 2^3, 0, 192, -216, \\
&\quad 24, 0, 3)) \qquad (38) \\
\pi^2\log 2 &= \frac{1}{32}P(3, 2^{12}, 24, (0, 9\cdot 2^{11}, -135\cdot 2^9, 9\cdot 2^{11}, 0, 99\cdot 2^8, 0, 27\cdot 2^{10}, 135\cdot 2^6, \\
&\quad 9\cdot 2^7, 0, 45\cdot 2^6, 0, 9\cdot 2^5, 135\cdot 2^3, 27\cdot 2^6, 0, 396, 0, 72, -135, 18, 0, 0)) \qquad (39) \\
\pi\log^2 2 &= \frac{1}{2^{56}}P(3, 2^{60}, 120, (7\cdot 2^{59}, -37\cdot 2^{60}, -63\cdot 2^{58}, 85\cdot 2^{59}, 3861\cdot 2^{56}, \\
&\quad -3357\cdot 2^{55}, 0, -655\cdot 2^{58}, 347\cdot 2^{54}, 79\cdot 2^{53}, 0, 4703\cdot 2^{52}, -7\cdot 2^{53}, 0, \\
&\quad -1687\cdot 2^{52}, -655\cdot 2^{54}, 7\cdot 2^{51}, -4067\cdot 2^{49}, 0, -6695\cdot 2^{48}, -347\cdot 2^{48}, \\
&\quad 0, 0, -7375\cdot 2^{46}, -3861\cdot 2^{46}, -37\cdot 2^{48}, -63\cdot 2^{46}, 85\cdot 2^{47}, -7\cdot 2^{45}, \\
&\quad -933\cdot 2^{45}, 0, -655\cdot 2^{46}, 347\cdot 2^{42}, -37\cdot 2^{44}, 875\cdot 2^{43}, 4703\cdot 2^{40}, \\
&\quad -7\cdot 2^{41}, 0, 63\cdot 2^{40}, -3105\cdot 2^{38}, 7\cdot 2^{39}, -4067\cdot 2^{37}, 0, 85\cdot 2^{39}, 441\cdot 2^{39}, \\
&\quad 0, 0, -7375\cdot 2^{34}, 7\cdot 2^{35}, 79\cdot 2^{33}, -63\cdot 2^{34}, 85\cdot 2^{35}, -7\cdot 2^{33}, \\
&\quad -3357\cdot 2^{31}, -875\cdot 2^{33}, -655\cdot 2^{34}, 347\cdot 2^{30}, -37\cdot 2^{32}, 0, -167\cdot 2^{32}, \\
&\quad -7\cdot 2^{29}, 0, 63\cdot 2^{28}, -655\cdot 2^{30}, -3861\cdot 2^{26}, -4067\cdot 2^{25}, 0, 85\cdot 2^{27}, \\
&\quad -347\cdot 2^{24}, -375\cdot 2^{23}, 0, -7375\cdot 2^{22}, 7\cdot 2^{23}, -37\cdot 2^{24}, 1687\cdot 2^{22}, \\
&\quad 85\cdot 2^{23}, -7\cdot 2^{21}, -3357\cdot 2^{19}, 0, -3105\cdot 2^{18}, 347\cdot 2^{18}, -37\cdot 2^{20}, 0, \\
&\quad 4703\cdot 2^{16}, 3861\cdot 2^{16}, 0, 63\cdot 2^{16}, -655\cdot 2^{18}, 7\cdot 2^{15}, -923\cdot 2^{15}, 0, \\
&\quad 85\cdot 2^{15}, -347\cdot 2^{12}, 0, -875\cdot 2^{13}, -7375\cdot 2^{10}, 7\cdot 2^{11}, -37\cdot 2^{12}, \\
&\quad -63\cdot 2^{10}, -6695\cdot 2^8, -7\cdot 2^9, -3357\cdot 2^7, 0, -655\cdot 2^{10}, -441\cdot 2^9, \\
&\quad -37\cdot 2^8, 0, 4703\cdot 2^4, -224, -375\cdot 2^3, 63\cdot 2^4, -655\cdot 2^6, 56, -8134, \\
&\quad 875\cdot 2^3, 85\cdot 2^3, -347, 0, 0, 0)) \qquad (40)
\end{aligned}
$$

$$
\begin{aligned}
\pi^3 \;=\; & \frac{1}{2^{54}} P(3, 2^{60}, 120, (5\cdot 2^{59}, -15\cdot 2^{60}, -225\cdot 2^{58}, 95\cdot 2^{59}, 4115\cdot 2^{56}, \\
& -3735\cdot 2^{55}, 0, -685\cdot 2^{58}, 505\cdot 2^{54}, 5\cdot 2^{53}, 0, 5485\cdot 2^{52}, -5\cdot 2^{53}, 0, \\
& -1775\cdot 2^{52}, -685\cdot 2^{54}, 5\cdot 2^{51}, -3945\cdot 2^{49}, 0, -7365\cdot 2^{48}, -505\cdot 2^{48}, \\
& 0, 0, -8125\cdot 2^{46}, -4115\cdot 2^{46}, -15\cdot 2^{48}, -225\cdot 2^{46}, 95\cdot 2^{47}, -5\cdot 2^{45}, \\
& -965\cdot 2^{45}, 0, -685\cdot 2^{46}, 505\cdot 2^{42}, -15\cdot 2^{44}, 125\cdot 2^{46}, 5485\cdot 2^{40}, \\
& -5\cdot 2^{41}, 0, 225\cdot 2^{40}, -2835\cdot 2^{38}, 5\cdot 2^{39}, -3945\cdot 2^{37}, 0, 95\cdot 2^{39}, \\
& 905\cdot 2^{38}, 0, 0, -8125\cdot 2^{34}, 5\cdot 2^{35}, 5\cdot 2^{33}, -225\cdot 2^{34}, 95\cdot 2^{35}, \\
& -5\cdot 2^{33}, -3735\cdot 2^{31}, -125\cdot 2^{36}, -685\cdot 2^{34}, 505\cdot 2^{30}, -15\cdot 2^{32}, 0, \\
& -165\cdot 2^{32}, -5\cdot 2^{29}, 0, 225\cdot 2^{28}, -685\cdot 2^{30}, -4115\cdot 2^{26}, -3945\cdot 2^{25}, \\
& 0, 95\cdot 2^{27}, -505\cdot 2^{24}, -125\cdot 2^{23}, 0, -8125\cdot 2^{22}, 5\cdot 2^{23}, -15\cdot 2^{24}, \\
& 1775\cdot 2^{22}, 95\cdot 2^{23}, -5\cdot 2^{21}, -3735\cdot 2^{19}, 0, -2835\cdot 2^{18}, 505\cdot 2^{18}, \\
& -15\cdot 2^{20}, 0, 5485\cdot 2^{16}, 4115\cdot 2^{16}, 0, 225\cdot 2^{16}, -685\cdot 2^{18}, 5\cdot 2^{15}, \\
& -955\cdot 2^{15}, 0, 95\cdot 2^{15}, -505\cdot 2^{12}, 0, -125\cdot 2^{16}, -8125\cdot 2^{10}, 5\cdot 2^{11}, \\
& -15\cdot 2^{12}, -225\cdot 2^{10}, -7365\cdot 2^{8}, -5\cdot 2^{9}, -3735\cdot 2^{7}, 0, -685\cdot 2^{10}, \\
& -905\cdot 2^{8}, -15\cdot 2^{8}, 0, 5485\cdot 2^{4}, -160, -125\cdot 2^{3}, 225\cdot 2^{4}, -685\cdot 2^{6}, 40, \\
& -7890, 125\cdot 2^{6}, 95\cdot 2^{3}, -505, 0, 0, 0)) \qquad (41)
\end{aligned}
$$

$$
35\zeta(3) - 2\pi^2 \log 2 = \frac{9}{4} P(3, 64, 6, (16, -24, -8, -6, 1, 0)) \qquad (42)
$$

$$
\begin{aligned}
\log^3 2 \;=\; & \frac{1}{3\cdot 2^8} P(3, 2^{12}, 24, (2^{13}, -5\cdot 2^{14}, -2^{12}, 17\cdot 2^{13}, -2^{11}, \\
& 5\cdot 19\cdot 2^9, 2^{10}, 9\cdot 2^{12}, 2^9, -5\cdot 2^{10}, -2^8, 2^6, -2^7, -5\cdot 2^8, 2^6, \\
& 9\cdot 2^8, 2^5, 5\cdot 19\cdot 2^3, -2^4, 17\cdot 2^5, -2^3, -5\cdot 2^4, 2^2, 9)) \qquad (43)
\end{aligned}
$$

$$
\begin{aligned}
\pi^2 \log 2 \;=\; & \frac{1}{3\cdot 2^6} P(3, 2^{12}, 24, (5\cdot 2^{11}, -41\cdot 2^{11}, -5\cdot 2^{10}, 49\cdot 2^{11}, \\
& -5\cdot 2^9, 67\cdot 2^9, 5\cdot 2^8, 27\cdot 2^{10}, 5\cdot 2^7, -41\cdot 2^7, -5\cdot 2^6, -5\cdot 2^7, \\
& -5\cdot 2^5, -41\cdot 2^5, 5\cdot 2^4, 27\cdot 2^6, 5\cdot 2^3, 67\cdot 2^3, -5\cdot 2^2, 49\cdot 2^3, \\
& -5\cdot 2^1, -41\cdot 2^1, 5, 0)) \qquad (44)
\end{aligned}
$$

$$
\begin{aligned}
\zeta(3) \;=\; & \frac{1}{21\cdot 2^5} P(3, 2^{12}, 24, (2^{11}, -11\cdot 2^{10}, -2^{10}, 23\cdot 2^9, -2^9, 2^{12}, \\
& 2^8, 27\cdot 2^7, 2^7, -11\cdot 2^6, -2^6, -2^7, -2^5, -11\cdot 2^4, 2^4, 27\cdot 2^3, 2^3, \\
& 2^6, -2^2, 23\cdot 2^1, -2^1, -11, 1, 0)) \qquad (45)
\end{aligned}
$$

$$\begin{aligned}\pi^3 &= \frac{5}{2^{55}}P(3, 2^{60}, 120, (2^{59}, -3\cdot 2^{60}, 11\cdot 2^{57}, 0, 23\cdot 2^{56}, 3\cdot 7\cdot 2^{56}, -2^{56}, 0,\\
&11\cdot 2^{54}, 13\cdot 2^{54}, 2^{54}, 0, -2^{53}, 3\cdot 2^{54}, 7\cdot 2^{52}, 0, 2^{51}, -3\cdot 7\cdot 2^{50}, 2^{50}, 0,\\
&-11\cdot 2^{48}, 3\cdot 2^{50}, -2^{48}, 0, -23\cdot 2^{46}, -3\cdot 2^{48}, 11\cdot 2^{45}, 0, -2^{45}, -2^{46}, -2^{44},\\
&0, 11\cdot 2^{42}, -3\cdot 2^{44}, -23\cdot 2^{41}, 0, -2^{41}, 3\cdot 2^{42}, -11\cdot 2^{39}, 0, 2^{39}, -3\cdot 7\cdot 2^{38},\\
&2^{38}, 0, 7\cdot 2^{37}, 3\cdot 2^{38}, -2^{36}, 0, 2^{35}, 13\cdot 2^{34}, 11\cdot 2^{33}, 0, -2^{33}, 3\cdot 7\cdot 2^{32},\\
&23\cdot 2^{31}, 0, 11\cdot 2^{30}, -3\cdot 2^{32}, 2^{30}, 0, -2^{29}, 3\cdot 2^{30}, -11\cdot 2^{27}, 0, -23\cdot 2^{26},\\
&-3\cdot 7\cdot 2^{26}, 2^{26}, 0, -11\cdot 2^{24}, -13\cdot 2^{24}, -2^{24}, 0, 2^{23}, -3\cdot 2^{24}, -7\cdot 2^{22}, 0,\\
&-2^{21}, 3\cdot 7\cdot 2^{20}, -2^{20}, 0, 11\cdot 2^{18}, -3\cdot 2^{20}, 2^{18}, 0, 23\cdot 2^{16}, 3\cdot 2^{18}, -11\cdot 2^{15},\\
&0, 2^{15}, 2^{16}, 2^{14}, 0, -11\cdot 2^{12}, 3\cdot 2^{14}, 23\cdot 2^{11}, 0, 2^{11}, -3\cdot 2^{12}, 11\cdot 2^{9}, 0, -2^{9},\\
&3\cdot 7\cdot 2^{8}, -2^{8}, 0, -7\cdot 2^{7}, -3\cdot 2^{8}, 2^{6}, 0, -2^{5}, -13\cdot 2^{4}, -11\cdot 2^{3}, 0, 2^{3},\\
&-3\cdot 7\cdot 2^{2}, -23\cdot 2, 0, -11, 3\cdot 2^{2}, -1, 0)) \qquad (46)\end{aligned}$$

