Equations for Euler Sums involving the harmonic function $H(2k)$

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Abstract

We present some new formulas for Euler sums involving the harmonic functions $H(2k)$ and $H(2k+1)$, by exploiting a duplication formula expressing $H(2k)$ in terms of $H(k)$ and a sum over $1/(2k(2k-1))$, combined with some advanced computational techniques. The sums listed involve denominators consisting of powers of $2k$ and $2k+1$, with summed powers ranging from 2 to 12. A single example of the many formulas we have found is:

$$\sum_{k=1}^{\infty} \frac{H(2k)}{(2k)^2(2k+1)^3} = \frac{1}{48}\left(-36\zeta(2) + 75\zeta(3) - 72\log(2)^2 - 48\mathrm{Li}_4(1/2)\right.$$
$$\left. -42\log(2)\zeta(3) + 51\zeta(4) - 2\log(2)^4 + 12\log(2)^2\zeta(2)\right). \qquad (1)$$

One key finding of this study is that for denominator orders 5, 7, 9 and 11, basic sums contain one or more Nielsen generalized polylogarithm terms, in addition to log and zeta function terms. We also present some formulas connecting other sums directly related to those evaluated here.

1 Introduction

Euler sums involving the harmonic number function $H(k) = 1 + 1/2 + 1/3 + \cdots + 1/k$, which arise in contexts as varied as number theory and quantum field theory, have a long history and a rich literature, old and new, e.g. [6]-[13]. In this paper, we add further elements to this topic, combining the methods of experimental mathematics with classical analytic techniques. The focus here is on sums containing the harmonic function $H(2k)$ as the numerator in the summand, with the denominator being of the form $(2k)^{m_1}(2k+1)^{m_2}$. Results will be presented for denominator orders ranging from 2 to 12. One key finding of this study is that for $m_1 + m_2 = 5, 7, 9, 11$ (and, conjecturally, higher odd values), basic Euler sum formulas, which are used to derive other formulas, involve Nielsen generalized polylogarithm terms, in addition to log and zeta function terms.

2 Basic formulas and general results

In the following, we will use the the following multiplication formula for $H_n = H(n)$:

$$H_{2n} = H_n + \sum_{k=1}^{n} \frac{1}{2k(2k-1)} = H_n + \mathcal{H}_1(n). \qquad (2)$$

Expressed in terms of the digamma function, this is:

$$\mathcal{H}_1(n) = \log(2) + \frac{1}{2}\left[\psi\left(n+\frac{1}{2}\right) - \psi(n+1)\right]. \tag{3}$$

An elementary proof of the middle form in (2) is:

$$\begin{aligned} H_{2n} - H_n &= \sum_{k=1}^{n}\frac{1}{2k} + \sum_{k=1}^{n}\frac{1}{2k-1} - \sum_{k=1}^{n}\frac{2}{2k} = \sum_{k=1}^{n}\left[\frac{1}{2k-1} - \frac{1}{2k}\right] \\ &= \sum_{k=1}^{n}\frac{1}{2k(2k-1)}. \end{aligned} \tag{4}$$

The argument in (4) follows similar lines to that in [6], and leads to the following recurrence relation for $\mathcal{H}_1(n)$:

$$\mathcal{H}_1(n+1) = \mathcal{H}_1(n) + \frac{1}{(2n+2)(2n+1)}. \tag{5}$$

This recurrence property is useful in manipulating sums involving $\mathcal{H}_1(n)$ by changes to the summation variable.

Another recurrence relation which is useful in linking analytic forms for sums is.

$$\sum_{k=1}^{\infty}\frac{F(k)}{(2k)^{m_1}(2k+1)^{m_2}} = \sum_{k=1}^{\infty}\frac{F(k)}{(2k)^{m_1}(2k+1)^{m_2-1}} - \sum_{k=1}^{\infty}\frac{F(k)}{(2k)^{m_1-1}(2k+1)^{m_2}}. \tag{6}$$

Here $F(k)$ is any function for which the sums in (6) converge and $m_1 \geq 1,\ m_2 \geq 1$. The relation is obtained by inserting a factor $2k+1-2k$ in the numerator of the left-hand side of (6). Further such relations can be obtained by sequentially using (6) or by replacing the numerator factor by $(2k+1-2k)^m$ with $m \geq \min(m_1, m_2)$.

We divide sums into two classes: basic sums and derived sums. The basic sums will be discussed first, and consist of the following three in this paper:

$$\sum_{k=1}^{\infty}\frac{H(2k)}{(2k)^n},\ \sum_{k=1}^{\infty}\frac{H(2k)}{(2k+1)^n},\ \sum_{k=1}^{\infty}\frac{H(2k)}{(2k)^n(2k+1)^n}. \tag{7}$$

These three are linked, with the first two being connected by Euler's expression for the sum of $H(k)/k^n$, which can be split into sums over even and odd values of k. This gives for $n \geq 2$:

$$\sum_{k=1}^{\infty} H(2k)\left(\frac{1}{(2k)^n} + \frac{1}{(2k+1)^n}\right) = \left(\frac{n}{2} + \frac{1}{2^{n+1}}\right)\zeta(n+1) - \frac{1}{2}\sum_{q=1}^{n-2}\zeta(n-q)\zeta(q+1). \tag{8}$$

The first two basic sums can be determined if the alternating sum over $(-1)^k H(k)/k^n$ is known, since:

$$\sum_{k=1}^{\infty} H(2k)\left(\frac{1}{(2k)^n} - \frac{1}{(2k+1)^n}\right) = \left(1 - \frac{1}{2^{n+1}}\right)\zeta(n+1) - \sum_{k=1}^{\infty}\frac{(-1)^{k-1}H(k)}{k^n}. \tag{9}$$

The alternating sum over $(-1)^k H(k)/k^n$ is known for n of even order and can in fact be evaluated explicitly using a result given by R. Sitaramachandrarao [12]. The result is given as an expression for H7 in Theorem 3.7:

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}H(k)}{k^{2b}} =$$

$$2^{-2b-1}\left(2\sum_{l=1}^{b-1}\left(\left(2^{2b-2l+1}-1\right)\zeta(2l)\bar{\zeta}(2b-2l+1) - \left(2^{2b}-1\right)\zeta(l+1)\zeta(2b-l)\right) + \left(2^{2b+1}b-2b-1\right)\zeta(2b+1)\right) \tag{10}$$

where $\overline{\zeta}(s) = (2^s - 1)\zeta(s)$ for $\Re(s) > 1$.

The explicit expressions for sums of denominator order 2, 3, 4 presented in following sections can be obtained from the literature cited, the recurrence relation (6) or via high-precision computation with an integer relation algorithm. Sadly, in the initial research leading to this paper, we found that these approaches fail for the basic sums of denominator order 5 and for higher odd denominator orders. Our initial approach then was to use one basic sum as an unknown for each case of odd denominator orders starting with 5. What we refer to as dot diagrams were developed as a graphical representation of the procedure followed.

To exhibit the structure of the recurrence solutions for the Euler sums under study for denominators of order 5 or greater, we show dots with colors depending on the basic sums of odd denominator order needed. With a basic sum known for denominator orders 5, 7, 9 the recurrence process permits expressions to be obtained for orders of denominators ranging up to 10, as illustrated in Figure 1 and Figure 2. Note that the connections between series implicit in the dot diagrams are exact for truncated series, as well as for series summed to infinity. Also, the sums corresponding to points in diagrams increase in value moving to the right along horizontal lines. Note that the recurrence relation (6) corresponds to right-angled triangles: the value at the apex is the value at the right-angle below it less the value at the point to the left.

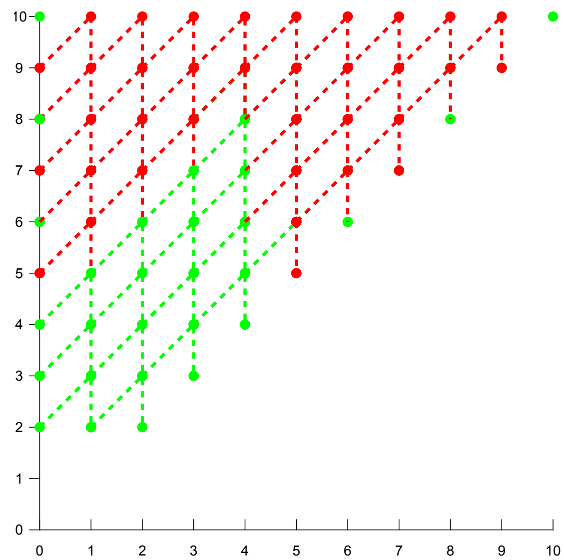

Figure 1: The green dots represent points in the $m_1, m_1 + m_2$ plane for which the Euler sum has a known solution in terms of ζ function values, while for the red dots no solution is known. The green dashed lines show successful connections via the recurrence relation, while for the red lines there is no successful connection.

