Jonathan Borwein: Mathematician extraordinaire

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1 Introduction

Many of us were shocked when our dear colleague Jonathan M. Borwein of the University of Newcastle, Australia, died in August 2016. After his passing, one immediate priority was to gather together as many of his works as possible. Accordingly, the present author and Nelson H. F. Beebe of the University of Utah began collecting as many of Borwein's published papers, books, reports and talks as possible, together with book reviews and articles written by others about Jon and his work. Our current catalog [6] lists nearly 2000 items. Even if one focuses only on formal, published, peer-reviewed articles, there are over 500 such items. These works are heavily cited — the Google citation tracker finds over 22,000 citations.

What is most striking about this catalogue is the breadth of topics. One bane of modern academic research in general, and of the field of mathematics in particular, is that most researchers today focus on a single specialized niche, seldom attempting to branch out into other specialties and disciplines or to forge potentially fruitful collaborations with researchers in other fields. In contrast, Borwein not only learned about numerous different specialities, but in fact did significant research in a wide range of fields, including experimental mathematics, optimization, convex analysis, applied mathematics, computer science, scientific visualization, biomedical imaging and mathematical finance.

It is hard to think of a single mathematician of the modern era who has published notable research in so many different arenas.

2 Optimization

Some of Jon's most significant contributions were in the area of optimization; indeed, papers in the area of optimization and convex analysis are the single most numerous category in the catalog [6].

One notable paper in the optimization arena is [5], which presents what is now known as the Barzilai-Borwein algorithm for large-scale unconstrained optimization. This paper has been cited over 1300 times. There are numerous techniques for this type of problem (unconstrained optimization) in the literature. The standard gradient method, namely to iterate $x_{k+1} = x_k - \alpha_k g_k(x_k)$, where α_k is typically calculated based on a fixed line search procedure, is fairly simple to use, but it makes no use of second order information and sometimes zig-zags rather than converges. Newton's method is to iterate $x_{k+1} = x_k - (F_k(x_k))^{-1} g_k(x_k)$, where $F_k = \nabla^2 f(x_k)$ is the Hessian of the system. It utilizes second-order information and typically converges quite rapidly near the solution, but it requires the expensive computation of the

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matrix $(F_k(x_k))^{-1}$, and for some applications the scheme requires additional custom modifications to ensure convergence.

The Barzilai-Borwein method mimics the gradient method, in that it selects α_k so that $\alpha_k g_k(x_k)$ approximates $(F_k(x_k))^{-1} g_k(x_k)$, but it does not require that one actually compute $(F_k(x_k))^{-1}$. As a result, this scheme often converges nearly as fast as the Newton method, but at significantly lower computational cost. Due to its simplicity and efficiency, variations of this method have been applied in a variety of applications, including sparse optimization, image analysis and signal processing.

3 Experimental mathematics

Jon is perhaps best known for deriving, with his brother Peter, quadratically and higher order convergent algorithms for π, including p-th order convergent algorithms for any prime p, and similar quadratically convergent algorithms for certain other fundamental constants and functions [7, 8, 9]. Here "quadratically convergent" means that each iteration of the algorithm approximately *doubles* the number of correct digits in the result, with a similar definition for higher-order convergence; similarly, p-th order convergent means that the number of correct digits increases approximately by a factor of p with each iteration.

One of their best-known algorithms is the following: Set $a_0 = 6 - 4\sqrt{2}$ and $y_0 = \sqrt{2} - 1$. Then iterate

$$\begin{aligned} y_{k+1} &= \frac{1 - (1 - y_k^4)^{1/4}}{1 + (1 - y_k^4)^{1/4}} \\ a_{k+1} &= a_k (1 + y_{k+1})^4 - 2^{2k+3} y_{k+1} (1 + y_{k+1} + y_{k+1}^2). \end{aligned}$$

Then a_k converge *quartically* to $1/\pi$: each iteration approximately *quadruples* the number of correct digits. This algorithm, together with a quadratically convergent algorithm due to Brent and Salamin, were employed in several large computations of π by Kanada and others.

But an event more enduring legacy is his advocacy of *experimental mathematics*, in particular his championing of the usage of advanced computing technology to discover new principles and formulas of mathematics, not just verify them with mathematical software.