$$\begin{aligned}\pi\log^2 2 &= \frac{1}{2^{57}}P(3, 2^{60}, 120, (7\cdot 2^{59}, -37\cdot 2^{60}, 13\cdot 17\cdot 2^{57}, 0, 19^2\cdot 2^{56},\\
&5\cdot 71\cdot 2^{56}, -7\cdot 2^{56}, 0, 13\cdot 17\cdot 2^{54}, 227\cdot 2^{54}, 7\cdot 2^{54}, 0, -7\cdot 2^{53},\\
&37\cdot 2^{54}, 7\cdot 11\cdot 2^{52}, 0, 7\cdot 2^{51}, -5\cdot 71\cdot 2^{50}, 7\cdot 2^{50}, 0, -13\cdot 17\cdot 2^{48},\\
&37\cdot 2^{50}, -7\cdot 2^{48}, 0, -19^2\cdot 2^{46}, -37\cdot 2^{48}, 13\cdot 17\cdot 2^{45}, 0, -7\cdot 2^{45},\\
&-5\cdot 2^{46}, -7\cdot 2^{44}, 0, 13\cdot 17\cdot 2^{42}, -37\cdot 2^{44}, -19^2\cdot 2^{41}, 0, -7\cdot 2^{41},\\
&37\cdot 2^{42}, -13\cdot 17\cdot 2^{39}, 0, 7\cdot 2^{39}, -5\cdot 71\cdot 2^{38}, 7\cdot 2^{38}, 0, 7\cdot 11\cdot 2^{37},\\
&37\cdot 2^{38}, -7\cdot 2^{36}, 0, 7\cdot 2^{35}, 227\cdot 2^{34}, 13\cdot 17\cdot 2^{33}, 0, -7\cdot 2^{33},\\
&5\cdot 71\cdot 2^{32}, 19^2\cdot 2^{31}, 0, 13\cdot 17\cdot 2^{30}, -37\cdot 2^{32}, 7\cdot 2^{30}, 0, -7\cdot 2^{29},\\
&37\cdot 2^{30}, -13\cdot 17\cdot 2^{27}, 0, -19^2\cdot 2^{26}, -5\cdot 71\cdot 2^{26}, 7\cdot 2^{26}, 0, -13\cdot 17\cdot 2^{24},\\
&-227\cdot 2^{24}, -7\cdot 2^{24}, 0, 7\cdot 2^{23}, -37\cdot 2^{24}, -7\cdot 11\cdot 2^{22}, 0, -7\cdot 2^{21},\\
&5\cdot 71\cdot 2^{20}, -7\cdot 2^{20}, 0, 13\cdot 17\cdot 2^{18}, -37\cdot 2^{20}, 7\cdot 2^{18}, 0, 19^2\cdot 2^{16},\\
&37\cdot 2^{18}, -13\cdot 17\cdot 2^{15}, 0, 7\cdot 2^{15}, 5\cdot 2^{16}, 7\cdot 2^{14}, 0, -13\cdot 17\cdot 2^{12},\\
&37\cdot 2^{14}, 19^2\cdot 2^{11}, 0, 7\cdot 2^{11}, -37\cdot 2^{12}, 13\cdot 17\cdot 2^{9}, 0, -7\cdot 2^{9}, 5\cdot 71\cdot 2^{8},\\
&-7\cdot 2^{8}, 0, -7\cdot 11\cdot 2^{7}, -37\cdot 2^{8}, 7\cdot 2^{6}, 0, -7\cdot 2^{5}, -227\cdot 2^{4}, -13\cdot 17\cdot 2^{3},\\
&0, 7\cdot 2^{3}, -5\cdot 71\cdot 2^{2}, -19^2\cdot 2, 0, -17\cdot 13, 37\cdot 2^{2}, -7, 0)) \qquad (47)\end{aligned}$$

$$
\begin{aligned}
\zeta(3) &= \frac{1}{7\cdot 2^{55}} P(3, 2^{60}, 120, (2^{59}, 0, -83\cdot 2^{57}, 11\cdot 2^{59}, 3\cdot 41\cdot 2^{56}, 0, 2^{56}, -11\cdot 2^{57}, \\
&\quad 83\cdot 2^{54}, 0, -2^{54}, 5^3\cdot 2^{53}, -2^{53}, 0, -3\cdot 7\cdot 2^{52}, -11\cdot 2^{53}, 2^{51}, 0, -2^{50}, -3^4\cdot 2^{49}, \\
&\quad -83\cdot 2^{48}, 0, 2^{48}, -5^3\cdot 2^{47}, -3\cdot 41\cdot 2^{46}, 0, -83\cdot 2^{45}, 11\cdot 2^{47}, -2^{45}, 0, 2^{44}, \\
&\quad -11\cdot 2^{45}, 83\cdot 2^{42}, 0, 3\cdot 41\cdot 2^{41}, 5^3\cdot 2^{41}, -2^{41}, 0, 83\cdot 2^{39}, 3^4\cdot 2^{39}, 2^{39}, 0, -2^{38}, \\
&\quad 11\cdot 2^{39}, 3\cdot 7\cdot 2^{37}, 0, 2^{36}, -5^3\cdot 2^{35}, 2^{35}, 0, -83\cdot 2^{33}, 11\cdot 2^{35}, -2^{33}, 0, -3\cdot 41\cdot 2^{31}, \\
&\quad -11\cdot 2^{33}, 83\cdot 2^{30}, 0, -2^{30}, 0, -2^{29}, 0, 83\cdot 2^{27}, -11\cdot 2^{29}, -3\cdot 41\cdot 2^{26}, 0, -2^{26}, \\
&\quad 11\cdot 2^{27}, -83\cdot 2^{24}, 0, 2^{24}, -5^3\cdot 2^{23}, 2^{23}, 0, 3\cdot 7\cdot 2^{22}, 11\cdot 2^{23}, -2^{21}, 0, 2^{20}, 3^4\cdot 2^{19}, \\
&\quad 83\cdot 2^{18}, 0, -2^{18}, 5^3\cdot 2^{17}, 3\cdot 41\cdot 2^{16}, 0, 83\cdot 2^{15}, -11\cdot 2^{17}, 2^{15}, 0, -2^{14}, 11\cdot 2^{15}, \\
&\quad -83\cdot 2^{12}, 0, -3\cdot 41\cdot 2^{11}, -5^3\cdot 2^{11}, 2^{11}, 0, -83\cdot 2^{9}, -3^4\cdot 2^{9}, -2^{9}, 0, 2^{8}, -11\cdot 2^{9}, \\
&\quad -3\cdot 7\cdot 2^{7}, 0, -2^{6}, 5^3\cdot 2^{5}, -2^{5}, 0, 83\cdot 2^{3}, -11\cdot 2^{5}, 2^{3}, 0, 3\cdot 41\cdot 2, 11\cdot 2^{3}, \\
&\quad -83, 0, 1, 0)) \qquad (48)
\end{aligned}
$$

$$
\begin{aligned}
\pi^2 \log 2 &= \frac{3}{5\cdot 2^{56}} P(3, 2^{60}, 120, (7\cdot 2^{59}, 0, -1031\cdot 2^{57}, 19\cdot 2^{62}, 3^2\cdot 179\cdot 2^{56}, 0, 7\cdot 2^{56}, \\
&\quad -19\cdot 2^{60}, 1031\cdot 2^{54}, 0, -7\cdot 2^{54}, 5^3\cdot 13\cdot 2^{53}, -7\cdot 2^{53}, 0, -3^3\cdot 11\cdot 2^{52}, \\
&\quad -19\cdot 2^{56}, 7\cdot 2^{51}, 0, -7\cdot 2^{50}, -3^2\cdot 113\cdot 2^{49}, -1031\cdot 2^{48}, 0, 7\cdot 2^{48}, \\
&\quad -5^3\cdot 13\cdot 2^{47}, -3^2\cdot 179\cdot 2^{46}, 0, -1031\cdot 2^{45}, 19\cdot 2^{50}, -7\cdot 2^{45}, 0, 7\cdot 2^{44}, \\
&\quad -19\cdot 2^{48}, 1031\cdot 2^{42}, 0, 3^2\cdot 179\cdot 2^{41}, 5^3\cdot 13\cdot 2^{41}, -7\cdot 2^{41}, 0, 1031\cdot 2^{39}, \\
&\quad 3^2\cdot 113\cdot 2^{39}, 7\cdot 2^{39}, 0, -7\cdot 2^{38}, 19\cdot 2^{42}, 3^3\cdot 11\cdot 2^{37}, 0, 7\cdot 2^{36}, -5^3\cdot 13\cdot 2^{35}, \\
&\quad 7\cdot 2^{35}, 0, -1031\cdot 2^{33}, 19\cdot 2^{38}, -7\cdot 2^{33}, 0, -3^2\cdot 179\cdot 2^{31}, -19\cdot 2^{36}, 1031\cdot 2^{30}, \\
&\quad 0, -7\cdot 2^{30}, 0, -7\cdot 2^{29}, 0, 1031\cdot 2^{27}, -19\cdot 2^{32}, -3^2\cdot 179\cdot 2^{26}, 0, -7\cdot 2^{26}, \\
&\quad 19\cdot 2^{30}, -1031\cdot 2^{24}, 0, 7\cdot 2^{24}, -5^3\cdot 13\cdot 2^{23}, 7\cdot 2^{23}, 0, 3^3\cdot 11\cdot 2^{22}, 19\cdot 2^{26}, \\
&\quad -7\cdot 2^{21}, 0, 7\cdot 2^{20}, 3^2\cdot 113\cdot 2^{19}, 1031\cdot 2^{18}, 0, -7\cdot 2^{18}, 5^3\cdot 13\cdot 2^{17}, 3^2\cdot 179\cdot 2^{16}, \\
&\quad 0, 1031\cdot 2^{15}, -19\cdot 2^{20}, 7\cdot 2^{15}, 0, -7\cdot 2^{14}, 19\cdot 2^{18}, -1031\cdot 2^{12}, 0, -3^2\cdot 179\cdot 2^{11}, \\
&\quad -5^3\cdot 13\cdot 2^{11}, 7\cdot 2^{11}, 0, -1031\cdot 2^{9}, -3^2\cdot 113\cdot 2^{9}, -7\cdot 2^{9}, 0, 7\cdot 2^{8}, -19\cdot 2^{12}, \\
&\quad -3^3\cdot 11\cdot 2^{7}, 0, -7\cdot 2^{6}, 5^3\cdot 13\cdot 2^{5}, -7\cdot 2^{5}, 0, 1031\cdot 2^{3}, -19\cdot 2^{8}, 7\cdot 2^{3}, 0, \\
&\quad 3^2\cdot 179\cdot 2, 19\cdot 2^{6}, -1031, 0, 7, 0)) \qquad (49)
\end{aligned}
$$

$$\begin{aligned}
\log^3 2 \;=\; & \frac{3}{2^{58}} P(3, 2^{60}, 120, (2^{59}, 0, -11 \cdot 19 \cdot 2^{57}, 5 \cdot 2^{62}, 373 \cdot 2^{56}, 0, 2^{56}, -5 \cdot 2^{60}, \\
& 11 \cdot 19 \cdot 2^{54}, 0, -2^{54}, 367 \cdot 2^{53}, -2^{53}, 0, -83 \cdot 2^{52}, -5 \cdot 2^{56}, 2^{51}, 0, -2^{50}, \\
& -5 \cdot 43 \cdot 2^{49}, -11 \cdot 19 \cdot 2^{48}, 0, 2^{48}, -367 \cdot 2^{47}, -373 \cdot 2^{46}, 0, -11 \cdot 19 \cdot 2^{45}, 5 \cdot 2^{50}, \\
& -2^{45}, 0, 2^{44}, -5 \cdot 2^{48}, 11 \cdot 19 \cdot 2^{42}, 0, 373 \cdot 2^{41}, 367 \cdot 2^{41}, -2^{41}, 0, 11 \cdot 19 \cdot 2^{39}, \\
& 5 \cdot 43 \cdot 2^{39}, 2^{39}, 0, -2^{38}, 5 \cdot 2^{42}, 83 \cdot 2^{37}, 0, 2^{36}, -367 \cdot 2^{35}, 2^{35}, 0, -11 \cdot 19 \cdot 2^{33}, \\
& 5 \cdot 2^{38}, -2^{33}, 0, -373 \cdot 2^{31}, -5 \cdot 2^{36}, 11 \cdot 19 \cdot 2^{30}, 0, -2^{30}, -2^{32}, -2^{29}, 0, \\
& 11 \cdot 19 \cdot 2^{27}, -5 \cdot 2^{32}, -373 \cdot 2^{26}, 0, -2^{26}, 5 \cdot 2^{30}, -11 \cdot 19 \cdot 2^{24}, 0, 2^{24}, -367 \cdot 2^{23}, \\
& 2^{23}, 0, 83 \cdot 2^{22}, 5 \cdot 2^{26}, -2^{21}, 0, 2^{20}, 5 \cdot 43 \cdot 2^{19}, 11 \cdot 19 \cdot 2^{18}, 0, -2^{18}, 367 \cdot 2^{17}, \\
& 373 \cdot 2^{16}, 0, 11 \cdot 19 \cdot 2^{15}, -5 \cdot 2^{20}, 2^{15}, 0, -2^{14}, 5 \cdot 2^{18}, -11 \cdot 19 \cdot 2^{12}, 0, -373 \cdot 2^{11}, \\
& -367 \cdot 2^{11}, 2^{11}, 0, -11 \cdot 19 \cdot 2^{9}, -5 \cdot 43 \cdot 2^{9}, -2^{9}, 0, 2^{8}, -5 \cdot 2^{12}, -83 \cdot 2^{7}, 0, -2^{6}, \\
& 367 \cdot 2^{5}, -2^{5}, 0, 11 \cdot 19 \cdot 2^{3}, -5 \cdot 2^{8}, 2^{3}, 0, 373 \cdot 2, 5 \cdot 2^{6}, -19 \cdot 11, 0, 1, 2^{2})) \qquad (50)
\end{aligned}$$

$$\frac{1}{3}\log^3 2 - \frac{5}{12}\pi^2 \log 2 + \frac{35}{4}\zeta(3) \;=\; P(3, 16, 8, (8, 0, -4, -4, -2, 0, 1, 1)) \qquad (51)$$

The existence of BBP formulas for these constants was originally established by Broadhurst [19]. However, except for 37, which appeared in [15], the specific explicit formulas listed here were produced by the author's PSLQ program. Formula 39 is proved by subtracting 5/192×Relation 123 from Formula 44. Formula 40=Formula 47+37/75/2^{57}× Relation 124+133/75/2^{56}× Relation 125. Formula 41=Formula 46+1/5/2^{55}× Relation 124+ 9/5/2^{54}× Relation 125. The results for $\pi \log^2 2$ and π^3 were produced by a special parallel version of this program, running on the IBM SP parallel computer system in the NERSC supercomputer facility at the Lawrence Berkeley National Laboratory. Formulas 42 through 51 were proven by Gery Huvent.