We then found, after some experimentation using in part the recently-released version 15.0 of Wolfram Mathematica, that full solutions can be obtained for many of these sums, from denominator order 5 onwards, using generalized Nielsen polylogarithms $S_{n,p}(z)$ [14, 15]. For $n \geq 1, p \geq 0$, these are defined as

$$S_{n,p}(z) = \frac{(-1)^{n+p-1}}{(n-1)!p!} \int_0^1 \frac{(\log t)^{n-1}(\log(1-zt))^p}{t} \mathrm{d}t. \tag{11}$$

For some small n, p, these reduce to simple expressions involving logs and the zeta function. A few examples include $S_{0,2}(-1) = \log(2)^2/2$, $S_{1,2}(-1) = \zeta(3)/8$ and $S_{3,2}(-1) = \pi^2\zeta(3)/12 - 29\zeta(5)/32$.

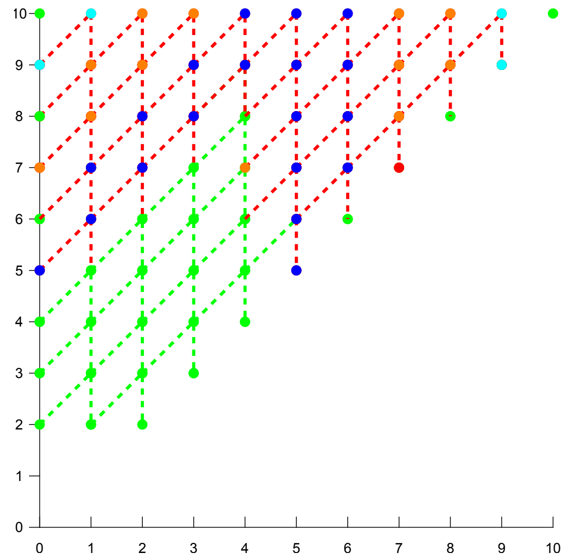

Figure 2: The green dots represent points in the m_1, m_1+m_2 plane for which the Euler sum has a known solution, while for $m_1 + m_2 \leq 10$ there are no longer red dots for which no solution is known. Newly solved dot points involve the value of the sum $H(2k)/((2k)^{m_1}(2k+1)^{m_2})$ with $m_1 = m_2 = 5$ (blue), 7 (orange) and cyan (9).

The Nielsen polylogarithms appearing in this paper all have $p = 2$. Using results in A002547, A001008 and A002805 [16], we have established the following result:

$$S_{n,2}(z) = \sum_{q=1}^{\infty} \frac{H(q)}{(q+1)^{n+1}} z^{q+1}. \tag{12}$$

Then

$$S_{n,2}(1) = \sum_{q=1}^{\infty} \frac{H(q)}{q^{n+1}} - \sum_{q=1}^{\infty} \frac{1}{q^{n+2}}. \tag{13}$$

This can always be expressed analytically in terms of ζ function values. On the other hand, $S_{n,2}(-1)$ involves alternating sums which can be split into two contributions:

$$S_{n,2}(-1) = \sum_{q=1}^{\infty} \frac{H(2q)}{(2q)^{n+1}} - \sum_{q=1}^{\infty} \frac{H(2q-1)}{(2q-1)^{n+1}} - \sum_{q=1}^{\infty} \frac{(-1)^q}{q^{n+2}}. \tag{14}$$

The sums of this type in analytic form given below are, for $n > 2$, restricted to odd values of n.

For a good summary of the properties of these functions, see Kölbig [14]. Kölbig comments that these generalized polynomials, as needed here, which involve $z = -1$ and $n + p$ even, are unlikely to be representable in terms of zeta function values. Note that the representations we give below for sums over $H(2k)/((2k)^n(2k+1)^n)$ can be derived through the use of recurrence relations of the type (6) and are typically much more compact than results given, say, by Wolfram Mathematica 15.0.

3 Connections between harmonic sums

We suppose we have assembled a complete set of sums over $H(2k)/((2k)^{m_1}(2k+1)^{m_2})$ for denominator order $m_1+m_2 \geq 2$. Then we can obtain from these the complete set of sums involving $\mathcal{H}_1(k)$ using

$$\sum_{k=1}^{\infty} \frac{\mathcal{H}_1(k)}{(2k)^{m_1}(2k+1)^{m_2}} = \sum_{k=1}^{\infty} \frac{H(2k)}{(2k)^{m_1}(2k+1)^{m_2}} - \sum_{k=1}^{\infty} \frac{H(k)}{(2k)^{m_1}(2k+1)^{m_2}}. \tag{15}$$

For values of the second sum on the right-hand side see for example [10].

Using (3), (15) gives

$$\sum_{k=1}^{\infty} \frac{\psi(k+1/2)-\psi(k+1)}{(2k)^{m_1}(2k+1)^{m_2}} = 2\sum_{k=1}^{\infty} \frac{\mathcal{H}_1(k)}{(2k)^{m_1}(2k+1)^{m_2}} - 2\log(2)\sum_{k=1}^{\infty} \frac{1}{(2k)^{m_1}(2k+1)^{m_2}}. \tag{16}$$

The second sum on the right-hand side can be evaluated by downwards recurrence or using Mathematica.

Sums involving the finite sum over reciprocals of odd integers $O(n)$ can be calculated, where

$$O(n) = H(2n) - \frac{1}{2}H(n). \tag{17}$$

Hence:

$$\sum_{k=1}^{\infty} \frac{O(k)}{(2k)^{m_1}(2k+1)^{m_2}} = \sum_{k=1}^{\infty} \frac{H(2k)}{(2k)^{m_1}(2k+1)^{m_2}} - \frac{1}{2}\sum_{k=1}^{\infty} \frac{H(k)}{(2k)^{m_1}(2k+1)^{m_2}}. \tag{18}$$

4 Computational methods

As mentioned above, one key technique employed in this research is to utilize very high-precision numerical computations, combined with an integer relation algorithm, to experimentally discover formulas, which then can be proven using more conventional techniques. These computations by themselves do not constitute formal proof, but if done carefully are very reliable.

Here we briefly summarize these methods; see [9, 5] for more details. A key tool here is the Euler-Maclaurin summation formula [1, pg. 285], which approximates a summation as an integral with high-order corrections (here $f(t)$ is assumed to have $(2s+2)$-th order derivatives on $[a,b]$):

$$\sum_{j=a}^{b} f(j) = \int_a^b f(t)\,\mathrm{d}t \;+\; \frac{1}{2}\left(f(a)+f(b)\right) \;+\; \sum_{j=1}^{s} \frac{B_{2j}\left(D^{2j-1}f(b) - D^{2j-1}f(a)\right)}{(2j)!} \;+\; R_s(a,b), \tag{19}$$

where B_k is the k-th Bernoulli number [7], $D^k f(a)$ is the k-th derivative of $f(t)$ evaluated at $t=a$, and $R_s(a,b)$ is a bounded error term. Applying (19) to the harmonic function $H(t) = \sum_{j=1}^{t} 1/j$ yields the rather complicated approximation $\hat{H}(t)$:

$$\begin{aligned}\hat{H}(t) = \gamma + \log(t) &+ \frac{1}{2t} - \frac{1}{12t^2} + \frac{1}{120t^4} - \frac{1}{252t^6} + \frac{1}{240t^8} - \frac{1}{132t^{10}} + \frac{691}{32760t^{12}} - \frac{1}{12t^{14}} \\ &+ \frac{3617}{8160t^{16}} - \frac{43867}{14364t^{18}} + \frac{174611}{6600t^{20}} - \frac{77683}{276t^{22}} + \frac{236364091}{65520t^{24}} - \frac{657931}{12t^{26}} + \frac{3392780147}{3480t^{28}} \\ &- \frac{1723168255201}{85932t^{30}} + \frac{7709321041217}{16320t^{32}} - \frac{151628697551}{12t^{34}} + \frac{26315271553053477373}{69090840t^{36}},\end{aligned} \tag{20}$$

which approximates $H(t)$ to within roughly t^{-38} for large t. This can be produced in Mathematica by typing `Series[HarmonicNumber[t], {t,Infinity,36}]`.

Given an Euler sum such as $\sum_{k=1}^{\infty} H(2k)/((2k)^m(2k+1)^n)$, let $\hat{G}(t) = \hat{H}(2t)/((2t)^m(2t+1)^n)$. Using the Euler-Maclaurin summation formula (19) once again, one can write

$$\sum_{j=1}^{\infty} \frac{H(2j)}{(2j)^m(2j+1)^n} = \sum_{j=1}^{k} \frac{H(2j)}{(2j)^m(2j+1)^n} + \sum_{j=k+1}^{\infty} \frac{H(2j)}{(2j)^m(2j+1)^n} \approx \sum_{j=1}^{k} \frac{H(2j)}{(2j)^m(2j+1)^n} + \sum_{j=k+1}^{\infty} \hat{G}(j)$$
$$\approx \sum_{j=1}^{k} \frac{H(2j)}{(2j)^m(2j+1)^n} + \int_{k+1}^{\infty} \hat{G}(t)\,\mathrm{d}t + \frac{1}{2}\hat{G}(k+1) - \sum_{j=1}^{s} \frac{B_{2j} D^{2j-1}\hat{G}(k+1)}{(2j)!}, \quad (21)$$

where $s = 18$, which is accurate to within roughly k^{-38}. In our runs, we set $k = 10^8 = 100,000,000$, so the approximation in the second line of (21) is correct to within roughly 10^{-350}.