One of many examples of this methodology in action was his analysis (in conjunction with the present author and the late Richard Crandall) of the following three classes of integrals that arise in mathematical physics: C_n are connected to quantum field theory, D_n arise in Ising theory, while the E_n integrands are derived from D_n:

$$\begin{aligned} C_n &:= \frac{4}{n!} \int_0^\infty \cdots \int_0^\infty \frac{1}{\left(\sum_{j=1}^n (u_j + 1/u_j)\right)^2} \frac{\mathrm{d}u_1}{u_1} \cdots \frac{\mathrm{d}u_n}{u_n} \\ D_n &:= \frac{4}{n!} \int_0^\infty \cdots \int_0^\infty \frac{\prod_{i<j} \left(\frac{u_i - u_j}{u_i + u_j}\right)^2}{\left(\sum_{j=1}^n (u_j + 1/u_j)\right)^2} \frac{\mathrm{d}u_1}{u_1} \cdots \frac{\mathrm{d}u_n}{u_n} \\ E_n &= 2 \int_0^1 \cdots \int_0^1 \left(\prod_{1 \le j < k \le n} \frac{u_k - u_j}{u_k + u_j} \right)^2 \mathrm{d}t_2 \, dt_3 \cdots \mathrm{d}t_n, \end{aligned}$$

where in the last line $u_k = t_1 t_2 \cdots t_k$ [1].

One early observation was that the C_n integrals can be converted to one-dimensional integrals involving the modified Bessel function $K_0(t)$:

$$C_n = \frac{2^n}{n!} \int_0^\infty t K_0^n(t)\ \mathrm{d}t.$$

It was quickly evident that high-precision numerical values of this sequence, computed using tanh-sinh quadrature, approach a limit. For example:

$$C_{1024} = 0.6304735033743867961220401927108789043545870787\ldots$$

When the first 50 digits of this constant were copied into the online Inverse Symbolic Calculator-2 (ISC-2) at `https://isc.carma.newcastle.edu.au` (which Jon was instrumental in developing and deploying), the result was:

$$\lim_{n \to \infty} C_n = 2e^{-2\gamma},$$

where γ denotes Euler's constant, a result which was then proved.

Subsequently high-precision computations, in conjunction with Ferguson's PSLQ algorithm [10, 4], were applied to find experimental evaluations of numerous other specific instances of these integrals, including:

$$\begin{aligned}
D_3 &= 8 + 4\pi^2/3 - 27\,\mathrm{L}_{-3}(2) \\
D_4 &= 4\pi^2/9 - 1/6 - 7\zeta(3)/2 \\
E_2 &= 6 - 8\log 2 \\
E_3 &= 10 - 2\pi^2 - 8\log 2 + 32\log^2 2 \\
E_4 &= 22 - 82\zeta(3) - 24\log 2 + 176\log^2 2 - 256(\log^3 2)/3 + 16\pi^2\log 2 - 22\pi^2/3 \\
E_5 &= 42 - 1984\,\mathrm{Li}_4(1/2) + 189\pi^4/10 - 74\zeta(3) - 1272\zeta(3)\log 2 + 40\pi^2\log^2 2 \\
&\quad - 62\pi^2/3 + 40(\pi^2\log 2)/3 + 88\log^4 2 + 464\log^2 2 - 40\log 2,
\end{aligned}$$

where $\mathrm{L}_{-3}(2)$ is a Dirichlet L-function constant, $\zeta(x)$ is the Riemann zeta function and $\mathrm{Li}_n(x)$ is the polylogarithm function [1]. The formula for E_5, which was initially found by Borwein (and which he was quite proud of), remained a numerically discovered but open conjecture for several years, but was finally proven in 2014 by Erik Panzer [13]. Resolution of the general case is still open.

4 Mathematical finance

A notable example of how Jon ventured into arenas quite far afield from his core research in optimization and computational mathematics is his work in mathematical finance. This began in 2013, when the present author mentioned to Jon some research he had been doing with Marcos Lopez de Prado, a financial mathematician in New York City. We were concerned about the yawning gap between state-of-the-art mathematical techniques that were being successfully applied in leading quantitative investment funds, on one hand, and the mathematically and statistically naive schemes and practices that were often being promoted to the public and even being presented in presumably peer-reviewed journals. It had become clear to us, based on our preliminary research, that "backtest overfitting," namely the statistical overfitting of historical market data, was rampant in the finance field, and is arguably the principal reason why so many financial strategies and investment fund designs, which look great on paper and in promotional literature, fall flat when actually fielded. We were also concerned with the many pseudoscientific techniques and strategies that are mentioned on a daily basis in the financial press.