7 Degree 4 binary formulas

$$\pi^4 = \frac{1}{164}P(4, 2^{12}, 24, (27 \cdot 2^{11}, -513 \cdot 2^{11}, 135 \cdot 2^{14}, -27 \cdot 2^{11}, -27 \cdot 2^9, -621 \cdot 2^{10}, 27 \cdot 2^8, -729 \cdot 2^{10}, -135 \cdot 2^{11}, -513 \cdot 2^7, -27 \cdot 2^6, -189 \cdot 2^9, -27 \cdot 2^5, -513 \cdot 2^5, -135 \cdot 2^8, -729 \cdot 2^6, 216, -621 \cdot 2^4, -108, -216, 135 \cdot 2^5, -1026, 27, 0)) \quad (52)$$

$$\log^4 2 = \frac{1}{205 \cdot 2^5}P(4, 2^{12}, 24, (73 \cdot 2^{12}, -2617 \cdot 2^{12}, 8455 \cdot 2^{12}, -2533 \cdot 2^{12}, -73 \cdot 2^{10}, -25781 \cdot 2^9, 73 \cdot 2^9, -6891 \cdot 2^{11}, -8455 \cdot 2^9, -2617 \cdot 2^8, -73 \cdot 2^7, -23551 \cdot 2^6, -73 \cdot 2^6, -2617 \cdot 2^6, -8455 \cdot 2^6, -6891 \cdot 2^7, 73 \cdot 2^4, -25781 \cdot 2^3, -73 \cdot 2^3, -2533 \cdot 2^4, 8455 \cdot 2^3, -10468, 146, -615)) \quad (53)$$

$$\pi^2 \log^2 2 = \frac{1}{41 \cdot 2^5}P(4, 2^{12}, 24, (121 \cdot 2^{11}, -3775 \cdot 2^{11}, 10375 \cdot 2^{11}, -1597 \cdot 2^{11}, -121 \cdot 2^9, -3421 \cdot 2^{11}, 121 \cdot 2^8, -7695 \cdot 2^{10}, -10375 \cdot 2^8, -3775 \cdot 2^7, -121 \cdot 2^6, -3539 \cdot 2^8, -121 \cdot 2^5, -3775 \cdot 2^5, -10375 \cdot 2^5, -7695 \cdot 2^6, 121 \cdot 2^3, -3421 \cdot 2^5, -484, -1597 \cdot 2^3, 41500, -7550, 121, 0)) \quad (54)$$

$$\pi^4 = \frac{675}{7 \cdot 71 \cdot 2^{51}}P(4, 2^{60}, 120, (2^{59}, -5 \cdot 2^{61}, 11 \cdot 17 \cdot 2^{57}, -2^{61}, -127 \cdot 2^{56}, -5 \cdot 2^{59}, 2^{56}, -2^{61}, -11 \cdot 17 \cdot 2^{54}, -5 \cdot 2^{57}, -2^{54}, -5 \cdot 41 \cdot 2^{53}, -2^{53}, -5 \cdot 2^{55}, -31 \cdot 2^{52}, -2^{57}, 2^{51}, -5 \cdot 2^{53}, -2^{50}, 109 \cdot 2^{49}, 11 \cdot 17 \cdot 2^{48}, -5 \cdot 2^{51}, 2^{48}, 5^3 \cdot 2^{47}, 127 \cdot 2^{46}, -5 \cdot 2^{49}, 11 \cdot 17 \cdot 2^{45}, -2^{49}, -2^{45}, -5 \cdot 2^{47}, 2^{44}, -2^{49}, -11 \cdot 17 \cdot 2^{42}, -5 \cdot 2^{45}, -127 \cdot 2^{41}, -5 \cdot 41 \cdot 2^{41}, -2^{41}, -5 \cdot 2^{43}, -11 \cdot 17 \cdot 2^{39}, -3^3 \cdot 7 \cdot 2^{39}, 2^{39}, -5 \cdot 2^{41}, -2^{38}, -2^{41}, 31 \cdot 2^{37}, -5 \cdot 2^{39}, 2^{36}, 5^3 \cdot 2^{35}, 2^{35}, -5 \cdot 2^{37}, 11 \cdot 17 \cdot 2^{33}, -2^{37}, -2^{33}, -5 \cdot 2^{35}, 127 \cdot 2^{31}, -2^{37}, -11 \cdot 17 \cdot 2^{30}, -5 \cdot 2^{33}, -2^{30}, -5 \cdot 2^{33}, -2^{29}, -5 \cdot 2^{31}, -11 \cdot 17 \cdot 2^{27}, -2^{33}, 127 \cdot 2^{26}, -5 \cdot 2^{29}, -2^{26}, -2^{29}, 11 \cdot 17 \cdot 2^{24}, -5 \cdot 2^{27}, 2^{24}, 5^3 \cdot 2^{23}, 2^{23}, -5 \cdot 2^{25}, 31 \cdot 2^{22}, -2^{25}, -2^{21}, -5 \cdot 2^{23}, 2^{20}, -3^3 \cdot 7 \cdot 2^{19}, -11 \cdot 17 \cdot 2^{18}, -5 \cdot 2^{21}, -2^{18}, -5 \cdot 41 \cdot 2^{17}, -127 \cdot 2^{16}, -5 \cdot 2^{19}, -11 \cdot 17 \cdot 2^{15}, -2^{21}, 2^{15}, -5 \cdot 2^{17}, -2^{14}, -2^{17}, 11 \cdot 17 \cdot 2^{12}, -5 \cdot 2^{15}, 127 \cdot 2^{11}, 5^3 \cdot 2^{11}, 2^{11}, -5 \cdot 2^{13}, 11 \cdot 17 \cdot 2^9, 109 \cdot 2^9, -2^9, -5 \cdot 2^{11}, 2^8, -2^{13}, -31 \cdot 2^7, -5 \cdot 2^9, -2^6, -5 \cdot 41 \cdot 2^5, -2^5, -5 \cdot 2^7, -11 \cdot 17 \cdot 2^3, -2^9, 2^3, -5 \cdot 2^5, -127 \cdot 2, -2^5, 17 \cdot 11, -5 \cdot 2^3, 1, 0)) \quad (55)$$

$$
\begin{aligned}
\log^4 2 \;=\;& \frac{1}{7\cdot 71\cdot 2^{54}} P(4, 2^{60}, 120, (823\cdot 2^{59}, -5\cdot 3137\cdot 2^{60}, 11\cdot 40829\cdot 2^{57}, -18047\cdot 2^{60},\\
& -277\cdot 1723\cdot 2^{56}, -5\cdot 3137\cdot 2^{58}, 823\cdot 2^{56}, 1181\cdot 2^{59}, -11\cdot 40829\cdot 2^{54},\\
& -5\cdot 3137\cdot 2^{56}, -823\cdot 2^{54}, -595141\cdot 2^{53}, -823\cdot 2^{53}, -5\cdot 3137\cdot 2^{54},\\
& 29\cdot 457\cdot 2^{52}, 1181\cdot 2^{55}, 823\cdot 2^{51}, -5\cdot 3137\cdot 2^{52}, -823\cdot 2^{50},\\
& 331249\cdot 2^{49}, 11\cdot 40829\cdot 2^{48}, -5\cdot 3137\cdot 2^{50}, 823\cdot 2^{48}, 19^2\cdot 1301\cdot 2^{47},\\
& 277\cdot 1723\cdot 2^{46}, -5\cdot 3137\cdot 2^{48}, 11\cdot 40829\cdot 2^{45}, -18047\cdot 2^{48}, -823\cdot 2^{45},\\
& -5\cdot 3137\cdot 2^{46}, 823\cdot 2^{44}, 1181\cdot 2^{47}, -11\cdot 40829\cdot 2^{42}, -5\cdot 3137\cdot 2^{44},\\
& -277\cdot 1723\cdot 2^{41}, -595141\cdot 2^{41}, -823\cdot 2^{41}, -5\cdot 3137\cdot 2^{42}, -11\cdot 40829\cdot 2^{39},\\
& -3\cdot 7^2\cdot 13\cdot 239\cdot 2^{39}, 823\cdot 2^{39}, -5\cdot 3137\cdot 2^{40}, -823\cdot 2^{38}, -18047\cdot 2^{40},\\
& -29\cdot 457\cdot 2^{37}, -5\cdot 3137\cdot 2^{38}, 823\cdot 2^{36}, 19^2\cdot 1301\cdot 2^{35}, 823\cdot 2^{35},\\
& -5\cdot 3137\cdot 2^{36}, 11\cdot 40829\cdot 2^{33}, -18047\cdot 2^{36}, -823\cdot 2^{33}, -5\cdot 3137\cdot 2^{34},\\
& 277\cdot 1723\cdot 2^{31}, 1181\cdot 2^{35}, -11\cdot 40829\cdot 2^{30}, -5\cdot 3137\cdot 2^{32}, -823\cdot 2^{30},\\
& -29879\cdot 2^{31}, -823\cdot 2^{29}, -5\cdot 3137\cdot 2^{30}, -11\cdot 40829\cdot 2^{27}, 1181\cdot 2^{31},\\
& 277\cdot 1723\cdot 2^{26}, -5\cdot 3137\cdot 2^{28}, -823\cdot 2^{26}, -18047\cdot 2^{28}, 11\cdot 40829\cdot 2^{24},\\
& -5\cdot 3137\cdot 2^{26}, 823\cdot 2^{24}, 19^2\cdot 1301\cdot 2^{23}, 823\cdot 2^{23}, -5\cdot 3137\cdot 2^{24},\\
& -29\cdot 457\cdot 2^{22}, -18047\cdot 2^{24}, -823\cdot 2^{21}, -5\cdot 3137\cdot 2^{22}, 823\cdot 2^{20},\\
& -3\cdot 7^2\cdot 13\cdot 239\cdot 2^{19}, -11\cdot 40829\cdot 2^{18}, -5\cdot 3137\cdot 2^{20}, -823\cdot 2^{18},\\
& -595141\cdot 2^{17}, -277\cdot 1723\cdot 2^{16}, -5\cdot 3137\cdot 2^{18}, -11\cdot 40829\cdot 2^{15},\\
& 1181\cdot 2^{19}, 823\cdot 2^{15}, -5\cdot 3137\cdot 2^{16}, -823\cdot 2^{14}, -18047\cdot 2^{16},\\
& 11\cdot 40829\cdot 2^{12}, -5\cdot 3137\cdot 2^{14}, 277\cdot 1723\cdot 2^{11}, 19^2\cdot 1301\cdot 2^{11},\\
& 823\cdot 2^{11}, -5\cdot 3137\cdot 2^{12}, 11\cdot 40829\cdot 2^{9}, 331249\cdot 2^{9}, -823\cdot 2^{9},\\
& -5\cdot 3137\cdot 2^{10}, 823\cdot 2^{8}, 1181\cdot 2^{11}, 29\cdot 457\cdot 2^{7}, -5\cdot 3137\cdot 2^{8},\\
& -823\cdot 2^{6}, -595141\cdot 2^{5}, -823\cdot 2^{5}, -5\cdot 3137\cdot 2^{6}, -11\cdot 40829\cdot 2^{3},\\
& 1181\cdot 2^{7}, 823\cdot 2^{3}, -5\cdot 3137\cdot 2^{4}, -277\cdot 1723\cdot 2, -18047\cdot 2^{4},\\
& 40829\cdot 11, -5\cdot 3137\cdot 2^{2}, 823, -3\cdot 7\cdot 71\cdot 2)) \qquad (56)
\end{aligned}
$$