Formula (21) suggests a four-step process, each performed with a precision of at least 300 digits (we used the arbitrary precision package [4]): (a) explicitly sum the series to $k = 10^8$ terms; (b) use a numerical quadrature program (we used the exp-sinh algorithm [2]) to compute the definite integral; (c) compute the third term (straightforward); and (d) evaluate the last term, using a symbolic package such as Wolfram Mathematica to evaluate the high-order derivatives of the complicated $\hat{G}(t)$ function. The present authors have used this basic process successfully in several previous studies. But the last step, which must be done symbolically, off-line, with Mathematica, is both awkward and expensive in computer run time. Is there a better way?

For this study, we employed a new technique, based on the following formula for the n-th derivative:

$$D^n[f(x)] = \lim_{h \to 0} \frac{1}{h^n} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(x+kh). \quad (22)$$

A d-digit approximation can be written as

$$D^n[f(x)] \approx \frac{1}{h^n} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(x+kh), \quad (23)$$

where h is chosen to be less than $1/10^d$. However, formula (23) hides an extreme numerical difficulty: the function values $f(x+kh)$ must be evaluated, and the summation performed, using at least $(n+1)d$-digit precision, or else the result will be completely inaccurate. For the present application, namely evaluating the last term of formula (21), if one requires derivatives accurate to 300 digits, then one must employ least $2sd$ = 10,800 digits in formula (23); in practice even higher precision is required (we employed 12,000 digits). Surprisingly, however, in our tests this absurd-looking scheme is actually very efficient, many times faster on challenging problems than using Mathematica symbolic processing to evaluate the high-order derivatives; for details, see [5].

With a 300- or 350-digit value for a given Euler sum in hand, we employed the multipair PSLQ algorithm [3] to search for integer relations with known constants, thus producing a formula. For example, for the order-10 formulas presented below in Section 14, the integer relation step employed 300-digit numerical values of these 35 constants in addition to the numerically-computed sum (these constants were included based on initial investigations suggesting that they may be involved in the right-hand sides of order-10 formulas):

$$\zeta(2), \log(2), \zeta(3), \log(2)^2, \mathrm{Li}_4(1/2), \log(2)\zeta(2), \log(2)\zeta(3), \zeta(4), \log(2)^4, \log(2)^2\zeta(2), \zeta(5), \zeta(2)\zeta(3), \zeta(6),$$
$$\zeta(3)^2, S_{4,2}(-1), \zeta(7), \zeta(2)\zeta(5), \zeta(3)\zeta(4), \zeta(8), \zeta(3)\zeta(5), S_{6,2}(-1), \zeta(9), \zeta(2)\zeta(7), \zeta(3)\zeta(6), \zeta(4)\zeta(5),$$
$$\zeta(10), \zeta(3), \zeta(7), \zeta(5)^2, \log(2)\zeta(9), S_{8,2}(-1), \zeta(11), \zeta(2)\zeta(9), \zeta(3)\zeta(8), \zeta(4)\zeta(7), \zeta(5)\zeta(6) \quad (24)$$

Integer relation detection requires very high precision (at least $n \cdot \max_i \log_{10}|a_i|$ digits, where n is the length and a_i are the integer coefficients) to produce reliable results. An effective check of numerical reliability is to note the dynamic range of the entries of the reduced vector at detection. In the computer runs employed for results presented below, this dynamic range always exceeded 10^{50}, and in most cases exceeded 10^{100}. This means, in effect, that the resulting formulas hold to at least 50 digits, and in most cases to at least 100 digits, beyond the level of precision required to discover them.

5 Validity checking

In the following sections, we present the results we have obtained for sums with denominator orders 2 through 12. We do not present rigorous proofs for most of the individual formulas below, but we have checked them analytically and numerically. Needless to say, in presenting results of this scope there is significant possibility of error, so we have employed several measures to ensure validity.

To begin with, in each section below the main set of results were generated automatically by a computer program, using techniques described above in Section 4. We then included the LaTeX code generated by this program into this manuscript without any alterations. Secondly, we then checked these formulas by a computer program that read the actual LaTeX source of this manuscript, parsed each numbered equation in Sections 6 through 16, separately translated the left-hand and right-hand sides to Mathematica script evaluated with 60-digit arithmetic; then this script was executed by Wolfram Mathematica 15.0. This process found no errors.

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6 Sums of denominator order two

We now present a catalog of results obtained by these methods, listed by denominator order (the sum of the degrees of the denominator terms of the left-hand side summation), starting with order 2.

$$\sum_{k=1}^{\infty} \frac{H(2k)}{(2k+1)^2} = \frac{-1}{16}\left(-7\zeta(3)\right) \tag{25}$$

$$\sum_{k=1}^{\infty} \frac{H(2k)}{(2k)(2k+1)} = \frac{1}{4}\left(\zeta(2) + 2\log(2)^2\right) \tag{26}$$

$$\sum_{k=1}^{\infty} \frac{H(2k)}{(2k)^2} = \frac{1}{16}\left(11\zeta(3)\right) \tag{27}$$

One additional formula of interest is:

$$\sum_{k=1}^{\infty} \frac{H_{2k}}{(2k-1)(2k+1)} = \frac{3}{8}\zeta(2) + \frac{1}{2}\log(2) \tag{28}$$

The relevant Nielsen polylogarithm for this order is

$$S_{1,2}(-1) = \frac{\zeta(3)}{8} \tag{29}$$

7 Sums of denominator order three

$$\sum_{k=1}^{\infty} \frac{H(2k)}{(2k+1)^3} = \frac{1}{48}\Big(-48\mathrm{Li}_4(1/2) - 42\log(2)\zeta(3) + 51\zeta(4) - 2\log(2)^4 \\ +12\log(2)^2\zeta(2)\Big) \tag{30}$$

$$\sum_{k=1}^{\infty} \frac{H(2k)}{(2k)(2k+1)^2} = \frac{-1}{16}\left(-4\zeta(2) + 7\zeta(3) - 8\log(2)^2\right) \tag{31}$$

$$\sum_{k=1}^{\infty} \frac{H(2k)}{(2k)^2(2k+1)} = \frac{-1}{16}\left(4\zeta(2) - 11\zeta(3) + 8\log(2)^2\right) \tag{32}$$

$$\sum_{k=1}^{\infty} \frac{H(2k)}{(2k)^3} = \frac{-1}{24}\Big(-24\mathrm{Li}_4(1/2) - 21\log(2)\zeta(3) + 18\zeta(4) - 1\log(2)^4 \\ +6\log(2)^2\zeta(2)\Big) \tag{33}$$

Three additional formulas are:

$$\sum_{k=1}^{\infty} \frac{H(2k)}{2k(2k-1)(2k+1)} = -\frac{1}{2}\log(2)^2 + \frac{1}{2}\log(2) + \frac{1}{8}\zeta(2) \tag{34}$$

$$\sum_{k=1}^{\infty} \frac{H(2k+1)}{(2k+1)^3} = -1 - \mathrm{Li}_4\left(\frac{1}{2}\right) - \frac{7}{8}\zeta(3)\log(2) + 2\zeta(4) - \frac{\log(2)^4}{24} + \frac{1}{4}\zeta(2)\log(2)^2 \tag{35}$$

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}H_k}{k^3} = -2\mathrm{Li}_4\left(\frac{1}{2}\right) - \frac{7}{4}\zeta(3)\log(2) + \frac{11}{4}\zeta(4) - \frac{\log(2)^4}{12} + \frac{1}{2}\zeta(2)\log(2)^2 \tag{36}$$

The relevant Nielsen polylogarithm for this order is

$$S_{2,2}(-1) = 2\mathrm{Li}_4\left(\frac{1}{2}\right) + \frac{7}{4}\zeta(3)\log(2) - \frac{\pi^4}{48} + \frac{\log(2)^4}{12} - \frac{1}{12}\pi^2\log(2)^2 \tag{37}$$

This identity could be employed, for example, to replace $\mathrm{Li}_4\left(\frac{1}{2}\right)$ by $S_{2,2}(-1)$ in this and later sections.