When we presented some of our findings and thoughts on the topic to Jon, he immediately understood the technical issues, appreciated their gravity and concurred with us that these issues deserved rigorous treatment. Four of us, namely Bailey, Borwein, Lopez de Prado and Jim Zhu (one of the editors of this volume), then co-authored a pair of papers with full details. The first paper, provocatively entitled "Pseudo-mathematics and financial charlatanism: The effects of backtest overfitting on out-of-sample performance" (a title that Borwein proposed), was published as a feature article in the *Notices of the American Mathematical Society* [2], and has been circulated widely in the financial community. The second paper addressed the probability of backtest overfitting in more technical depth [3].

Jon also urged us to start a blog to present many of these related issues for an even broader audience. The result was the *Mathematical Investor* blog [11], with the provocative subtitle (also proposed by Jon) "Mathematicians against fraudulent financial and investment advice (MAFFIA)." Its mission is to identify and draw attention to abuses of mathematics and statistics in the financial field, and to also call out the failure of many in the financial mathematics community for their silence on these abuses, such as

1. Failing to disclose the number of models or variations that were used to develop an investment strategy or fund (which failure makes the strategy or fund highly susceptible to backtest overfitting).
2. Making vague predictions that do not permit rigorous testing and falsification.
3. Misusing probability theory, statistics and stochastic calculus.
4. Suggesting in press reports and promotions that investors can achieve above-market returns via unsophisticated chart-watching techniques (e.g., "technical analysis," "Elliott waves," etc.).
5. Using pseudomathematical technical jargon: "stochastic oscillators," "Fibonacci ratios," "cycles," "waves," "golden ratios," "parabolic SAR," "pivot point," "momentum," etc., none of which has any rigorous scientific basis.

As we explained, "Our silence is consent, making us accomplices in these abuses." [2].

5 Mathematical education and public communication

Jon's passion for sharing the joy of mathematical research and communicating this joy to the public was central to his career. He personally mentored scores of graduate students, and taught hundreds of others. Many of these students have in turn become notable mathematicians and computer scientists themselves. This alone would be an achievement worthy of acclaim.

Along this line, Jon specifically selected many of his research topics based on their potential for public appeal and inspiring students. This is particularly clear with his interest in π, formulas for π and experimental mathematics in general, which he saw as a powerful vehicle to convey the excitement of modern mathematics to the younger, tech-savvy crowd, and yet basic enough to be comprehensible even to high school and undergraduate students.

As mentioned above, Jon was an avid blogger, which again was rooted in his passion for communicating with students and the public at large. The present author is deeply grateful to have been a part of this effort with Jon. Beginning in 2009, when we founded the "Math Drudge" blog [12], he and I co-authored over 200 articles on a wide range of topics, covering virtually every facet of modern mathematics, computing and science. A few of the topics we addressed in these blogs include:

1. The psychology of mathematics.
2. Pseudoscience and anti-science.
3. The fallacies of creationism and intelligent design.
4. The sad state of math and science education.
5. Global warming and global warming denial.
6. The computation of π.
7. New developments in physics and cosmology.
8. New developments in computer science.
9. Fermi's paradox.
10. Artificial intelligence.
11. Computer games vs. humans.
12. Supercomputers.
13. Ancient Indian mathematics.
14. The origins of decimal arithmetic.
15. Moore's law and the future of mathematics.
16. The discovery of the Higgs boson.
17. New ways to visualize the digits of π.
18. DNA and evolution.
19. Pseudoscience from the political left and right.
20. New energy technologies, including LENR and fusion.
21. Fields medalists, Abel Prize recipients and Breakthrough Prize recipients.

It should be emphasized that Jon personally proposed, co-wrote and co-edited virtually every one of these blogs. They reflect both his interests and his passions to communicate better with the public. Some of these blogs were subsequently published in venues such as *The Conversation* and *The Huffington Post*.

6 Summary

Jonathan Borwein's prodigious output in optimization and experimental mathematics is certainly his singular contribution to modern mathematics. But beyond his technical accomplishments, he was a master of mathematical communication (his lectures were always paragons of well-organized and visually appealing mathematics and graphics), mathematical education (part of his interest in π was to bring the joy of mathematical discovery to students), and in promoting science, mathematics and computing to the general public. To this end, he wrote and lectured tirelessly. By one reckoning he presented an average of one lecture per week for decades, and wrote hundreds of articles targeted to the general public. His death is a loss to all those who treasure modern mathematics, science and clear thinking.

References

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