$$
\begin{aligned}
\pi^2\log^2 2 \;=\;& \frac{1}{7\cdot 71\cdot 2^{54}}P(4,2^{60},120,(13^2\cdot 19\cdot 2^{59},-37\cdot 179\cdot 2^{63},5\cdot 13\cdot 19973\cdot 2^{57},\\
& -5\cdot 1663\cdot 2^{62},-17\cdot 179\cdot 379\cdot 2^{56},-37\cdot 179\cdot 2^{61},13^2\cdot 19\cdot 2^{56},-4931\cdot 2^{60},\\
& -5\cdot 13\cdot 19973\cdot 2^{54},-37\cdot 179\cdot 2^{59},-13^2\cdot 19\cdot 2^{54},-43\cdot 36529\cdot 2^{53},-13^2\cdot 19\cdot 2^{53},\\
& -37\cdot 179\cdot 2^{57},-5\cdot 15137\cdot 2^{52},-4931\cdot 2^{56},13^2\cdot 19\cdot 2^{51},-37\cdot 179\cdot 2^{55},\\
& -13^2\cdot 19\cdot 2^{50},5\cdot 176159\cdot 2^{49},5\cdot 13\cdot 19973\cdot 2^{48},-37\cdot 179\cdot 2^{53},13^2\cdot 19\cdot 2^{48},\\
& 5^5\cdot 367\cdot 2^{47},17\cdot 179\cdot 379\cdot 2^{46},-37\cdot 179\cdot 2^{51},5\cdot 13\cdot 19973\cdot 2^{45},-5\cdot 1663\cdot 2^{50},\\
& -13^2\cdot 19\cdot 2^{45},-37\cdot 179\cdot 2^{49},13^2\cdot 19\cdot 2^{44},-4931\cdot 2^{48},-5\cdot 13\cdot 19973\cdot 2^{42},\\
& -37\cdot 179\cdot 2^{47},-17\cdot 179\cdot 379\cdot 2^{41},-43\cdot 36529\cdot 2^{41},-13^2\cdot 19\cdot 2^{41},-37\cdot 179\cdot 2^{45},\\
& -5\cdot 13\cdot 19973\cdot 2^{39},-3^5\cdot 7\cdot 13\cdot 59\cdot 2^{39},13^2\cdot 19\cdot 2^{39},-37\cdot 179\cdot 2^{43},-13^2\cdot 19\cdot 2^{38},\\
& -5\cdot 1663\cdot 2^{42},5\cdot 15137\cdot 2^{37},-37\cdot 179\cdot 2^{41},13^2\cdot 19\cdot 2^{36},5^5\cdot 367\cdot 2^{35},\\
& 13^2\cdot 19\cdot 2^{35},-37\cdot 179\cdot 2^{39},5\cdot 13\cdot 19973\cdot 2^{33},-5\cdot 1663\cdot 2^{38},-13^2\cdot 19\cdot 2^{33},\\
& -37\cdot 179\cdot 2^{37},17\cdot 179\cdot 379\cdot 2^{31},-4931\cdot 2^{36},-5\cdot 13\cdot 19973\cdot 2^{30},-37\cdot 179\cdot 2^{35},\\
& -13^2\cdot 19\cdot 2^{30},-37\cdot 179\cdot 2^{35},-13^2\cdot 19\cdot 2^{29},-37\cdot 179\cdot 2^{33},-5\cdot 13\cdot 19973\cdot 2^{27},\\
& -4931\cdot 2^{32},17\cdot 179\cdot 379\cdot 2^{26},-37\cdot 179\cdot 2^{31},-13^2\cdot 19\cdot 2^{26},-5\cdot 1663\cdot 2^{30},\\
& 5\cdot 13\cdot 19973\cdot 2^{24},-37\cdot 179\cdot 2^{29},13^2\cdot 19\cdot 2^{24},5^5\cdot 367\cdot 2^{23},13^2\cdot 19\cdot 2^{23},\\
& -37\cdot 179\cdot 2^{27},5\cdot 15137\cdot 2^{22},-5\cdot 1663\cdot 2^{26},-13^2\cdot 19\cdot 2^{21},-37\cdot 179\cdot 2^{25},\\
& 13^2\cdot 19\cdot 2^{20},-3^5\cdot 7\cdot 13\cdot 59\cdot 2^{19},-5\cdot 13\cdot 19973\cdot 2^{18},-37\cdot 179\cdot 2^{23},\\
& -13^2\cdot 19\cdot 2^{18},-43\cdot 36529\cdot 2^{17},-17\cdot 179\cdot 379\cdot 2^{16},-37\cdot 179\cdot 2^{21},-5\cdot 13\cdot 19973\cdot 2^{15},\\
& -4931\cdot 2^{20},13^2\cdot 19\cdot 2^{15},-37\cdot 179\cdot 2^{19},-13^2\cdot 19\cdot 2^{14},-5\cdot 1663\cdot 2^{18},\\
& 5\cdot 13\cdot 19973\cdot 2^{12},-37\cdot 179\cdot 2^{17},17\cdot 179\cdot 379\cdot 2^{11},5^5\cdot 367\cdot 2^{11},13^2\cdot 19\cdot 2^{11},\\
& -37\cdot 179\cdot 2^{15},5\cdot 13\cdot 19973\cdot 2^{9},5\cdot 176159\cdot 2^{9},-13^2\cdot 19\cdot 2^{9},-37\cdot 179\cdot 2^{13},\\
& 13^2\cdot 19\cdot 2^{8},-4931\cdot 2^{12},-5\cdot 15137\cdot 2^{7},-37\cdot 179\cdot 2^{11},-13^2\cdot 19\cdot 2^{6},\\
& -43\cdot 36529\cdot 2^{5},-13^2\cdot 19\cdot 2^{5},-37\cdot 179\cdot 2^{9},-5\cdot 13\cdot 19973\cdot 2^{3},-4931\cdot 2^{8},\\
& 13^2\cdot 19\cdot 2^{3},-37\cdot 179\cdot 2^{7},-17\cdot 179\cdot 379\cdot 2,-5\cdot 1663\cdot 2^{6},13\cdot 19973\cdot 5,\\
& -37\cdot 179\cdot 2^{5},19\cdot 13^2,0)) \qquad (57)
\end{aligned}
$$

$$
\begin{aligned}
20\,\mathrm{Cl}_4\left(\frac{\pi}{2}\right)+\frac{3\pi\log^3 2}{32}-\frac{27\pi^3\log 2}{128} \;=\;& \frac{9}{2^{10}}P(4,2^{12},24,(2^{11},-2^{12},-7\cdot 2^{9},0,-2^{9},\\
& -5\cdot 2^{8},-2^{8},0,-7\cdot 2^{6},-2^{8},2^{6},0,-2^{5},2^{6},7\cdot 2^{3},0,\\
& 2^{3},5\cdot 2^{2},2^{2},0,7,2^{2},-1,0)) \qquad (58)
\end{aligned}
$$

$$12\,\mathrm{Cl}_4\left(\frac{\pi}{2}\right)+\frac{29\pi\log^3 2}{192}-\frac{47\pi^3\log 2}{256}$$
$$=\frac{1}{2^{58}}P(4,2^{60},120,(2^{60},23\cdot 2^{60},$$
$$-239\cdot 2^{57},0,-3\cdot 211\cdot 2^{55},-5\cdot 67\cdot 2^{56},-2^{57},0,-239\cdot 2^{54},$$
$$-3^2\cdot 7^2\cdot 2^{53},2^{55},0,-2^{54},-23\cdot 2^{54},-3\cdot 7^2\cdot 2^{50},0,$$
$$2^{52},5\cdot 67\cdot 2^{50},2^{51},0,239\cdot 2^{48},-23\cdot 2^{50},-2^{49},0,$$
$$3\cdot 211\cdot 2^{45},23\cdot 2^{48},-239\cdot 2^{45},0,-2^{46},-3^2\cdot 5\cdot 2^{43},$$
$$-2^{45},0,-239\cdot 2^{42},23\cdot 2^{44},3\cdot 211\cdot 2^{40},0,-2^{42},-23\cdot 2^{42},239\cdot 2^{39},$$
$$0,2^{40},5\cdot 67\cdot 2^{38},2^{39},0,-3\cdot 7^2\cdot 2^{35},-23\cdot 2^{38},-2^{37},0,2^{36},$$
$$-3^2\cdot 7^2\cdot 2^{33},-239\cdot 2^{33},0,-2^{34},-5\cdot 67\cdot 2^{32},-3\cdot 211\cdot 2^{30},0,$$
$$-239\cdot 2^{30},23\cdot 2^{32},2^{31},0,-2^{30},-23\cdot 2^{30},239\cdot 2^{27},0,3\cdot 211\cdot 2^{25},$$
$$5\cdot 67\cdot 2^{26},2^{27},0,239\cdot 2^{24},3^2\cdot 7^2\cdot 2^{23},-2^{25},0,2^{24},23\cdot 2^{24},$$
$$3\cdot 7^2\cdot 2^{20},0,-2^{22},-5\cdot 67\cdot 2^{20},-2^{21},0,-239\cdot 2^{18},23\cdot 2^{20},2^{19},0,$$
$$-3\cdot 211\cdot 2^{15},-23\cdot 2^{18},239\cdot 2^{15},0,2^{16},3^2\cdot 5\cdot 2^{13},2^{15},0,239\cdot 2^{12},$$
$$-23\cdot 2^{14},-3\cdot 211\cdot 2^{10},0,2^{12},23\cdot 2^{12},-239\cdot 2^{9},0,-2^{10},$$
$$-5\cdot 67\cdot 2^{8},-2^{9},0,3\cdot 7^2\cdot 2^{5},23\cdot 2^{8},2^{7},0,-2^{6},3^2\cdot 7^2\cdot 2^{3},$$
$$239\cdot 2^{3},0,2^{4},5\cdot 67\cdot 2^{2},211\cdot 3,0,239,-23\cdot 2^{2},-2,0)) \tag{59}$$

The existence of BBP-type formulas for these constants was originally established by Broadhurst [19], although the explicit formulas given here were found by the author's PSLQ program. Formulas 52 to 54 were subsequently proved by Kunle Adegoke [2] who also found formulas 55 to 59. Here Cl_4 is a Clausen function (see section 9).

8 Degree 5 binary formulas

$$
\begin{aligned}
\zeta(5) \quad = \quad & \frac{1}{62651 \cdot 2^{49}} P(5, 2^{60}, 120, (279 \cdot 2^{59}, -7263 \cdot 2^{60}, 293715 \cdot 2^{57}, \\
& -13977 \cdot 2^{60}, -1153683 \cdot 2^{56}, 28377 \cdot 2^{60}, 279 \cdot 2^{56}, 83871 \cdot 2^{59}, \\
& -293715 \cdot 2^{54}, -7263 \cdot 2^{56}, -279 \cdot 2^{54}, -889173 \cdot 2^{53}, -279 \cdot 2^{53}, \\
& -7263 \cdot 2^{54}, 429705 \cdot 2^{52}, 83871 \cdot 2^{55}, 279 \cdot 2^{51}, 28377 \cdot 2^{54}, \\
& -279 \cdot 2^{50}, 1041309 \cdot 2^{49}, 293715 \cdot 2^{48}, -7263 \cdot 2^{50}, 279 \cdot 2^{48}, \\
& 1153125 \cdot 2^{47}, 1153683 \cdot 2^{46}, -7263 \cdot 2^{48}, 293715 \cdot 2^{45}, -13977 \cdot 2^{48}, \\
& -279 \cdot 2^{45}, 28377 \cdot 2^{48}, 279 \cdot 2^{44}, 83871 \cdot 2^{47}, -293715 \cdot 2^{42}, \\
& -7263 \cdot 2^{44}, -1153683 \cdot 2^{41}, -889173 \cdot 2^{41}, -279 \cdot 2^{41}, -7263 \cdot 2^{42}, \\
& -293715 \cdot 2^{39}, 188811 \cdot 2^{39}, 279 \cdot 2^{39}, 28377 \cdot 2^{42}, -279 \cdot 2^{38}, \\
& -13977 \cdot 2^{40}, -429705 \cdot 2^{37}, -7263 \cdot 2^{38}, 279 \cdot 2^{36}, 1153125 \cdot 2^{35}, \\
& 279 \cdot 2^{35}, -7263 \cdot 2^{36}, 293715 \cdot 2^{33}, -13977 \cdot 2^{36}, -279 \cdot 2^{33}, \\
& 28377 \cdot 2^{36}, 1153683 \cdot 2^{31}, 83871 \cdot 2^{35}, -293715 \cdot 2^{30}, -7263 \cdot 2^{32}, \\
& -279 \cdot 2^{30}, 16497 \cdot 2^{33}, -279 \cdot 2^{29}, -7263 \cdot 2^{30}, -293715 \cdot 2^{27}, \\
& 83871 \cdot 2^{31}, 1153683 \cdot 2^{26}, 28377 \cdot 2^{30}, -279 \cdot 2^{26}, -13977 \cdot 2^{28}, \\
& 293715 \cdot 2^{24}, -7263 \cdot 2^{26}, 279 \cdot 2^{24}, 1153125 \cdot 2^{23}, 279 \cdot 2^{23}, \\
& -7263 \cdot 2^{24}, -429705 \cdot 2^{22}, -13977 \cdot 2^{24}, -279 \cdot 2^{21}, 28377 \cdot 2^{24}, \\
& 279 \cdot 2^{20}, 188811 \cdot 2^{19}, -293715 \cdot 2^{18}, -7263 \cdot 2^{20}, -279 \cdot 2^{18}, \\
& -889173 \cdot 2^{17}, -1153683 \cdot 2^{16}, -7263 \cdot 2^{18}, -293715 \cdot 2^{15}, 83871 \cdot 2^{19}, \\
& 279 \cdot 2^{15}, 28377 \cdot 2^{18}, -279 \cdot 2^{14}, -13977 \cdot 2^{16}, 293715 \cdot 2^{12}, \\
& -7263 \cdot 2^{14}, 1153683 \cdot 2^{11}, 1153125 \cdot 2^{11}, 279 \cdot 2^{11}, -7263 \cdot 2^{12}, \\
& 293715 \cdot 2^{9}, 1041309 \cdot 2^{9}, -279 \cdot 2^{9}, 28377 \cdot 2^{12}, 279 \cdot 2^{8}, \\
& 83871 \cdot 2^{11}, 429705 \cdot 2^{7}, -7263 \cdot 2^{8}, -279 \cdot 2^{6}, -889173 \cdot 2^{5}, \\
& -279 \cdot 2^{5}, -7263 \cdot 2^{6}, -293715 \cdot 2^{3}, 83871 \cdot 2^{7}, 279 \cdot 2^{3}, \\
& 28377 \cdot 2^{6}, -2307366, -13977 \cdot 2^{4}, 293715, -29052, 279, 0)) \qquad (60)
\end{aligned}
$$