8 Sums of denominator order four

$$\sum_{k=1}^{\infty} \frac{H(2k)}{(2k+1)^4} = \frac{-1}{64}\left(-93\zeta(5) + 48\zeta(2)\zeta(3)\right) \tag{38}$$

$$\sum_{k=1}^{\infty} \frac{H(2k)}{(2k)(2k+1)^3} = \frac{-1}{48}\big(-12\zeta(2) + 21\zeta(3) - 24\log(2)^2 - 48\mathrm{Li}_4(1/2)$$
$$-42\log(2)\zeta(3) + 51\zeta(4) - 2\log(2)^4 + 12\log(2)^2\zeta(2)\big) \tag{39}$$

$$\sum_{k=1}^{\infty} \frac{H(2k)}{(2k)^2(2k+1)^2} = \frac{-1}{8}\left(4\zeta(2) - 9\zeta(3) + 8\log(2)^2\right) \tag{40}$$

$$\sum_{k=1}^{\infty} \frac{H(2k)}{(2k)^3(2k+1)} = \frac{-1}{48}\big(-12\zeta(2) + 33\zeta(3) - 24\log(2)^2 - 48\mathrm{Li}_4(1/2)$$
$$-42\log(2)\zeta(3) + 36\zeta(4) - 2\log(2)^4 + 12\log(2)^2\zeta(2)\big) \tag{41}$$

$$\sum_{k=1}^{\infty} \frac{H(2k)}{(2k)^4} = \frac{1}{64}\left(37\zeta(5) - 16\zeta(2)\zeta(3)\right) \tag{42}$$

Some additional formulas include:

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}H_k}{k^4} = \frac{118}{64}\zeta(5) - \frac{1}{2}\zeta(2)\zeta(3) \tag{43}$$

$$\sum_{k=1}^{\infty} \frac{H(2k)}{(2k)^3(2k-1)} = \frac{1}{240}$$
$$\left(-5\left(48\mathrm{Li}_4\left(\frac{1}{2}\right) + 33\zeta(3) + 42\zeta(3)\log(2) + 2\log(2)^4 - 24\log(2)^2 + 12\log(4)^2 - 24\log(4)\right) + 2\pi^4 + 10\pi^2\left(2 + \log(2)^2\right)\right) \tag{44}$$

The relevant Nielsen polylogarithm for this order is

$$S_{3,2}(-1) = \frac{\pi^2\zeta(3)}{12} - \frac{29\zeta(5)}{32} \tag{45}$$

9 Sums of denominator order five

$$\sum_{k=1}^{\infty} \frac{H(2k)}{(2k+1)^5} = \frac{1}{8}\left(3\zeta(6) - 2\zeta(3)^2 - 4S_{4,2}(-1)\right) \tag{46}$$

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{H(2k)}{(2k)(2k+1)^4} &= \frac{1}{192}\left(48\zeta(2) - 84\zeta(3) + 96\log(2)^2 + 192\mathrm{Li}_4(1/2)\right.\\ &\quad +168\log(2)\zeta(3) - 204\zeta(4) + 8\log(2)^4 - 48\log(2)^2\zeta(2) - 279\zeta(5)\\ &\quad \left.+144\zeta(2)\zeta(3)\right)\end{aligned} \tag{47}$$

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{H(2k)}{(2k)^2(2k+1)^3} &= \frac{1}{48}\left(-36\zeta(2) + 75\zeta(3) - 72\log(2)^2 - 48\mathrm{Li}_4(1/2)\right.\\ &\quad \left.-42\log(2)\zeta(3) + 51\zeta(4) - 2\log(2)^4 + 12\log(2)^2\zeta(2)\right)\end{aligned} \tag{48}$$

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{H(2k)}{(2k)^3(2k+1)^2} &= \frac{1}{48}\left(36\zeta(2) - 87\zeta(3) + 72\log(2)^2 + 48\mathrm{Li}_4(1/2)\right.\\ &\quad \left.+42\log(2)\zeta(3) - 36\zeta(4) + 2\log(2)^4 - 12\log(2)^2\zeta(2)\right)\end{aligned} \tag{49}$$

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{H(2k)}{(2k)^4(2k+1)} &= \frac{-1}{192}\left(48\zeta(2) - 132\zeta(3) + 96\log(2)^2 + 192\mathrm{Li}_4(1/2)\right.\\ &\quad +168\log(2)\zeta(3) - 144\zeta(4) + 8\log(2)^4 - 48\log(2)^2\zeta(2) - 111\zeta(5)\\ &\quad \left.+48\zeta(2)\zeta(3)\right)\end{aligned} \tag{50}$$

$$\sum_{k=1}^{\infty} \frac{H(2k)}{(2k)^5} = \frac{1}{64}\left(25\zeta(6) - 16\zeta(3)^2 + 32S_{4,2}(-1)\right) \tag{51}$$

Some additional formulas inclulde:

$$\sum_{k=1}^{\infty} H(2k)\left(\frac{1}{(2k)^5} + \frac{1}{(2k+1)^5}\right) = \frac{49}{64}\zeta(6) - \frac{1}{2}\zeta(3)^2 \tag{52}$$

As we remarked above, Wolfram 15.0 gives expressions for basic sums having a denominator of order 5 involving Nielsen generalzed polylogarithms, for example:

$$\sum_{k=1}^{\infty} \frac{H(2k)}{(2k)^5} = \frac{1}{2}\left(S_{4,2}(-1) - \frac{\zeta(3)^2}{2} + \frac{5\pi^6}{6048}\right) \tag{53}$$

The Wolfram 15.0 expression for the second basic sum is unduly complicated, in our view; it can be simplified using the following substitution for a harmonic polylogarithm:

$$H(-5,1;1) = S_{4,2}(-1) - 12\zeta(3)^2 - 31\zeta(5)\log(2) + \frac{13\pi^6}{2520} \tag{54}$$

The simplified form is then

$$\sum_{k=1}^{\infty} \frac{H(2k+1)}{(2k+1)^5} = -1 + \frac{-10080S_{4,2}(-1) - 5040\zeta(3)^2 + 29\pi^6}{20160} \tag{55}$$

$$\begin{aligned}\sum_{k=1}^{\infty} H(2k)\left(\frac{1}{(2k)^5} - \frac{1}{(2k+1)^5}\right) &= \sum_{k=1}^{\infty} H(2k)\left(\frac{5}{(2k)^3(2k+1)^3} + \frac{5}{(2k)^4(2k+1)^4}\right.\\ &\quad \left.+\frac{1}{(2k)^5(2k+1)^5}\right)\end{aligned} \tag{56}$$

Using equations (165) and (53), (55) the third basic sum for this case is:

$$\sum_{k=1}^{\infty} \frac{H(2k)}{(2k)^5(2k+1)^5} =$$
$$30\mathrm{Li}_4\left(\frac{1}{2}\right) + S_{4,2}(-1) - \frac{315\zeta(3)}{8} - \frac{325\zeta(5)}{32} - \frac{5}{12}\pi^2\left(-2\zeta(3) - 7 + 3\log(2)^2\right) + \frac{105}{4}\zeta(3)\log(2) - \frac{29\pi^4}{96} + \frac{\pi^6}{60480} + \frac{5\log(2)^4}{4} + 35\log(2)^2 \quad (57)$$

10 Sums of denominator order six

$$\sum_{k=1}^{\infty} \frac{H(2k)}{(2k+1)^6} = \frac{-1}{256}\left(-635\zeta(7) + 192\zeta(2)\zeta(5) + 240\zeta(3)\zeta(4)\right) \quad (58)$$

$$\sum_{k=1}^{\infty} \frac{H(2k)}{(2k)(2k+1)^5} = \frac{1}{192}\big(48\zeta(2) - 84\zeta(3) + 96\log(2)^2 + 192\mathrm{Li}_4(1/2) + 168\log(2)\zeta(3) - 204\zeta(4) + 8\log(2)^4 - 48\log(2)^2\zeta(2) - 279\zeta(5) + 144\zeta(2)\zeta(3) - 72\zeta(6) + 48\zeta(3)^2 + 96S_{4,2}(-1)\big) \quad (59)$$

$$\sum_{k=1}^{\infty} \frac{H(2k)}{(2k)^2(2k+1)^4} = \frac{1}{192}\big(-192\zeta(2) + 384\zeta(3) - 384\log(2)^2 - 384\mathrm{Li}_4(1/2) - 336\log(2)\zeta(3) + 408\zeta(4) - 16\log(2)^4 + 96\log(2)^2\zeta(2) + 279\zeta(5) - 144\zeta(2)\zeta(3)) \quad (60)$$

$$\sum_{k=1}^{\infty} \frac{H(2k)}{(2k)^3(2k+1)^3} = \frac{1}{48}\big(72\zeta(2) - 162\zeta(3) + 144\log(2)^2 + 96\mathrm{Li}_4(1/2) + 84\log(2)\zeta(3) - 87\zeta(4) + 4\log(2)^4 - 24\log(2)^2\zeta(2)\big) \quad (61)$$

$$\sum_{k=1}^{\infty} \frac{H(2k)}{(2k)^4(2k+1)^2} = \frac{-1}{192}\big(192\zeta(2) - 480\zeta(3) + 384\log(2)^2 + 384\mathrm{Li}_4(1/2) + 336\log(2)\zeta(3) - 288\zeta(4) + 16\log(2)^4 - 96\log(2)^2\zeta(2) - 111\zeta(5) + 48\zeta(2)\zeta(3)) \quad (62)$$

$$\sum_{k=1}^{\infty} \frac{H(2k)}{(2k)^5(2k+1)} = \frac{-1}{192}\big(-48\zeta(2) + 132\zeta(3) - 96\log(2)^2 - 192\mathrm{Li}_4(1/2) - 168\log(2)\zeta(3) + 144\zeta(4) - 8\log(2)^4 + 48\log(2)^2\zeta(2) + 111\zeta(5) - 48\zeta(2)\zeta(3) - 75\zeta(6) + 48\zeta(3)^2 - 96S_{4,2}(-1)\big) \quad (63)$$

$$\sum_{k=1}^{\infty} \frac{H(2k)}{(2k)^6} = \frac{1}{256}\left(135\zeta(7) - 64\zeta(2)\zeta(5) - 16\zeta(3)\zeta(4)\right) \quad (64)$$