$$
\begin{aligned}
\log^5 2 \quad = \quad & \frac{1}{2021 \cdot 2^{52}} P(5, 2^{60}, 120, (2783 \cdot 2^{59}, -32699 \cdot 2^{62}, 7171925 \cdot 2^{57}, \\
& -187547 \cdot 2^{61}, -41252441 \cdot 2^{56}, 9391097 \cdot 2^{57}, 2783 \cdot 2^{56}, \\
& 52183 \cdot 2^{65}, -7171925 \cdot 2^{54}, -32699 \cdot 2^{58}, -2783 \cdot 2^{54}, \\
& -29483621 \cdot 2^{53}, -2783 \cdot 2^{53}, -32699 \cdot 2^{56}, 17037475 \cdot 2^{52}, \\
& 52183 \cdot 2^{61}, 2783 \cdot 2^{51}, 9391097 \cdot 2^{51}, -2783 \cdot 2^{50}, \\
& 38246123 \cdot 2^{49}, 7171925 \cdot 2^{48}, -32699 \cdot 2^{52}, 2783 \cdot 2^{48}, \\
& 41307505 \cdot 2^{47}, 41252441 \cdot 2^{46}, -32699 \cdot 2^{50}, 7171925 \cdot 2^{45}, \\
& -187547 \cdot 2^{49}, -2783 \cdot 2^{45}, 9391097 \cdot 2^{45}, 2783 \cdot 2^{44}, \\
& 52183 \cdot 2^{53}, -7171925 \cdot 2^{42}, -32699 \cdot 2^{46}, -41252441 \cdot 2^{41}, \\
& -29483621 \cdot 2^{41}, -2783 \cdot 2^{41}, -32699 \cdot 2^{44}, -7171925 \cdot 2^{39}, \\
& 12188517 \cdot 2^{39}, 2783 \cdot 2^{39}, 9391097 \cdot 2^{39}, -2783 \cdot 2^{38}, \\
& -187547 \cdot 2^{41}, -17037475 \cdot 2^{37}, -32699 \cdot 2^{40}, 2783 \cdot 2^{36}, \\
& 41307505 \cdot 2^{35}, 2783 \cdot 2^{35}, -32699 \cdot 2^{38}, 7171925 \cdot 2^{33}, \\
& -187547 \cdot 2^{37}, -2783 \cdot 2^{33}, 9391097 \cdot 2^{33}, 41252441 \cdot 2^{31}, \\
& 52183 \cdot 2^{41}, -7171925 \cdot 2^{30}, -32699 \cdot 2^{34}, -2783 \cdot 2^{30}, \\
& 5881627 \cdot 2^{30}, -2783 \cdot 2^{29}, -32699 \cdot 2^{32}, -7171925 \cdot 2^{27}, \\
& 52183 \cdot 2^{37}, 41252441 \cdot 2^{26}, 9391097 \cdot 2^{27}, -2783 \cdot 2^{26}, \\
& -187547 \cdot 2^{29}, 7171925 \cdot 2^{24}, -32699 \cdot 2^{28}, 2783 \cdot 2^{24}, \\
& 41307505 \cdot 2^{23}, 2783 \cdot 2^{23}, -32699 \cdot 2^{26}, -17037475 \cdot 2^{22}, \\
& -187547 \cdot 2^{25}, -2783 \cdot 2^{21}, 9391097 \cdot 2^{21}, 2783 \cdot 2^{20}, \\
& 12188517 \cdot 2^{19}, -7171925 \cdot 2^{18}, -32699 \cdot 2^{22}, -2783 \cdot 2^{18}, \\
& -29483621 \cdot 2^{17}, -41252441 \cdot 2^{16}, -32699 \cdot 2^{20}, -7171925 \cdot 2^{15}, \\
& 52183 \cdot 2^{25}, 2783 \cdot 2^{15}, 9391097 \cdot 2^{15}, -2783 \cdot 2^{14}, -187547 \cdot 2^{17}, \\
& 7171925 \cdot 2^{12}, -32699 \cdot 2^{16}, 41252441 \cdot 2^{11}, 41307505 \cdot 2^{11}, 2783 \cdot 2^{11}, \\
& -32699 \cdot 2^{14}, 7171925 \cdot 2^{9}, 38246123 \cdot 2^{9}, -2783 \cdot 2^{9}, 9391097 \cdot 2^{9}, \\
& 2783 \cdot 2^{8}, 52183 \cdot 2^{17}, 17037475 \cdot 2^{7}, -32699 \cdot 2^{10}, -2783 \cdot 2^{6}, \\
& -29483621 \cdot 2^{5}, -2783 \cdot 2^{5}, -32699 \cdot 2^{8}, -7171925 \cdot 2^{3}, 52183 \cdot 2^{13}, \\
& 2783 \cdot 2^{3}, 9391097 \cdot 2^{3}, -82504882, -187547 \cdot 2^{5}, 7171925, \\
& -32699 \cdot 2^{4}, 2783, 30315)) \qquad (61)
\end{aligned}
$$

$$
\begin{aligned}
\pi^2\log^3 2 \;&=\; \frac{1}{2021\cdot 2^{53}}P(5,2^{60},120,(21345\cdot 2^{59},-464511\cdot 2^{61},47870835\cdot 2^{57},\\
&-1312971\cdot 2^{61},-236170815\cdot 2^{56},1579179\cdot 2^{62},21345\cdot 2^{56},\\
&286131\cdot 2^{65},-47870835\cdot 2^{54},-464511\cdot 2^{57},-21345\cdot 2^{54},\\
&-173704605\cdot 2^{53},-21345\cdot 2^{53},-464511\cdot 2^{55},94128645\cdot 2^{52},\\
&286131\cdot 2^{61},21345\cdot 2^{51},1579179\cdot 2^{56},-21345\cdot 2^{50},\\
&215120589\cdot 2^{49},47870835\cdot 2^{48},-464511\cdot 2^{51},21345\cdot 2^{48},\\
&236128125\cdot 2^{47},236170815\cdot 2^{46},-464511\cdot 2^{49},47870835\cdot 2^{45},\\
&-1312971\cdot 2^{49},-21345\cdot 2^{45},1579179\cdot 2^{50},21345\cdot 2^{44},\\
&286131\cdot 2^{53},-47870835\cdot 2^{42},-464511\cdot 2^{45},-236170815\cdot 2^{41},\\
&-173704605\cdot 2^{41},-21345\cdot 2^{41},-464511\cdot 2^{43},-47870835\cdot 2^{39},\\
&56870019\cdot 2^{39},21345\cdot 2^{39},1579179\cdot 2^{44},-21345\cdot 2^{38},\\
&-1312971\cdot 2^{41},-94128645\cdot 2^{37},-464511\cdot 2^{39},21345\cdot 2^{36},\\
&236128125\cdot 2^{35},21345\cdot 2^{35},-464511\cdot 2^{37},47870835\cdot 2^{33},\\
&-1312971\cdot 2^{37},-21345\cdot 2^{33},1579179\cdot 2^{38},236170815\cdot 2^{31},\\
&286131\cdot 2^{41},-47870835\cdot 2^{30},-464511\cdot 2^{33},-21345\cdot 2^{30},\\
&1950735\cdot 2^{34},-21345\cdot 2^{29},-464511\cdot 2^{31},-47870835\cdot 2^{27},\\
&286131\cdot 2^{37},236170815\cdot 2^{26},1579179\cdot 2^{32},-21345\cdot 2^{26},\\
&-1312971\cdot 2^{29},47870835\cdot 2^{24},-464511\cdot 2^{27},21345\cdot 2^{24},\\
&236128125\cdot 2^{23},21345\cdot 2^{23},-464511\cdot 2^{25},-94128645\cdot 2^{22},\\
&-1312971\cdot 2^{25},-21345\cdot 2^{21},1579179\cdot 2^{26},21345\cdot 2^{20},\\
&56870019\cdot 2^{19},-47870835\cdot 2^{18},-464511\cdot 2^{21},-21345\cdot 2^{18},\\
&-173704605\cdot 2^{17},-236170815\cdot 2^{16},-464511\cdot 2^{19},-47870835\cdot 2^{15},\\
&286131\cdot 2^{25},21345\cdot 2^{15},1579179\cdot 2^{20},-21345\cdot 2^{14},-1312971\cdot 2^{17},\\
&47870835\cdot 2^{12},-464511\cdot 2^{15},236170815\cdot 2^{11},236128125\cdot 2^{11},\\
&21345\cdot 2^{11},-464511\cdot 2^{13},47870835\cdot 2^{9},215120589\cdot 2^{9},-21345\cdot 2^{9},\\
&1579179\cdot 2^{14},21345\cdot 2^{8},286131\cdot 2^{17},94128645\cdot 2^{7},-464511\cdot 2^{9},\\
&-21345\cdot 2^{6},-173704605\cdot 2^{5},-21345\cdot 2^{5},-464511\cdot 2^{7},\\
&-47870835\cdot 2^{3},286131\cdot 2^{13},21345\cdot 2^{3},1579179\cdot 2^{8},-472341630,\\
&-1312971\cdot 2^{5},47870835,-464511\cdot 2^{3},21345,0)) \qquad (62)
\end{aligned}
$$

$$
\begin{aligned}
\pi^4 \log 2 \quad = \quad & \frac{1}{2021 \cdot 2^{50}} P(5, 2^{60}, 120, (5157 \cdot 2^{59}, -89127 \cdot 2^{61}, 7805295 \cdot 2^{57}, \\
& -195183 \cdot 2^{61}, -32325939 \cdot 2^{56}, 1621107 \cdot 2^{59}, 5157 \cdot 2^{56}, \\
& 37287 \cdot 2^{65}, -7805295 \cdot 2^{54}, -89127 \cdot 2^{57}, -5157 \cdot 2^{54}, \\
& -24620409 \cdot 2^{53}, -5157 \cdot 2^{53}, -89127 \cdot 2^{55}, 12255165 \cdot 2^{52}, \\
& 37287 \cdot 2^{61}, 5157 \cdot 2^{51}, 1621107 \cdot 2^{53}, -5157 \cdot 2^{50}, \\
& 29192697 \cdot 2^{49}, 7805295 \cdot 2^{48}, -89127 \cdot 2^{51}, 5157 \cdot 2^{48}, \\
& 32315625 \cdot 2^{47}, 32325939 \cdot 2^{46}, -89127 \cdot 2^{49}, 7805295 \cdot 2^{45}, \\
& -195183 \cdot 2^{49}, -5157 \cdot 2^{45}, 1621107 \cdot 2^{47}, 5157 \cdot 2^{44}, \\
& 37287 \cdot 2^{53}, -7805295 \cdot 2^{42}, -89127 \cdot 2^{45}, -32325939 \cdot 2^{41}, \\
& -24620409 \cdot 2^{41}, -5157 \cdot 2^{41}, -89127 \cdot 2^{43}, -7805295 \cdot 2^{39}, \\
& 5866263 \cdot 2^{39}, 5157 \cdot 2^{39}, 1621107 \cdot 2^{41}, -5157 \cdot 2^{38}, \\
& -195183 \cdot 2^{41}, -12255165 \cdot 2^{37}, -89127 \cdot 2^{39}, 5157 \cdot 2^{36}, \\
& 32315625 \cdot 2^{35}, 5157 \cdot 2^{35}, -89127 \cdot 2^{37}, 7805295 \cdot 2^{33}, \\
& -195183 \cdot 2^{37}, -5157 \cdot 2^{33}, 1621107 \cdot 2^{35}, 32325939 \cdot 2^{31}, \\
& 37287 \cdot 2^{41}, -7805295 \cdot 2^{30}, -89127 \cdot 2^{33}, -5157 \cdot 2^{30}, \\
& 480951 \cdot 2^{33}, -5157 \cdot 2^{29}, -89127 \cdot 2^{31}, -7805295 \cdot 2^{27}, \\
& 37287 \cdot 2^{37}, 32325939 \cdot 2^{26}, 1621107 \cdot 2^{29}, -5157 \cdot 2^{26}, \\
& -195183 \cdot 2^{29}, 7805295 \cdot 2^{24}, -89127 \cdot 2^{27}, 5157 \cdot 2^{24}, \\
& 32315625 \cdot 2^{23}, 5157 \cdot 2^{23}, -89127 \cdot 2^{25}, -12255165 \cdot 2^{22}, \\
& -195183 \cdot 2^{25}, -5157 \cdot 2^{21}, 1621107 \cdot 2^{23}, 5157 \cdot 2^{20}, \\
& 5866263 \cdot 2^{19}, -7805295 \cdot 2^{18}, -89127 \cdot 2^{21}, -5157 \cdot 2^{18}, \\
& -24620409 \cdot 2^{17}, -32325939 \cdot 2^{16}, -89127 \cdot 2^{19}, -7805295 \cdot 2^{15}, \\
& 37287 \cdot 2^{25}, 5157 \cdot 2^{15}, 1621107 \cdot 2^{17}, -5157 \cdot 2^{14}, -195183 \cdot 2^{17}, \\
& 7805295 \cdot 2^{12}, -89127 \cdot 2^{15}, 32325939 \cdot 2^{11}, 32315625 \cdot 2^{11}, 5157 \cdot 2^{11}, \\
& -89127 \cdot 2^{13}, 7805295 \cdot 2^{9}, 29192697 \cdot 2^{9}, -5157 \cdot 2^{9}, 1621107 \cdot 2^{11}, \\
& 5157 \cdot 2^{8}, 37287 \cdot 2^{17}, 12255165 \cdot 2^{7}, -89127 \cdot 2^{9}, -5157 \cdot 2^{6}, \\
& -24620409 \cdot 2^{5}, -5157 \cdot 2^{5}, -89127 \cdot 2^{7}, -7805295 \cdot 2^{3}, 37287 \cdot 2^{13}, \\
& 5157 \cdot 2^{3}, 1621107 \cdot 2^{5}, -64651878, -195183 \cdot 2^{5}, 7805295, \\
& -89127 \cdot 2^{3}, 5157, 0)) \qquad (63)
\end{aligned}
$$

As before, the existence of BBP-type formulas for these constants was originally established by Broadhurst [19], although the explicit formulas given here were found by the author's PSLQ program. Proofs for these formulas were subsequently found by Kunle Adegoke [7].

9 Ternary (base-3) formulas

No ternary BBP formulas (i.e. formulas with $b = 3^m$ for some integer $m > 0$) were presented in [13], but several have subsequently been discovered. Here are some that are now known:

$$\log 2 = \frac{2}{3} P(1, 9, 2, (1, 0)) \tag{64}$$

$$\sqrt{3} \tan^{-1}\left(\frac{\sqrt{3}}{7}\right) = \frac{1}{6} P(1, 27, 3, (3, -1, 0)) \tag{65}$$

$$\pi\sqrt{3} = \frac{1}{9} P(1, 3^6, 12, (81, -54, 0, -9, 0, -12, -3, -2, 0, -1, 0, 0)) \tag{66}$$

$$\log 3 = \frac{1}{9} P(1, 9, 2, (9, 1)) \tag{67}$$

$$\log 3 = \frac{1}{3^6} P(1, 3^6, 6, (729, 81, 81, 9, 9, 1)) \tag{68}$$

$$\log 5 = \frac{4}{27} P(1, 3^4, 4, (9, 3, 1, 0)) \tag{69}$$

$$\log 7 = \frac{1}{3^5} P(1, 3^6, 6, (405, 81, 72, 9, 5, 0)) \tag{70}$$

$$\log 11 = \frac{1}{2 \cdot 3^9} P(1, 3^{10}, 10, (85293, 10935, 9477, 1215, 648, 135, 117, 15, 13, 0)) \tag{71}$$

$$\log 13 = \frac{1}{3^5} P(1, 3^6, 6, (567, 81, 36, 9, 7, 0)) \tag{72}$$

$$\pi^2 = \frac{2}{27} P(2, 3^6, 12, (243, -405, 0, -81, -27, -72, -9, -9, 0, -5, 1, 0)) \tag{73}$$

$$\log^2 3 = \frac{1}{3^6} P(2, 3^6, 12, (4374, -13122, 0, -2106, -486, -1944, -162, -234, 0, -162, 18, -8)) \tag{74}$$

$$\pi\sqrt{3}\log 3 = \frac{2}{27} P(2, 3^6, 12, (243, -405, -486, -135, 27, 0, -9, 15, 18, 5, -1, 0)) \tag{75}$$

$$13\zeta(3) - \pi^2 \log 3 + \log^3 3 = \frac{2}{3} P(3, 9, 2, (9, 1)) \tag{76}$$

$$\frac{1}{\sqrt{3}} \left(\frac{29\pi^3}{1296} - \frac{\pi \log^2 3}{48} \right) = \frac{1}{2 \cdot 3^6} P(3, 3^6, 12, (3^6, -5 \cdot 3^5, -2 \cdot 3^6, -5 \cdot 3^4, 3^4, 0, -3^3, 5 \cdot 3^2, 2 \cdot 3^3, 5 \cdot 3, -3, 0))$$

$$\frac{13}{3}\zeta(3) + \frac{\log^3 3}{8} - \frac{5\pi^2 \log 3}{24} = \frac{1}{3^5} P(3, 3^6, 12, (3^6, 3^5, 0, -3^4, -3^4, -2 \cdot 3^3, -3^3, -3^2, 0, 3, 3, 2)) \tag{77}$$

$$\frac{127\pi^4}{5184} - \frac{\pi^2 \log^2 3}{32} + \frac{5 \log^4 3}{192} = \frac{1}{3^6} P(4, 3^6, 12, (3^7, -5 \cdot 3^7, 0, -19 \cdot 3^4, -3^5, -2 \cdot 3^6, -3^4, -19 \cdot 3^2, 0, -5 \cdot 3^3, 3^2, -10)) \tag{78}$$

$$\frac{1}{\sqrt{3}} \left(11\, \mathrm{Cl}_4 \left(\frac{\pi}{3} \right) - \frac{29\pi^3 \log 3}{288} + \frac{\pi \log^3 3}{32} \right) = \frac{1}{2} P(4, -27, 6, (9, -15, -18, -5, 1, 0)) \tag{79}$$

$$\frac{1573}{144}\zeta(5) - \frac{1}{128} \log^5 3 + \frac{1}{64}\pi^2 \log^3 3 - \frac{127}{3456}\pi^4 \log 3 = \frac{1}{3^5} P(5, 3^6, 12, (3^7, -5 \cdot 3^7, 0, -19 \cdot 3^4, -3^5, -2 \cdot 3^6, -3^4, -19 \cdot 3^2, 0, -5 \cdot 3^3, 3^2, -10)) \tag{80}$$

In Formula 79 above, Cl denotes the Clausen function: $\mathrm{Cl}_n(t) = \sum_{k \geq 1} \cos(kt)/k^n$ if n is odd, otherwise $\mathrm{Cl}_n(t) = \sum_{k \geq 1} \sin(kt)/k^n$.

Formulas 64 and 65 appeared in [15]. Alexander Povolotsky discovered the formula $\log 3 = 1/4 + 1/4 \sum_{k \geq 0} 1/9^{k+1}(27/(2k+1) + 4/(2k+2) + 1/(2k+3))$. Subsequently Jaume Oliver i Lafont simplified this to $\log 3 = \sum_{k \geq 0} 1/9^{k+1}(9/(2k+1) + 1/(2k+2))$, which after minor modification yields Formula 67 [26]. Formulas 69 through 72 are due to Oliver i Lafont. Formulas 66 and 73 through 75 are due to Broadhurst [18]. Formulas 76 through 80 are due to Kunle Adegoke [3]. Formula 79 was first found by Broadhurst [19], using PSLQ.

10 Other specific bases

Here are several interesting results in other bases. Here $\phi = (1+\sqrt{5})/2$ is the golden mean.

$$\log\left(\frac{9}{10}\right) = \frac{-1}{10}P(1,10,1,(1)) \tag{81}$$

$$\log\left(\frac{3}{2}\right) = \frac{2}{5}P(1,25,2,(1,0)) \tag{82}$$

$$\sqrt{5}\log\phi = P(1,5,2,(1,0)) \tag{83}$$

$$\frac{25}{2}\log\left(\frac{781}{256}\left(\frac{57-5\sqrt{5}}{57+5\sqrt{5}}\right)^{\sqrt{5}}\right) = P(1,5^5,5,(0,5,1,0,0)) \tag{84}$$

$$\begin{aligned}\frac{1}{\sqrt{\phi}}\tan^{-1}\left(\frac{5^{1/4}}{\sqrt{\phi}}\frac{233-329\sqrt{5}}{5938}\right) &+ \sqrt{\phi}\tan^{-1}\left(\frac{5^{1/4}}{\sqrt{\phi}}\frac{939+281\sqrt{5}}{5938}\right)\\ &= \frac{1}{2\cdot 5^{13/4}}P(1,5^5,5,(125,-25,5,-1,0))\end{aligned} \tag{85}$$

$$\log\left(\frac{1111111111}{387420489}\right) = \frac{1}{10^8}P(1,10^{10},10,(10^8,10^7,10^6,10^5,10^4,10^3,10^2,10^1,1,0)) \tag{86}$$

Formula 81 appeared in [13] (although it is an elementary observation). Formulas 82 and 83 are due to Jaume Oliver i Lafont. Formulas 84 through 86 appeared in [15].

11 General bases

$$b^2\log\left(\frac{b^2+b+1}{b^2-2b+1}\right) = 3P(1,b^3,3,(b,1,0)) \tag{87}$$

$$b^{b-2}\log\left(\frac{b^b-1}{(b-1)^b}\right) = P(1,b^b,b,(b^{b-2},b^{b-3},\cdots,b^2,b,1)) \tag{88}$$

$$\sqrt{b}\arctan(\frac{1}{\sqrt{b}}) = \frac{1}{b}P(1,b^2,4,(b,0,-1,0)) \tag{89}$$

$$b^2\sqrt{3}\arctan\left(\frac{\sqrt{3}}{2b-1}\right) = \frac{3}{2}P(1,-b^3,3,(b,1,0)) \tag{90}$$

$$b^2\sqrt{3}\arctan\left(\frac{\sqrt{3}}{2b+1}\right) = \frac{3}{2}P(1,b^3,3,(b,-1,0)) \tag{91}$$

$$b^7\arctan\left(\frac{1}{2b-1}\right) = \frac{1}{16}P(1,16b^8,8,(8b^6,8b^5,4b^4,0,-2b^2,-2b,-1,0)) \tag{92}$$

$$b^7\arctan\left(\frac{1}{2b+1}\right) = \frac{1}{16}P(1,16b^8,8,(8b^6,-8b^5,4b^4,0,-2b^2,2b,-1,0)) \tag{93}$$

$$b^3\sqrt{b}\sqrt{2}\arctan\left(\frac{\sqrt{b}}{b-1}\sqrt{2}\right) = 2P(1,b^4,8,(b^3,0,b^2,0,-b,0,-1,0)) \tag{94}$$

$$9b^5\sqrt{3}\arctan\left(\frac{1}{\sqrt{3}}\frac{1}{2b-1}\right) = \frac{1}{2}P(1,-27b^6,6,(9b^4,9b^3,6b^2,3b,1,0)) \tag{95}$$

$$9b^5\sqrt{3}\arctan\left(\frac{1}{\sqrt{3}}\frac{1}{2b+1}\right) = \frac{1}{2}P(1,-27b^6,6,(9b^4,-9b^3,6b^2,-3b,1,0)) \tag{96}$$

$$b^2\sqrt{b}\arctan\left(\frac{\sqrt{b}}{b-1}\right) = P(1,-b^3,6,(b^2,0,2b,0,1,0) \tag{97}$$

$$\log\left(\frac{b+1}{b}\right) = \frac{1}{b^2}P(1,b^2,2,(b,-1)) \tag{98}$$

$$\log\left(\frac{b-1}{b}\right) = -\frac{1}{b^2}P(1,b^2,2,(b,1)) \tag{99}$$

$$\sqrt{b}\log\left(\frac{\sqrt{b}+1}{\sqrt{b}-1}\right) = 2P(1,b,2,(1,0)) \tag{100}$$

$$\log\left(\frac{b^2+b+1}{b^2}\right) = \frac{1}{b^3}P(1, b^3, 3, (b^2, b, -2)) \tag{101}$$

$$\log\left(\frac{b^2-b+1}{b^2}\right) = -\frac{1}{b^3}P(1, -b^3, 3, (b^2, -b, -2)) \tag{102}$$

$$\log\left(\frac{2b^2-2b+1}{2b^2}\right) = -\frac{1}{2b^4}P(1, -4b^4, 4, (2b^3, 0, -b, -1)) \tag{103}$$

$$\log\left(\frac{2b^2+2b+1}{2b^2}\right) = \frac{1}{2b^4}P(1, -4b^4, 4, (2b^3, 0, -b, 1)) \tag{104}$$

$$\frac{b\sqrt{b}}{\sqrt{2}}\log\left(\frac{b+\sqrt{2}\sqrt{b}+1}{b-\sqrt{2}\sqrt{b}+1}\right) = 2P(1, -b^2, 4, (b, 0, -1, 0)) \tag{105}$$

$$\log\left(\frac{3b^2\pm 3b+1}{3b^2}\right) = \pm\frac{1}{27b^6}P(1, -27b^6, 6, (27b^5, \mp 9b^4, 0, \pm 3b^2, -3b, \pm 2)) \tag{106}$$

$$\frac{b^2\sqrt{b}}{\sqrt{3}}\log\left(\frac{b+\sqrt{3}\sqrt{b}+1}{b-\sqrt{3}\sqrt{b}+1}\right) = 2P(1, -b^3, 6, (b^2, 0, 0, 0, -1, 0)) \tag{107}$$

Formulas 87 and 88 appeared in [15]. Formulas 89 through 107 are due to Kunle Adegoke [5]. Formula 100 was first obtained by Jaume Oliver Lafont [29].

12 Zero relations

Below are some of the known BBP zero relations, or in other words BBP-type formulas that evaluate to zero. These have been discovered using the author's PSLQ program, and most are new with this compilation. For brevity, not all of the zero relations that have been found are listed here — some of the larger ones are omitted — although the author has a complete set. Further, zero relations that are merely a rewriting of another on the list, such as by expanding a relation with base b and length n to one with base b^r and length rn, are not included in these listings. For convenience, however, the total number of linearly independent zero relations for various choices of s, b and n, including rewritings and unlisted relations, are tabulated in Table 1.

Knowledge of these zero relations is essential for finding formulas such as those above using integer relation programs (such as PSLQ). This is because unless these zero relations are excluded from the search for a conjectured BBP-type formula, the search may only recover a zero relation. A zero relation may be excluded from a integer relation search by setting the input vector element whose position corresponds to the zero relation's smallest nonzero element to some value that is not linearly related to the other entries of the input vector.

For example, note in Table 1 below that there are five zero relations with $s = 1$, $b = 2^{12}$ and $n = 24$. These relations are given below as formulas 110 through 115. If one is searching for a conjectured formula with these parameters using PSLQ, then these five zero relations must be excluded. This can be done by setting entries 19 through 23 of the PSLQ input vector to e, e^2, e^3, e^4 and e^5, respectively, where e is the base of natural logarithms. Positions 19 through 23 are specified here because in relations 110 through 115 below, the smallest nonzero entries appear in positions 23, 22, 21, 20 and 19, respectively. Powers of e are specified here because, as far as anyone can tell (although this has not been rigorously proven), e is not a polylogarithmic constant in the sense of this paper, and thus it and its powers are not expected to satisfy BBP-type linear relations (this assumption is confirmed by extensive e xperience using the author's PSLQ programs). In any event, it is clear that many other sets of transcendental constants could be used here.

Note that by simply adding a rational multiple of one of these zero relations to one the formulas above (with matching arguments s, b and n), one can produce a valid variant of that formula. Clearly infinitely many variants can be produced in this manner.

Aside from the discussion in [20], these zero relations are somewhat mysterious — it is not understood why zero relations occur for certain s, b and n, but not others. It should also be noted that in most but not all cases where a zero relation has been found, nontrivial BBP-type formulas have been found with the same parameters. This suggests that significant BBP-type results may remain to be discovered. In any event, it is hoped that this compilation will spur some additional insight into these questions.