Some additional formulas include:

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}H(k)}{k^6} = \frac{377}{128}\zeta(7) - \frac{7}{8}\zeta(4)\zeta(3) - \frac{1}{2}\zeta(2)\zeta(5). \quad (65)$$

$$\sum_{k=1}^{\infty} \frac{H(2k-1)}{(2k-1)^6} = \frac{1}{768}\left(-8\pi^4\zeta(3) - 96\pi^2\zeta(5) + 2667\zeta(7)\right), \quad (66)$$

$$\sum_{k=1}^{\infty} H(2k)\left(\frac{1}{(2k)^6}-\frac{1}{(2k+1)^6}\right) = \sum_{k=1}^{\infty} H(2k)\left(\frac{6}{(2k)(2k+1)^6}+\frac{15}{(2k)^2(2k+1)^6}\right.$$
$$\left.+\frac{20}{(2k)^3(2k+1)^6}+\frac{15}{(2k)^4(2k+1)^6}+\frac{6}{(2k)^5(2k+1)^6}+\frac{1}{(2k)^6(2k+1)^6}\right) \quad (67)$$

We also observe that a formula of denominator order 12 can be obtained as follows:

$$\sum_{k=1}^{\infty}\frac{H(2k)}{(2k)^6(2k+1)^6} = -6\sum_{k=1}^{\infty}\frac{H(2k)}{(2k)^5(2k+1)^5}$$
$$+68\mathrm{Li}_4\left(\frac{1}{2}\right)-\frac{189\zeta(3)}{2}-\frac{1}{720}\pi^4(8\zeta(3)+493)-\frac{585\zeta(5)}{32}+\frac{385\zeta(7)}{128}-\frac{1}{6}\pi^2\left(-9\zeta(3)+\zeta(5)-42+17\log(2)^2\right)$$
$$+\frac{119}{2}\zeta(3)\log(2)+\frac{17\log(2)^4}{6}+84\log(2)^2 \quad (68)$$

Using (57) this becomes

$$\sum_{k=1}^{\infty}\frac{H(2k)}{(2k)^6(2k+1)^6} = -112\mathrm{Li}_4\left(\frac{1}{2}\right)-6S_{4,2}(-1)+\frac{567\zeta(3)}{4}-\frac{1}{180}\pi^4(2\zeta(3)-203)+\frac{1365\zeta(5)}{32}+\frac{385\zeta(7)}{128}$$
$$+\frac{1}{6}\pi^2\left(-21\zeta(3)-\zeta(5)-63+28\log(2)^2\right)-98\zeta(3)\log(2)-\frac{\pi^6}{10080}-\frac{14\log(2)^4}{3}-126\log(2)^2 \quad (69)$$

The relevant Nielsen polylogarithm for this order is

$$S_{5,2}(-1) = \frac{56\pi^4\zeta(3)+480\pi^2\zeta(5)-11295\zeta(7)}{5760} \quad (70)$$

11 Sums of denominator order seven

$$\sum_{k=1}^{\infty} \frac{H(2k)}{(2k+1)^7} = \frac{1}{8}\left(5\zeta(8) - 4\zeta(3)\zeta(5) - 4S_{6,2}(-1)\right) \tag{71}$$

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{H(2k)}{(2k)(2k+1)^6} = \frac{-1}{768}\big(&-192\zeta(2) + 336\zeta(3) - 384\log(2)^2 - 768\mathrm{Li}_4(1/2)\\ &-672\log(2)\zeta(3) + 816\zeta(4) - 32\log(2)^4 + 192\log(2)^2\zeta(2) + 1116\zeta(5) - 576\zeta(2)\zeta(3)\\ &+288\zeta(6) - 192\zeta(3)^2 - 384S_{4,2}(-1) + 1905\zeta(7) - 576\zeta(2)\zeta(5)\\ &-720\zeta(3)\zeta(4)\big)\end{aligned} \tag{72}$$

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{H(2k)}{(2k)^2(2k+1)^5} = \frac{1}{32}\big(&-40\zeta(2) + 78\zeta(3) - 80\log(2)^2 - 96\mathrm{Li}_4(1/2)\\ &-84\log(2)\zeta(3) + 102\zeta(4) - 4\log(2)^4 + 24\log(2)^2\zeta(2) + 93\zeta(5) - 48\zeta(2)\zeta(3)\\ &+12\zeta(6) - 8\zeta(3)^2 - 16S_{4,2}(-1)\big)\end{aligned} \tag{73}$$

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{H(2k)}{(2k)^3(2k+1)^4} = \frac{-1}{192}\big(&-480\zeta(2) + 1032\zeta(3) - 960\log(2)^2 - 768\mathrm{Li}_4(1/2)\\ &-672\log(2)\zeta(3) + 756\zeta(4) - 32\log(2)^4 + 192\log(2)^2\zeta(2) + 279\zeta(5)\\ &-144\zeta(2)\zeta(3)\big)\end{aligned} \tag{74}$$

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{H(2k)}{(2k)^4(2k+1)^3} = \frac{1}{192}\big(&-480\zeta(2) + 1128\zeta(3) - 960\log(2)^2 - 768\mathrm{Li}_4(1/2)\\ &-672\log(2)\zeta(3) + 636\zeta(4) - 32\log(2)^4 + 192\log(2)^2\zeta(2) + 111\zeta(5)\\ &-48\zeta(2)\zeta(3)\big)\end{aligned} \tag{75}$$

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{H(2k)}{(2k)^5(2k+1)^2} = \frac{-1}{64}\big(&-80\zeta(2) + 204\zeta(3) - 160\log(2)^2 - 192\mathrm{Li}_4(1/2)\\ &-168\log(2)\zeta(3) + 144\zeta(4) - 8\log(2)^4 + 48\log(2)^2\zeta(2) + 74\zeta(5) - 32\zeta(2)\zeta(3)\\ &-25\zeta(6) + 16\zeta(3)^2 - 32S_{4,2}(-1)\big)\end{aligned} \tag{76}$$

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{H(2k)}{(2k)^6(2k+1)} = \frac{1}{768}\big(&-192\zeta(2) + 528\zeta(3) - 384\log(2)^2 - 768\mathrm{Li}_4(1/2)\\ &-672\log(2)\zeta(3) + 576\zeta(4) - 32\log(2)^4 + 192\log(2)^2\zeta(2) + 444\zeta(5) - 192\zeta(2)\zeta(3)\\ &-300\zeta(6) + 192\zeta(3)^2 - 384S_{4,2}(-1) + 405\zeta(7) - 192\zeta(2)\zeta(5)\\ &-48\zeta(3)\zeta(4)\big)\end{aligned} \tag{77}$$

$$\sum_{k=1}^{\infty} \frac{H(2k)}{(2k)^7} = \frac{1}{256}\left(161\zeta(8) - 128\zeta(3)\zeta(5) + 128S_{6,2}(-1)\right) \tag{78}$$

Some additional formulas include:

$$\sum_{k=1}^{\infty} H(2k)\left(\frac{1}{(2k)^7} + \frac{1}{(2k+1)^7}\right) = \frac{107\pi^8}{806400} - \zeta(3)\zeta(5) \tag{79}$$

$$\sum_{k=1}^{\infty} H(2k)\left(\frac{1}{(2k)^7}-\frac{1}{(2k+1)^7}\right) = \sum_{k=1}^{\infty} H(2k)\left(\frac{7}{(2k)^4(2k+1)^4}+\frac{14}{(2k)^5(2k+1)^5}+\frac{7}{(2k)^6(2k+1)^6}+\frac{1}{(2k)^7(2k+1)^7}\right) \tag{80}$$

Let

$$T_{10}=\frac{1}{2}\left(420\mathrm{Li}_4\left(\frac{1}{2}\right)-\frac{2331\zeta(3)}{4}-\frac{7}{720}\pi^4(8\zeta(3)+435)-\frac{455\zeta(5)}{4}-\zeta(3)\zeta(5)+\frac{2695\zeta(7)}{128}\right.$$
$$\left.-\frac{7}{6}\pi^2\left(-8\zeta(3)+\zeta(5)-37+15\log(2)^2\right)+\frac{735}{2}\zeta(3)\log(2)+\frac{107\pi^8}{806400}+\frac{35\log(2)^4}{2}+518\log(2)^2\right) \tag{81}$$

and

$$T_{11}=\frac{1}{2}\left(-\frac{7}{128}\left(7680\mathrm{Li}_4\left(\frac{1}{2}\right)-10656\zeta(3)-2080\zeta(5)+385\zeta(7)+6720\zeta(3)\log(2)+320\log(2)^4+9472\log(2)^2\right)\right.$$
$$\left.+\frac{7}{720}\pi^4(8\zeta(3)+435)-\zeta(3)\zeta(5)+\frac{7}{6}\pi^2\left(-8\zeta(3)+\zeta(5)-37+15\log(2)^2\right)+\frac{107\pi^8}{806400}\right) \tag{82}$$