Note that all of these formulas except for the last two are binary formulas (i.e. $b = 2^m$ for some integer $m > 0$).

s	b	n	No. zero relations	s	b	n	No. zero relations
1	16	8	1	1	2^{48}	48	1
1	64	6	1	1	2^{48}	96	5
1	2^{8}	16	1	1	2^{52}	104	1
1	2^{12}	12	1	1	2^{54}	54	1
1	2^{12}	24	5	1	2^{56}	112	1
1	2^{16}	32	1	1	2^{60}	60	1
1	2^{18}	18	1	1	2^{60}	120	7
1	2^{20}	40	3	2	2^{12}	24	2
1	2^{24}	24	1	2	2^{20}	40	1
1	2^{24}	48	5	2	2^{24}	48	2
1	2^{28}	56	1	2	2^{36}	72	2
1	2^{30}	30	1	2	2^{40}	80	1
1	2^{30}	60	1	2	2^{48}	96	2
1	2^{32}	64	1	2	2^{60}	120	4
1	2^{36}	36	1	3	2^{12}	24	1
1	2^{36}	72	5	3	2^{24}	48	1
1	2^{40}	80	3	3	2^{36}	72	1
1	2^{42}	42	1	3	2^{48}	96	1
1	2^{42}	84	1	3	2^{60}	120	2
1	2^{44}	88	1	4	2^{60}	120	1
1	3^{6}	12	2				

Table 1: Zero relation counts for various parameters

$$0 = P(1,16,8,(-8,8,4,8,2,2,-1,0)) \quad (108)$$

$$0 = P(1,64,6,(16,-24,-8,-6,1,0)) \quad (109)$$

$$0 = P(1,2^{12},24,(0,0,2^{11},-2^{11},0,-2^9,256,-3\cdot 2^8,0,0,-64,-128,0,-32,-32, -48,0,-24,-4,-8,0,-2,1,0)) \quad (110)$$

$$0 = P(1,2^{12},24,(2^{11},-2^{11},-2^{11},0,-2^9,-2^{10},-2^8,0, -2^8,-2^7,2^6,0,-32,32,32,0,8,16,4,0,4,2,-1,0)) \quad (111)$$

$$0 = P(1,2^{12},24,(-2^9,-2^{10},2^{10},7\cdot 2^8,256,3\cdot 2^8,64,3\cdot 2^7,0,0,0,0,8,-32,-16, 12,-4,4,-1,8,0,-1,0,0)) \quad (112)$$

$$0 = P(1,2^{12},24,(2^9,-2^{10},-2^9,256,0,256,64,3\cdot 2^7,64,0,0,0,-8,-16,8,12,0, 4,-1,2,-1,0,0,0)) \quad (113)$$

$$0 = P(1,2^{12},24,(3\cdot 2^9,-3\cdot 2^{10},0,-256,0,0,192,3\cdot 2^7,0,0,0,-64,-24,-48, 0,-12,0,0,-3,2,0,0,0,0)) \quad (114)$$

$$0 = P(1,2^{12},24,(-2^{10},3\cdot 2^9,2^9,256,128,128,-64,-192,0,32,0,32,16,16, -8,0,-2,-2,1,0,0,0,0,0)) \quad (115)$$

$$0 = P(1,2^{20},40,(0,2^{18},-2^{18},2^{17},0,-5\cdot 2^{16},2^{16},-5\cdot 2^{15},0,-2^{16},-2^{14},2^{13}, 0,-5\cdot 2^{12},-2^{14},-5\cdot 2^{11},0,2^{10},-2^{10},-2^{11},0,-5\cdot 2^8,256,-5\cdot 2^7,0,64,-64, 32,0,0,16,-40,0,4,16,2,0,-5,1,0)) \quad (116)$$

$$0 = P(1,2^{20},40,(2^{18},-2^{19},0,-2^{17},3\cdot 2^{15},2^{16},0,0,2^{14},2^{13},0,-2^{13},-2^{12},2^{12}, 5\cdot 2^{10},0,2^{10},-2^{11},0,-2^9,-256,256,0,0,-96,-128,0,-32,-16,-24, 0,0,4,-8,-5,-2,-1,1,0,0)) \quad (117)$$

$$0 = P(1,2^{20},40,(-2^{18},3\cdot 2^{18},0,-2^{18},-13\cdot 2^{15},0,0,5\cdot 2^{15},-2^{14},2^{13},0,-2^{14}, 2^{12},0,5\cdot 2^{10},5\cdot 2^{11},-2^{10},3\cdot 2^{10},0,3\cdot 2^9,256,0,0,5\cdot 2^7,13\cdot 2^5, 192,0,-64,16,40,0,40,-4,12,-5,-4,1,0,0,0)) \quad (118)$$

$$0 = P(1,2^{20},40,(2^{19},-3\cdot 2^{19},2^{18},0,2^{19},3\cdot 2^{17}, -2^{16},0,2^{15},2^{16},2^{14},0,-2^{13},3\cdot 2^{13},2^{14},0, 2^{11},-3\cdot 2^{11},2^{10},0,-2^9,3\cdot 2^9,-2^8,0,-2^9,-3\cdot 2^7, 2^6,0,-2^5,-2^6,-2^4,0,2^3,-3\cdot 2^3,-2^4,0,-2,6,-1,0)) \quad (119)$$

$$0 = P(2,2^{12},24,(0,2^{10},-3\cdot 2^{10},2^9,0,2^{10},0,9\cdot 2^7,3\cdot 2^7,64,0,128,0,16,48,72, 0,16,0,2,-6,1,0,0)) \quad (120)$$

$$
\begin{aligned}
0 &= P(2,2^{12},24,(-2^{11},0,17\cdot 2^{11},-17\cdot 2^{10},2^{9},-15\cdot 2^{10},-256,-63\cdot 2^{8},\\
&\quad -17\cdot 2^{8},0,64,-5\cdot 2^{8},32,0,-17\cdot 2^{5},-63\cdot 2^{4},-8,-240,4,-68,68,\\
&\quad 0,-1,0)) \qquad (121)\\
0 &= P(2,2^{20},40,(2^{19},-3\cdot 2^{20},-2^{18},13\cdot 2^{18},3\cdot 2^{20},-3\cdot 2^{18},2^{16},\\
&\quad -25\cdot 2^{16},2^{15},-3\cdot 2^{16},-2^{14},13\cdot 2^{14},-2^{13},-3\cdot 2^{14},-3\cdot 2^{15},-25\cdot 2^{12},\\
&\quad 2^{11},-3\cdot 2^{12},-2^{10},-3\cdot 2^{12},-2^{9},-3\cdot 2^{10},256,-25\cdot 2^{8},-3\cdot 2^{10},\\
&\quad -3\cdot 2^{8},-64,13\cdot 2^{6},-32,-192,16,-25\cdot 2^{4},8,-48,96,52,-2,-12,1,0)) \qquad (122)\\
0 &= P(3,2^{12},24,(2^{11},-19\cdot 2^{11},5\cdot 2^{14},-2^{11},-2^{9},-23\cdot 2^{10},256,-27\cdot 2^{10},\\
&\quad -5\cdot 2^{11},-19\cdot 2^{7},-64,-7\cdot 2^{9},-32,-19\cdot 2^{5},-5\cdot 2^{8},-27\cdot 2^{6},8,\\
&\quad -23\cdot 2^{4},-4,-8,160,-38,1,0)) \qquad (123)
\end{aligned}
$$

$$
\begin{aligned}
0 &= P(3,2^{60},120,(7\cdot 2^{59},-3\cdot 5^{2}\cdot 2^{60},1579\cdot 2^{57},-29\cdot 2^{60},-3\cdot 23\cdot 31\cdot 2^{56},\\
&\quad 3\cdot 5\cdot 2^{60},7\cdot 2^{56},67\cdot 2^{59},-1579\cdot 2^{54},-3\cdot 5^{2}\cdot 2^{56},-7\cdot 2^{54},-5\cdot 11\cdot 43\cdot 2^{53},\\
&\quad -7\cdot 2^{53},-3\cdot 5^{2}\cdot 2^{54},3\cdot 7\cdot 13\cdot 2^{52},67\cdot 2^{55},7\cdot 2^{51},3\cdot 5\cdot 2^{54},-7\cdot 2^{50},\\
&\quad 3\cdot 631\cdot 2^{49},1579\cdot 2^{48},-3\cdot 5^{2}\cdot 2^{50},7\cdot 2^{48},5^{3}\cdot 17\cdot 2^{47},3\cdot 23\cdot 31\cdot 2^{46},\\
&\quad -3\cdot 5^{2}\cdot 2^{48},1579\cdot 2^{45},-29\cdot 2^{48},-7\cdot 2^{45},3\cdot 5\cdot 2^{48},7\cdot 2^{44},67\cdot 2^{47},\\
&\quad -1579\cdot 2^{42},-3\cdot 5^{2}\cdot 2^{44},-3\cdot 23\cdot 31\cdot 2^{41},-5\cdot 11\cdot 43\cdot 2^{41},-7\cdot 2^{41},\\
&\quad -3\cdot 5^{2}\cdot 2^{42},-1579\cdot 2^{39},-3^{4}\cdot 13\cdot 2^{39},7\cdot 2^{39},3\cdot 5\cdot 2^{42},-7\cdot 2^{38},\\
&\quad -29\cdot 2^{40},-3\cdot 7\cdot 13\cdot 2^{37},-3\cdot 5^{2}\cdot 2^{38},7\cdot 2^{36},5^{3}\cdot 17\cdot 2^{35},7\cdot 2^{35},\\
&\quad -3\cdot 5^{2}\cdot 2^{36},1579\cdot 2^{33},-29\cdot 2^{36},-7\cdot 2^{33},3\cdot 5\cdot 2^{36},3\cdot 23\cdot 31\cdot 2^{31},\\
&\quad 67\cdot 2^{35},-1579\cdot 2^{30},-3\cdot 5^{2}\cdot 2^{32},-7\cdot 2^{30},-3\cdot 5\cdot 2^{33},-7\cdot 2^{29},\\
&\quad -3\cdot 5^{2}\cdot 2^{30},-1579\cdot 2^{27},67\cdot 2^{31},3\cdot 23\cdot 31\cdot 2^{26},3\cdot 5\cdot 2^{30},-7\cdot 2^{26},\\
&\quad -29\cdot 2^{28},1579\cdot 2^{24},-3\cdot 5^{2}\cdot 2^{26},7\cdot 2^{24},5^{3}\cdot 17\cdot 2^{23},7\cdot 2^{23},\\
&\quad -3\cdot 5^{2}\cdot 2^{24},-3\cdot 7\cdot 13\cdot 2^{22},-29\cdot 2^{24},-7\cdot 2^{21},3\cdot 5\cdot 2^{24},7\cdot 2^{20},\\
&\quad -3^{4}\cdot 13\cdot 2^{19},-1579\cdot 2^{18},-3\cdot 5^{2}\cdot 2^{20},-7\cdot 2^{18},-5\cdot 11\cdot 43\cdot 2^{17},\\
&\quad -3\cdot 23\cdot 31\cdot 2^{16},-3\cdot 5^{2}\cdot 2^{18},-1579\cdot 2^{15},67\cdot 2^{19},7\cdot 2^{15},3\cdot 5\cdot 2^{18},\\
&\quad -7\cdot 2^{14},-29\cdot 2^{16},1579\cdot 2^{12},-3\cdot 5^{2}\cdot 2^{14},3\cdot 23\cdot 31\cdot 2^{11},5^{3}\cdot 17\cdot 2^{11},\\
&\quad 7\cdot 2^{11},-3\cdot 5^{2}\cdot 2^{12},1579\cdot 2^{9},3\cdot 631\cdot 2^{9},-7\cdot 2^{9},3\cdot 5\cdot 2^{12},7\cdot 2^{8},\\
&\quad 67\cdot 2^{11},3\cdot 7\cdot 13\cdot 2^{7},-3\cdot 5^{2}\cdot 2^{8},-7\cdot 2^{6},-5\cdot 11\cdot 43\cdot 2^{5},-7\cdot 2^{5},\\
&\quad -3\cdot 5^{2}\cdot 2^{6},-1579\cdot 2^{3},67\cdot 2^{7},7\cdot 2^{3},3\cdot 5\cdot 2^{6},-3\cdot 23\cdot 31\cdot 2,\\
&\quad -29\cdot 2^{4},1579,-3\cdot 5^{2}\cdot 2^{2},7,0)) \qquad (124)
\end{aligned}
$$

$$
\begin{aligned}
0 \quad = \quad & P(3, 2^{60}, 120, (2^{59}, 0, -353 \cdot 2^{57}, 7 \cdot 2^{62}, 3 \cdot 7 \cdot 113 \cdot 2^{56}, -3^3 \cdot 5 \cdot 2^{59}, 2^{56}, \\
& -97 \cdot 2^{60}, 353 \cdot 2^{54}, 0, -2^{54}, 5 \cdot 331 \cdot 2^{53}, -2^{53}, 0, -3 \cdot 337 \cdot 2^{52}, -97 \cdot 2^{56}, 2^{51}, \\
& -3^3 \cdot 5 \cdot 2^{53}, -2^{50}, -3^2 \cdot 239 \cdot 2^{49}, -353 \cdot 2^{48}, 0, 2^{48}, -5^3 \cdot 19 \cdot 2^{47}, -3 \cdot 7 \cdot 113 \cdot 2^{46}, \\
& 0, -353 \cdot 2^{45}, 7 \cdot 2^{50}, -2^{45}, -3^3 \cdot 5 \cdot 2^{47}, 2^{44}, -97 \cdot 2^{48}, 353 \cdot 2^{42}, 0, 3 \cdot 7 \cdot 113 \cdot 2^{41}, \\
& 5 \cdot 331 \cdot 2^{41}, -2^{41}, 0, 353 \cdot 2^{39}, -3^6 \cdot 2^{39}, 2^{39}, -3^3 \cdot 5 \cdot 2^{41}, -2^{38}, 7 \cdot 2^{42}, \\
& 3 \cdot 337 \cdot 2^{37}, 0, 2^{36}, -5^3 \cdot 19 \cdot 2^{35}, 2^{35}, 0, -353 \cdot 2^{33}, 7 \cdot 2^{38}, -2^{33}, -3^3 \cdot 5 \cdot 2^{35}, \\
& -3 \cdot 7 \cdot 113 \cdot 2^{31}, -97 \cdot 2^{36}, 353 \cdot 2^{30}, 0, -2^{30}, -3^2 \cdot 5 \cdot 2^{33}, -2^{29}, 0, 353 \cdot 2^{27}, \\
& -97 \cdot 2^{32}, -3 \cdot 7 \cdot 113 \cdot 2^{26}, -3^3 \cdot 5 \cdot 2^{29}, -2^{26}, 7 \cdot 2^{30}, -353 \cdot 2^{24}, 0, 2^{24}, \\
& -5^3 \cdot 19 \cdot 2^{23}, 2^{23}, 0, 3 \cdot 337 \cdot 2^{22}, 7 \cdot 2^{26}, -2^{21}, -3^3 \cdot 5 \cdot 2^{23}, 2^{20}, -3^6 \cdot 2^{19}, \\
& 353 \cdot 2^{18}, 0, -2^{18}, 5 \cdot 331 \cdot 2^{17}, 3 \cdot 7 \cdot 113 \cdot 2^{16}, 0, 353 \cdot 2^{15}, -97 \cdot 2^{20}, 2^{15}, \\
& -3^3 \cdot 5 \cdot 2^{17}, -2^{14}, 7 \cdot 2^{18}, -353 \cdot 2^{12}, 0, -3 \cdot 7 \cdot 113 \cdot 2^{11}, -5^3 \cdot 19 \cdot 2^{11}, 2^{11}, 0, \\
& -353 \cdot 2^{9}, -3^2 \cdot 239 \cdot 2^{9}, -2^{9}, -3^3 \cdot 5 \cdot 2^{11}, 2^{8}, -97 \cdot 2^{12}, -3 \cdot 337 \cdot 2^{7}, 0, -2^{6}, \\
& 5 \cdot 331 \cdot 2^{5}, -2^{5}, 0, 353 \cdot 2^{3}, -97 \cdot 2^{8}, 2^{3}, -3^3 \cdot 5 \cdot 2^{5}, \\
& 3 \cdot 7 \cdot 113 \cdot 2, 7 \cdot 2^{6}, -353, 0, 1, 0)) \qquad (125)
\end{aligned}
$$

$$
\begin{aligned}
0 \;=\;& P(4, 2^{60}, 120, (-31 \cdot 2^{59}, 3 \cdot 269 \cdot 2^{60}, -5 \cdot 61 \cdot 107 \cdot 2^{57}, \\
& 1553 \cdot 2^{60}, 3^2 \cdot 14243 \cdot 2^{56}, -3 \cdot 1051 \cdot 2^{60}, -31 \cdot 2^{56}, \\
& -9319 \cdot 2^{59}, 5 \cdot 61 \cdot 107 \cdot 2^{54}, 3 \cdot 269 \cdot 2^{56}, 31 \cdot 2^{54}, \\
& 31 \cdot 3187 \cdot 2^{53}, 31 \cdot 2^{53}, 3 \cdot 269 \cdot 2^{54}, -3^2 \cdot 5 \cdot 1061 \cdot 2^{52}, \\
& -9319 \cdot 2^{55}, -31 \cdot 2^{51}, -3 \cdot 1051 \cdot 2^{54}, 31 \cdot 2^{50}, -3 \cdot 38567 \cdot 2^{49}, \\
& -5 \cdot 61 \cdot 107 \cdot 2^{48}, 3 \cdot 269 \cdot 2^{50}, -31 \cdot 2^{48}, -5^5 \cdot 41 \cdot 2^{47}, \\
& -3^2 \cdot 14243 \cdot 2^{46}, 3 \cdot 269 \cdot 2^{48}, -5 \cdot 61 \cdot 107 \cdot 2^{45}, 1553 \cdot 2^{48}, \\
& 31 \cdot 2^{45}, -3 \cdot 1051 \cdot 2^{48}, -31 \cdot 2^{44}, -9319 \cdot 2^{47}, 5 \cdot 61 \cdot 107 \cdot 2^{42}, \\
& 3 \cdot 269 \cdot 2^{44}, 3^2 \cdot 14243 \cdot 2^{41}, 31 \cdot 3187 \cdot 2^{41}, 31 \cdot 2^{41}, \\
& 3 \cdot 269 \cdot 2^{42}, 5 \cdot 61 \cdot 107 \cdot 2^{39}, -3^4 \cdot 7 \cdot 37 \cdot 2^{39}, -31 \cdot 2^{39}, \\
& -3 \cdot 1051 \cdot 2^{42}, 31 \cdot 2^{38}, 1553 \cdot 2^{40}, 3^2 \cdot 5 \cdot 1061 \cdot 2^{37}, \\
& 3 \cdot 269 \cdot 2^{38}, -31 \cdot 2^{36}, -5^5 \cdot 41 \cdot 2^{35}, -31 \cdot 2^{35}, \\
& 3 \cdot 269 \cdot 2^{36}, -5 \cdot 61 \cdot 107 \cdot 2^{33}, 1553 \cdot 2^{36}, 31 \cdot 2^{33}, \\
& -3 \cdot 1051 \cdot 2^{36}, -3^2 \cdot 14243 \cdot 2^{31}, -9319 \cdot 2^{35}, \\
& 5 \cdot 61 \cdot 107 \cdot 2^{30}, 3 \cdot 269 \cdot 2^{32}, 31 \cdot 2^{30}, -3 \cdot 13 \cdot 47 \cdot 2^{33}, \\
& 31 \cdot 2^{29}, 3 \cdot 269 \cdot 2^{30}, 5 \cdot 61 \cdot 107 \cdot 2^{27}, -9319 \cdot 2^{31}, \\
& -3^2 \cdot 14243 \cdot 2^{26}, -3 \cdot 1051 \cdot 2^{30}, 31 \cdot 2^{26}, 1553 \cdot 2^{28}, \\
& -5 \cdot 61 \cdot 107 \cdot 2^{24}, 3 \cdot 269 \cdot 2^{26}, -31 \cdot 2^{24}, -5^5 \cdot 41 \cdot 2^{23}, \\
& -31 \cdot 2^{23}, 3 \cdot 269 \cdot 2^{24}, 3^2 \cdot 5 \cdot 1061 \cdot 2^{22}, 1553 \cdot 2^{24}, \\
& 31 \cdot 2^{21}, -3 \cdot 1051 \cdot 2^{24}, -31 \cdot 2^{20}, -3^4 \cdot 7 \cdot 37 \cdot 2^{19}, \\
& 5 \cdot 61 \cdot 107 \cdot 2^{18}, 3 \cdot 269 \cdot 2^{20}, 31 \cdot 2^{18}, 31 \cdot 3187 \cdot 2^{17}, \\
& 3^2 \cdot 14243 \cdot 2^{16}, 3 \cdot 269 \cdot 2^{18}, 5 \cdot 61 \cdot 107 \cdot 2^{15}, -9319 \cdot 2^{19}, \\
& -31 \cdot 2^{15}, -3 \cdot 1051 \cdot 2^{18}, 31 \cdot 2^{14}, 1553 \cdot 2^{16}, \\
& -5 \cdot 61 \cdot 107 \cdot 2^{12}, 3 \cdot 269 \cdot 2^{14}, -3^2 \cdot 14243 \cdot 2^{11}, \\
& -5^5 \cdot 41 \cdot 2^{11}, -31 \cdot 2^{11}, 3 \cdot 269 \cdot 2^{12}, \\
& -5 \cdot 61 \cdot 107 \cdot 2^9, -3 \cdot 38567 \cdot 2^9, 31 \cdot 2^9, -3 \cdot 1051 \cdot 2^{12}, \\
& -31 \cdot 2^8, -9319 \cdot 2^{11}, -3^2 \cdot 5 \cdot 1061 \cdot 2^7, 3 \cdot 269 \cdot 2^8, \\
& 31 \cdot 2^6, 31 \cdot 3187 \cdot 2^5, 31 \cdot 2^5, 3 \cdot 269 \cdot 2^6, \\
& 5 \cdot 61 \cdot 107 \cdot 2^3, -9319 \cdot 2^7, -31 \cdot 2^3, -3 \cdot 1051 \cdot 2^6, \\
& 3^2 \cdot 14243 \cdot 2, 1553 \cdot 2^4, -61 \cdot 107 \cdot 5, 3 \cdot 269 \cdot 2^2, -31, 0)) \qquad (126)
\end{aligned}
$$

$$
\begin{aligned}
0 \;=\;& P(1, 729, 12, (0, 81, -162, 0, 27, 36, 0, 9, 6, 4, -1, 0)) \qquad (127) \\
0 \;=\;& P(1, 729, 12, (243, -324, -162, -81, 0, -36, -9, 0, 6, -1, 0, 0)) \qquad (128)
\end{aligned}
$$

Relation 108 appeared in [13]. Relation 109 and 110 were given in [15]. Relations 111 and 119 are due to Jaume Oliver Lafont [27]. Lafont subsequently proved 110 through 118 [28]. Relations 112 through 118 and 120 through 123 and 127 and 128 were found experimentally by the author using his PSLQ program, but in the wake of the other relations in this list are now proven. Relation 123 has been proved by Kunle Adegoke as the difference of Formulas 37 and 45. Relations 124 through 126 are due to Kunle Adegoke [2], who also proved 127 and 128 [3].

13 Curiosities

There are two other formulas worth mentioning, although neither, technically speaking, is a BBP-type formula. The first formula employs the irrational base $b = 2/\phi = 2\phi - 2$, where ϕ is the golden mean (see Section 9):

$$\frac{3\pi\sqrt{\phi}}{5^{5/4}} = \frac{1}{2^9} P(1, 2/\phi, 10, (256\phi, 128\phi^3, 64\phi^4, 32\phi^4, 0, -8\phi^6, -4\phi^8, -2\phi^9, 0)). \quad (129)$$

The second example of this class is the formula

$$\frac{1}{\sqrt{19}} \cos^{-1}\left(\frac{9}{10}\right) = \frac{1}{10} \sum_{k=0}^{\infty} \frac{D_k}{10^k} \left(\frac{1}{k+1}\right), \quad (130)$$

where the D coefficients satisfy the recurrence $D_0 = D_1 = 1$, and $D_{k+1} = D_k - 5D_{k-1}$ for $k \geq 2$. It is possible that a variant of the original BBP algorithm can be fashioned for this case, on the idea that the D_k comprise a Lucas sequence, and as is known, evaluations of sequence elements mod n can be effected via exponential-ladder methods. These two formulas appeared in [15].

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References

[1] Kunle Adegoke, "BBP formula for Catalan's constant," private communication, Sept. 2010, available from http://adegoke.atwebpages.com/.

[2] Kunle Adegoke, "A Novel Approach to the Discovery of Binary BBP-type Formulas for Polylogarithm Constants," *Integers*, vol. 12 (2012), pg. 345–371, available from http://www.integers-ejcnt.org/I66/I66.pdf.

[3] Kunle Adegoke, "A Novel Approach to the Discovery of Ternary BBP-type Formulas for Polylogarithm Constants," *Notes on Number Theory and Discrete Mathematics*, vol. 17 (2011), no. 1, pg. 4–20, available from http://www.nntdm.net/volume17-2011.html.

[4] Kunle Adegoke, "Symbolic Routes to BBP-type Formulas of any Degree in Arbitrary Bases," *Applied Mathematics and Information Sciences*, vol. 5 (2011), no. 2, pg. 264–275.

[5] Kunle Adegoke, "A Non-PSLQ Route to BBP-type Formulas," *Journal of Mathematics Research*, vol. 2 (2010), no. 2, pg. 56–64, available from http://www.ccsenet.org/journal/index.php/jmr/article/download/3853/4736.

[6] Kunle Adegoke, "New Binary and Ternary Digit Extraction (BBP-type) Formulas for Trilogarithm Constants," *New York Journal of Mathematics*, vol. 16 (2010), pg. 361–367, available from http://nyjm.albany.edu/j/2010/16-14v.pdf.

[7] Kunle Adegoke, "Formal Proofs of Degree 5 Binary BBP-type Formulas," *Functiones et Approximatio*, vol. 48 (2013), no. 1, pg. 19–27, available from http://adegoke.atwebpages.com/.

[8] Kunle Adegoke, "A Binary BBP-type Formula for $\pi\sqrt{5}$," manuscript (2013), available from http://adegoke.atwebpages.com/bbp-type/pi_root5.pdf.

[9] Kunle Adegoke, "Binary BBP type formulas for the arctangents of powers of the golden ratio," manuscript (2013), available from http://adegoke.atwebpages.com/bbp-type/golden_arctan.pdf.

[10] Kunle Adegoke, Jaume Oliver Lafont and Olawanle Layeni, "A class of digit extraction BBP-type formulas in general binary bases," *Notes on Number Theory and Discrete Mathematics*, vol. 17 (2012), no. 4, pg. 18–32, available from http://www.nntdm.net/papers/nntdm-17/NNTDM-17-4-18-32.pdf.

[11] Victor Adamchik and Stan Wagon, "A Simple Formula for Pi," *American Mathematical Monthly*, Nov. 1997, pg. 852–855.

[12] Victor Adamchik and Stan Wagon, "Pi: A 2000-Year-Old Search Changes Direction," *Mathematica in Science and Education*, vol. 5 (1996), no. 1, pg. 11–19.

[13] David H. Bailey, Peter B. Borwein and Simon Plouffe, "On The Rapid Computation of Various Polylogarithmic Constants," *Mathematics of Computation*, vol. 66, no. 218, 1997, pp. 903–913.

[14] David H. Bailey and David J. Broadhurst, "Parallel Integer Relation Detection: Techniques and Applications," *Mathematics of Computation*, to appear, 2000.

[15] David H. Bailey and Richard E. Crandall, "On the Random Character of Fundamental Constant Expansions," Experimental Mathematics, vol. 10, no. 2 (Jun 2001), pg. 175–190.

[16] Jonathan M. Borwein and Peter B. Borwein, *Pi and the AGM*, John Wiley and Sons, New York, 1987.

[17] Jonathan M. Borwein, William F. Galway and David Borwein, "Finding and Excluding b-ary Machin-Type BBP Formulae," *Canadian Journal of Mathematics*, vol. 56 (2004), pg. 1339–1342.

[18] David J. Broadhurst, "Massive 3-loop Feynman Diagrams Reducible to SC* Primitives of Algebras of the Sixth Root of Unity," preprint, Mar. 1998, available from `http://www.arxiv.org/abs/hep-th/9803091`.

[19] David J. Broadhurst, "Polylogarithmic Ladders, Hypergeometric Series and the Ten Millionth Digits of $\zeta(3)$ and $\zeta(5)$," March 1998, available from `http://www.arxiv.org/abs/math/9803067`.

[20] David J. Broadhurst, "Vanishing Polylogarithmic Ladders," manuscript, March 2000, available from author.

[21] Helaman R. P. Ferguson, David H. Bailey and Stephen Arno, "Analysis of PSLQ, An Integer Relation Finding Algorithm," *Mathematics of Computation*, vol. 68, no. 225 (Jan. 1999), pg. 351–369.

[22] Robert Harley, personal communication to Peter Borwein, 1995.

[23] Donald E. Knuth, *The Art of Computer Programming*, vol. 2, third edition, Addison-Wesley, 1998.

[24] Jeffrey C. Lagarias, "On the Normality of Arithmetical Constants," manuscript, Sept. 2000.

[25] N. Nielsen, *Der Eulersche Dilogarithmus*, Halle, Leipzig, 1909.

[26] See `http://www.research.att.com/~njas/sequences/A002391`.

[27] See `http://oeis.org/wiki/User:Jaume_Oliver_Lafont/BBP#Obtention_of_a_zero_relation_of_the_form_0.3DP.281.2C2.5E20.2C40.2CA.29`.

[28] Jaume Oliver Lafont, "Degree 1 BBP-Type Zero Relations," manuscript, 27 Jan 2011.

[29] See http://oeis.org/A171220.