Then

$$\sum_{k=1}^{\infty}\frac{H(2k)}{(2k)^7}=\frac{1}{2}\sum_{k=1}^{\infty}\frac{H(2k)}{(2k)^7(2k+1)^7}-14\sum_{k=1}^{\infty}\frac{H(2k)}{(2k)^5(2k+1)^5}+T_{10} \tag{83}$$

and

$$\sum_{k=1}^{\infty}\frac{H(2k)}{(2k+1)^7}=-\frac{1}{2}\sum_{k=1}^{\infty}\frac{H(2k)}{(2k)^7(2k+1)^7}+14\sum_{k=1}^{\infty}\frac{H(2k)}{(2k)^5(2k+1)^5}+T_{11} \tag{84}$$

From (83),

$$\sum_{k=1}^{\infty}\frac{H(2k)}{(2k)^7(2k+1)^7}=420\mathrm{Li}_4\left(\frac{1}{2}\right)+28S_{4,2}(-1)+S_{6,2}(-1)-\frac{2079\zeta(3)}{4}+\frac{7}{720}\pi^4(8\zeta(3)-435)-\frac{1365\zeta(5)}{8}-\frac{2695\zeta(7)}{128}$$
$$-\frac{7}{6}\pi^2\left(-12\zeta(3)-\zeta(5)-33+15\log(2)^2\right)+\frac{735}{2}\zeta(3)\log(2)+\frac{\pi^6}{2160}+\frac{\pi^8}{2419200}+\frac{35\log(2)^4}{2}+462\log(2)^2 \tag{85}$$

Let

$$T_5=\frac{1}{480}[-12\pi^4-20\pi^2(-5+3\log(2)^2-2\zeta(3))$$
$$+15(80\log(2)^2+4\log(2)^4+96\mathrm{Li}_4(1/2)-102\zeta(3)+84\log(2)\zeta(3)-37\zeta(5))] \tag{86}$$

and

$$T_6=\frac{1}{480}[17\pi^4+20\pi^2(-5+3\log(2)^2-6\zeta(3))$$
$$-15(80\log(2)^2+4\log(2)^4+96\mathrm{Li}_4(1/2)-78\zeta(3)+84\log(2)\zeta(3)-93\zeta(5))] \tag{87}$$

Then one obtains these two formulas that involve a sum of denominator order 10:

$$\sum_{k=1}^{\infty}\frac{H(2k)}{(2k)^5(2k+1)^2}=T_5+\sum_{k=1}^{\infty}\frac{H(2k)}{(2k)^5}=T_5+T_3+\frac{1}{2}\sum_{k=1}^{\infty}\frac{H(2k)}{(2k)^5(2k+1)^5} \tag{88}$$

and

$$\sum_{k=1}^{\infty}\frac{H(2k)}{(2k)^2(2k+1)^5}=T_6+\sum_{k=1}^{\infty}\frac{H(2k)}{(2k+1)^5}=T_6+T_4-\frac{1}{2}\sum_{k=1}^{\infty}\frac{H(2k)}{(2k)^5(2k+1)^5} \tag{89}$$

12 Sums of denominator order eight

$$\sum_{k=1}^{\infty} \frac{H(2k)}{(2k+1)^8} = \frac{1}{1024}\left(3577\zeta(9) - 768\zeta(2)\zeta(7) - 1008\zeta(3)\zeta(6)\right.$$
$$\left. -960\zeta(4)\zeta(5)\right) \tag{90}$$

$$\sum_{k=1}^{\infty} \frac{H(2k)}{(2k)(2k+1)^7} = \frac{1}{768}\left(192\zeta(2) - 336\zeta(3) + 384\log(2)^2 + 768\mathrm{Li}_4(1/2)\right.$$
$$+672\log(2)\zeta(3) - 816\zeta(4) + 32\log(2)^4 - 192\log(2)^2\zeta(2) - 1116\zeta(5) + 576\zeta(2)\zeta(3)$$
$$-288\zeta(6) + 192\zeta(3)^2 + 384S_{4,2}(-1) - 1905\zeta(7) + 576\zeta(2)\zeta(5) + 720\zeta(3)\zeta(4)$$
$$\left. -480\zeta(8) + 384\zeta(3)\zeta(5) + 384S_{6,2}(-1)\right) \tag{91}$$

$$\sum_{k=1}^{\infty} \frac{H(2k)}{(2k)^2(2k+1)^6} = \frac{1}{768}\left(-1152\zeta(2) + 2208\zeta(3) - 2304\log(2)^2 - 3072\mathrm{Li}_4(1/2)\right.$$
$$-2688\log(2)\zeta(3) + 3264\zeta(4) - 128\log(2)^4 + 768\log(2)^2\zeta(2) + 3348\zeta(5)$$
$$-1728\zeta(2)\zeta(3) + 576\zeta(6) - 384\zeta(3)^2 - 768S_{4,2}(-1) + 1905\zeta(7) - 576\zeta(2)\zeta(5)$$
$$\left. -720\zeta(3)\zeta(4)\right) \tag{92}$$

$$\sum_{k=1}^{\infty} \frac{H(2k)}{(2k)^3(2k+1)^5} = \frac{1}{192}\left(720\zeta(2) - 1500\zeta(3) + 1440\log(2)^2 + 1344\mathrm{Li}_4(1/2)\right.$$
$$+1176\log(2)\zeta(3) - 1368\zeta(4) + 56\log(2)^4 - 336\log(2)^2\zeta(2) - 837\zeta(5) + 432\zeta(2)\zeta(3)$$
$$\left. -72\zeta(6) + 48\zeta(3)^2 + 96S_{4,2}(-1)\right) \tag{93}$$

$$\sum_{k=1}^{\infty} \frac{H(2k)}{(2k)^4(2k+1)^4} = \frac{-1}{96}\left(480\zeta(2) - 1080\zeta(3) + 960\log(2)^2 + 768\mathrm{Li}_4(1/2)\right.$$
$$+672\log(2)\zeta(3) - 696\zeta(4) + 32\log(2)^4 - 192\log(2)^2\zeta(2) - 195\zeta(5)$$
$$\left. +96\zeta(2)\zeta(3)\right) \tag{94}$$

$$\sum_{k=1}^{\infty} \frac{H(2k)}{(2k)^5(2k+1)^3} = \frac{1}{192}\left(720\zeta(2) - 1740\zeta(3) + 1440\log(2)^2 + 1344\mathrm{Li}_4(1/2)\right.$$
$$+1176\log(2)\zeta(3) - 1068\zeta(4) + 56\log(2)^4 - 336\log(2)^2\zeta(2) - 333\zeta(5) + 144\zeta(2)\zeta(3)$$
$$\left. +75\zeta(6) - 48\zeta(3)^2 + 96S_{4,2}(-1)\right) \tag{95}$$

$$\sum_{k=1}^{\infty} \frac{H(2k)}{(2k)^6(2k+1)^2} = \frac{1}{768}\left(-1152\zeta(2) + 2976\zeta(3) - 2304\log(2)^2 - 3072\mathrm{Li}_4(1/2)\right.$$
$$-2688\log(2)\zeta(3) + 2304\zeta(4) - 128\log(2)^4 + 768\log(2)^2\zeta(2) + 1332\zeta(5)$$
$$-576\zeta(2)\zeta(3) - 600\zeta(6) + 384\zeta(3)^2 - 768S_{4,2}(-1) + 405\zeta(7) - 192\zeta(2)\zeta(5)$$
$$\left. -48\zeta(3)\zeta(4)\right) \tag{96}$$

$$\sum_{k=1}^{\infty} \frac{H(2k)}{(2k)^7(2k+1)} = \frac{1}{768}\left(192\zeta(2) - 528\zeta(3) + 384\log(2)^2 + 768\mathrm{Li}_4(1/2)\right.$$
$$+672\log(2)\zeta(3) - 576\zeta(4) + 32\log(2)^4 - 192\log(2)^2\zeta(2) - 444\zeta(5) + 192\zeta(2)\zeta(3)$$
$$+300\zeta(6) - 192\zeta(3)^2 + 384S_{4,2}(-1) - 405\zeta(7) + 192\zeta(2)\zeta(5) + 48\zeta(3)\zeta(4)$$
$$\left. +483\zeta(8) - 384\zeta(3)\zeta(5) + 384S_{6,2}(-1)\right) \tag{97}$$

$$\sum_{k=1}^{\infty} \frac{H(2k)}{(2k)^8} = \frac{1}{1024} \left(521\zeta(9) - 256\zeta(2)\zeta(7) - 16\zeta(3)\zeta(6)\right.$$
$$\left. -64\zeta(4)\zeta(5)\right) \tag{98}$$

Some additional formulas include:

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1} H(k)}{k^8} = \frac{-496\pi^6\zeta(3) - 4704\pi^4\zeta(5) - 40320\pi^2\zeta(7) + 1926855\zeta(9)}{483840} \tag{99}$$

$$\sum_{k=1}^{\infty} \frac{H(2k)}{(2k)(2k+1)^7} = T_{13} + \frac{1}{2}\sum_{k=1}^{\infty} \frac{H(2k)}{(2k)^7(2k+1)^7} - \frac{27}{2}\sum_{k=1}^{\infty} \frac{H(2k)}{(2k)^5(2k+1)^5} \tag{100}$$

The relevant Nielsen polylogarithm for this order is

$$S_{7,2}(-1) = \frac{496\pi^6\zeta(3) + 4704\pi^4\zeta(5) + 40320\pi^2\zeta(7) - 1444905\zeta(9)}{483840} \tag{101}$$

13 Sums of denominator order nine

$$\sum_{k=1}^{\infty}\frac{H(2k)}{(2k+1)^9}=\frac{1}{8}\left(7\zeta(10)-4\zeta(3)\zeta(7)-2\zeta(5)^2-4S_{8,2}(-1)\right) \tag{102}$$

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{H(2k)}{(2k)(2k+1)^8}=&\frac{1}{3072}\big(768\zeta(2)-1344\zeta(3)+1536\log(2)^2+3072\mathrm{Li}_4(1/2)\\&+2688\log(2)\zeta(3)-3264\zeta(4)+128\log(2)^4-768\log(2)^2\zeta(2)-4464\zeta(5)\\&+2304\zeta(2)\zeta(3)-1152\zeta(6)+768\zeta(3)^2+1536S_{4,2}(-1)-7620\zeta(7)\\&+2304\zeta(2)\zeta(5)+2880\zeta(3)\zeta(4)-1920\zeta(8)+1536\zeta(3)\zeta(5)+1536S_{6,2}(-1)\\&-10731\zeta(9)+2304\zeta(2)\zeta(7)+3024\zeta(3)\zeta(6)+2880\zeta(4)\zeta(5))\end{aligned} \tag{103}$$

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{H(2k)}{(2k)^2(2k+1)^7}=&\frac{-1}{384}\big(672\zeta(2)-1272\zeta(3)+1344\log(2)^2+1920\mathrm{Li}_4(1/2)\\&+1680\log(2)\zeta(3)-2040\zeta(4)+80\log(2)^4-480\log(2)^2\zeta(2)-2232\zeta(5)\\&+1152\zeta(2)\zeta(3)-432\zeta(6)+288\zeta(3)^2+576S_{4,2}(-1)-1905\zeta(7)+576\zeta(2)\zeta(5)\\&+720\zeta(3)\zeta(4)-240\zeta(8)+192\zeta(3)\zeta(5)+192S_{6,2}(-1))\end{aligned} \tag{104}$$

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{H(2k)}{(2k)^3(2k+1)^6}=&\frac{1}{768}\big(4032\zeta(2)-8208\zeta(3)+8064\log(2)^2+8448\mathrm{Li}_4(1/2)\\&+7392\log(2)\zeta(3)-8736\zeta(4)+352\log(2)^4-2112\log(2)^2\zeta(2)-6696\zeta(5)\\&+3456\zeta(2)\zeta(3)-864\zeta(6)+576\zeta(3)^2+1152S_{4,2}(-1)-1905\zeta(7)+576\zeta(2)\zeta(5)\\&+720\zeta(3)\zeta(4))\end{aligned} \tag{105}$$

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{H(2k)}{(2k)^4(2k+1)^5}=&\frac{-1}{64}\big(560\zeta(2)-1220\zeta(3)+1120\log(2)^2+960\mathrm{Li}_4(1/2)\\&+840\log(2)\zeta(3)-920\zeta(4)+40\log(2)^4-240\log(2)^2\zeta(2)-409\zeta(5)+208\zeta(2)\zeta(3)\\&-24\zeta(6)+16\zeta(3)^2+32S_{4,2}(-1)\big)\end{aligned} \tag{106}$$

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{H(2k)}{(2k)^5(2k+1)^4}=&\frac{-1}{64}\big(-560\zeta(2)+1300\zeta(3)-1120\log(2)^2-960\mathrm{Li}_4(1/2)\\&-840\log(2)\zeta(3)+820\zeta(4)-40\log(2)^4+240\log(2)^2\zeta(2)+241\zeta(5)-112\zeta(2)\zeta(3)\\&-25\zeta(6)+16\zeta(3)^2-32S_{4,2}(-1)\big)\end{aligned} \tag{107}$$

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{H(2k)}{(2k)^6(2k+1)^3}=&\frac{-1}{768}\big(4032\zeta(2)-9936\zeta(3)+8064\log(2)^2+8448\mathrm{Li}_4(1/2)\\&+7392\log(2)\zeta(3)-6576\zeta(4)+352\log(2)^4-2112\log(2)^2\zeta(2)-2664\zeta(5)\\&+1152\zeta(2)\zeta(3)+900\zeta(6)-576\zeta(3)^2+1152S_{4,2}(-1)-405\zeta(7)+192\zeta(2)\zeta(5)\\&+48\zeta(3)\zeta(4))\end{aligned} \tag{108}$$

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{H(2k)}{(2k)^7(2k+1)^2}=&\frac{1}{768}\big(1344\zeta(2)-3504\zeta(3)+2688\log(2)^2+3840\mathrm{Li}_4(1/2)\\&+3360\log(2)\zeta(3)-2880\zeta(4)+160\log(2)^4-960\log(2)^2\zeta(2)-1776\zeta(5)\\&+768\zeta(2)\zeta(3)+900\zeta(6)-576\zeta(3)^2+1152S_{4,2}(-1)-810\zeta(7)+384\zeta(2)\zeta(5)\\&+96\zeta(3)\zeta(4)+483\zeta(8)-384\zeta(3)\zeta(5)+384S_{6,2}(-1))\end{aligned} \tag{109}$$

$$\sum_{k=1}^{\infty}\frac{H(2k)}{(2k)^8(2k+1)}=\frac{1}{3072}\big(-768\zeta(2)+2112\zeta(3)-1536\log(2)^2-3072\mathrm{Li}_4(1/2)$$
$$-2688\log(2)\zeta(3)+2304\zeta(4)-128\log(2)^4+768\log(2)^2\zeta(2)+1776\zeta(5)$$
$$-768\zeta(2)\zeta(3)-1200\zeta(6)+768\zeta(3)^2-1536S_{4,2}(-1)+1620\zeta(7)-768\zeta(2)\zeta(5)$$
$$-192\zeta(3)\zeta(4)-1932\zeta(8)+1536\zeta(3)\zeta(5)-1536S_{6,2}(-1)+1563\zeta(9)$$
$$-768\zeta(2)\zeta(7)-48\zeta(3)\zeta(6)-192\zeta(4)\zeta(5)\big) \tag{110}$$

$$\sum_{k=1}^{\infty}\frac{H(2k)}{(2k)^9}=\frac{-1}{1024}\left(-897\zeta(10)+512\zeta(3)\zeta(7)+256\zeta(5)^2-512S_{8,2}(-1)\right) \tag{111}$$

Some additional formulas include:

$$\sum_{k=1}^{\infty}H(2k)\left(\frac{1}{(2k)^9}+\frac{1}{(2k+1)^9}\right)=-\frac{\zeta(5)^2}{2}-\zeta(3)\zeta(7)+\frac{163\pi^{10}}{8709120} \tag{112}$$

Let

$$T_9=-\frac{\pi^2\zeta(3)}{4}+\frac{1}{8}\left(-4\zeta(3)^2-10\zeta(3)+21\zeta(5)\right)+\frac{7\pi^6}{8640}+\frac{5\pi^4}{288} \tag{113}$$

Then one obtains these formulas involving a sum of denominator order 10:

$$\sum_{k=1}^{\infty}\frac{H(2k)}{(2k)^5(2k+1)^4}=\frac{1}{2}\left(\sum_{k=1}^{\infty}\frac{H(2k)}{(2k)^5(2k+1)^5}+T_9\right), \tag{114}$$

and

$$\sum_{k=1}^{\infty}\frac{H(2k)}{(2k)^4(2k+1)^5}=\frac{1}{2}\left(-\sum_{k=1}^{\infty}\frac{H(2k)}{(2k)^5(2k+1)^5}+T_9\right). \tag{115}$$

$$\sum_{k=1}^{\infty}\frac{H(2k)}{(2k)(2k+1)^8}=\frac{1}{2}\sum_{k=1}^{\infty}\frac{H(2k)}{(2k)^7(2k+1)^7}-\frac{27}{2}\sum_{k=1}^{\infty}\frac{H(2k)}{(2k)^5(2k+1)^5}+T_{14} \tag{116}$$

$$\sum_{k=1}^{\infty}\frac{H(2k)}{(2k)^8(2k+1)}=-\frac{1}{2}\sum_{k=1}^{\infty}\frac{H(2k)}{(2k)^7(2k+1)^7}+\frac{27}{2}\sum_{k=1}^{\infty}\frac{H(2k)}{(2k)^5(2k+1)^5}+T_{15} \tag{117}$$

Let

$$T_{16}=170\mathrm{Li}_4\left(\frac{1}{2}\right)-\frac{1895\zeta(3)}{8}+\frac{43\pi^2\zeta(3)}{12}-\frac{3\pi^4\zeta(3)}{80}-\frac{3\zeta(3)^2}{4}-\frac{2813\zeta(5)}{64}-\frac{\pi^2\zeta(5)}{2}-\frac{\zeta(3)\zeta(5)}{2}+\frac{2425\zeta(7)}{256}$$
$$+\frac{595}{4}\zeta(3)\log(2)+\frac{35\pi^2}{2}-\frac{109\pi^4}{64}+\frac{7\pi^6}{5760}+\frac{107\pi^8}{1612800}+\frac{85\log(2)^4}{12}+210\log(2)^2-\frac{85}{12}\pi^2\log(2)^2 \tag{118}$$

and

$$T_{17}=-170\mathrm{Li}_4\left(\frac{1}{2}\right)+\frac{1885\zeta(3)}{8}-\frac{47\pi^2\zeta(3)}{12}+\frac{13\pi^4\zeta(3)}{720}-\frac{3\zeta(3)^2}{4}+\frac{3037\zeta(5)}{64}+\frac{\pi^2\zeta(5)}{3}-\frac{\zeta(3)\zeta(5)}{2}-\frac{1425\zeta(7)}{256}$$
$$-\frac{595}{4}\zeta(3)\log(2)-\frac{35\pi^2}{2}+\frac{991\pi^4}{576}+\frac{7\pi^6}{5760}+\frac{107\pi^8}{1612800}-\frac{85\log(2)^4}{12}-210\log(2)^2+\frac{85}{12}\pi^2\log(2)^2 \tag{119}$$

Then one obtains the following formulas involving sums of denominator order 14 and 10:

$$\sum_{k=1}^{\infty}\frac{H(2k)}{(2k)^7(2k+1)^2}=\frac{1}{2}\sum_{k=1}^{\infty}\frac{H(2k)}{(2k)^7(2k+1)^7}-\frac{25}{2}\sum_{k=1}^{\infty}\frac{H(2k)}{(2k)^5(2k+1)^5}+T_{16}, \tag{120}$$

and

$$\sum_{k=1}^{\infty} \frac{H(2k)}{(2k)^2(2k+1)^7} = -\frac{1}{2}\sum_{k=1}^{\infty} \frac{H(2k)}{(2k)^7(2k+1)^7} + \frac{25}{2}\sum_{k=1}^{\infty} \frac{H(2k)}{(2k)^5(2k+1)^5} + T_{17} \tag{121}$$

14 Sums of denominator order ten

$$\sum_{k=1}^{\infty} \frac{H(2k)}{(2k+1)^{10}} = \frac{-1}{4096}\left(-18423\zeta(11) + 3072\zeta(2)\zeta(9) + 4080\zeta(3)\zeta(8)\right.$$
$$\left.+3840\zeta(4)\zeta(7) + 4032\zeta(5)\zeta(6)\right) \quad (122)$$

$$\sum_{k=1}^{\infty} \frac{H(2k)}{(2k)(2k+1)^{9}} = \frac{1}{3072}\big(768\zeta(2) - 1344\zeta(3) + 1536\log(2)^2 + 3072\mathrm{Li}_4(1/2)$$
$$+2688\log(2)\zeta(3) - 3264\zeta(4) + 128\log(2)^4 - 768\log(2)^2\zeta(2) - 4464\zeta(5)$$
$$+2304\zeta(2)\zeta(3) - 1152\zeta(6) + 768\zeta(3)^2 + 1536S_{4,2}(-1) - 7620\zeta(7)$$
$$+2304\zeta(2)\zeta(5) + 2880\zeta(3)\zeta(4) - 1920\zeta(8) + 1536\zeta(3)\zeta(5) + 1536S_{6,2}(-1)$$
$$-10731\zeta(9) + 2304\zeta(2)\zeta(7) + 3024\zeta(3)\zeta(6) + 2880\zeta(4)\zeta(5) - 2688\zeta(10)$$
$$+1536\zeta(3)\zeta(7) + 768\zeta(5)^2 + 1536S_{8,2}(-1)\big) \quad (123)$$

$$\sum_{k=1}^{\infty} \frac{H(2k)}{(2k)^2(2k+1)^{8}} = \frac{-1}{1024}\big(2048\zeta(2) - 3840\zeta(3) + 4096\log(2)^2 + 6144\mathrm{Li}_4(1/2)$$
$$+5376\log(2)\zeta(3) - 6528\zeta(4) + 256\log(2)^4 - 1536\log(2)^2\zeta(2) - 7440\zeta(5)$$
$$+3840\zeta(2)\zeta(3) - 1536\zeta(6) + 1024\zeta(3)^2 + 2048S_{4,2}(-1) - 7620\zeta(7)$$
$$+2304\zeta(2)\zeta(5) + 2880\zeta(3)\zeta(4) - 1280\zeta(8) + 1024\zeta(3)\zeta(5) + 1024S_{6,2}(-1)$$
$$-3577\zeta(9) + 768\zeta(2)\zeta(7) + 1008\zeta(3)\zeta(6) + 960\zeta(4)\zeta(5)\big) \quad (124)$$

$$\sum_{k=1}^{\infty} \frac{H(2k)}{(2k)^3(2k+1)^{7}} = \frac{-1}{768}\big(-5376\zeta(2) + 10752\zeta(3) - 10752\log(2)^2 - 12288\mathrm{Li}_4(1/2)$$
$$-10752\log(2)\zeta(3) + 12816\zeta(4) - 512\log(2)^4 + 3072\log(2)^2\zeta(2) + 11160\zeta(5)$$
$$-5760\zeta(2)\zeta(3) + 1728\zeta(6) - 1152\zeta(3)^2 - 2304S_{4,2}(-1) + 5715\zeta(7)$$
$$-1728\zeta(2)\zeta(5) - 2160\zeta(3)\zeta(4) + 480\zeta(8) - 384\zeta(3)\zeta(5) - 384S_{6,2}(-1)\big) \quad (125)$$

$$\sum_{k=1}^{\infty} \frac{H(2k)}{(2k)^4(2k+1)^{6}} = \frac{1}{768}\big(-10752\zeta(2) + 22848\zeta(3) - 21504\log(2)^2 - 19968\mathrm{Li}_4(1/2)$$
$$-17472\log(2)\zeta(3) + 19776\zeta(4) - 832\log(2)^4 + 4992\log(2)^2\zeta(2) + 11604\zeta(5)$$
$$-5952\zeta(2)\zeta(3) + 1152\zeta(6) - 768\zeta(3)^2 - 1536S_{4,2}(-1) + 1905\zeta(7) - 576\zeta(2)\zeta(5)$$
$$-720\zeta(3)\zeta(4)\big) \quad (126)$$

$$\sum_{k=1}^{\infty} \frac{H(2k)}{(2k)^5(2k+1)^{5}} = \frac{-1}{64}\big(-1120\zeta(2) + 2520\zeta(3) - 2240\log(2)^2 - 1920\mathrm{Li}_4(1/2)$$
$$-1680\log(2)\zeta(3) + 1740\zeta(4) - 80\log(2)^4 + 480\log(2)^2\zeta(2) + 650\zeta(5) - 320\zeta(2)\zeta(3)$$
$$-\zeta(6) - 64S_{4,2}(-1)\big) \quad (127)$$

$$\sum_{k=1}^{\infty} \frac{H(2k)}{(2k)^6(2k+1)^{4}} = \frac{1}{768}\big(-10752\zeta(2) + 25536\zeta(3) - 21504\log(2)^2 - 19968\mathrm{Li}_4(1/2)$$
$$-17472\log(2)\zeta(3) + 16416\zeta(4) - 832\log(2)^4 + 4992\log(2)^2\zeta(2) + 5556\zeta(5)$$
$$-2496\zeta(2)\zeta(3) - 1200\zeta(6) + 768\zeta(3)^2 - 1536S_{4,2}(-1) + 405\zeta(7) - 192\zeta(2)\zeta(5)$$
$$-48\zeta(3)\zeta(4)\big) \quad (128)$$

$$\sum_{k=1}^{\infty}\frac{H(2k)}{(2k)^7(2k+1)^3} = \frac{1}{768}\big(5376\zeta(2) - 13440\zeta(3) + 10752\log(2)^2 + 12288\mathrm{Li}_4(1/2)$$
$$+10752\log(2)\zeta(3) - 9456\zeta(4) + 512\log(2)^4 - 3072\log(2)^2\zeta(2) - 4440\zeta(5)$$
$$+1920\zeta(2)\zeta(3) + 1800\zeta(6) - 1152\zeta(3)^2 + 2304S_{4,2}(-1) - 1215\zeta(7)$$
$$+576\zeta(2)\zeta(5) + 144\zeta(3)\zeta(4) + 483\zeta(8) - 384\zeta(3)\zeta(5) + 384S_{6,2}(-1)) \qquad (129)$$

$$\sum_{k=1}^{\infty}\frac{H(2k)}{(2k)^8(2k+1)^2} = \frac{1}{1024}\big(-2048\zeta(2) + 5376\zeta(3) - 4096\log(2)^2 - 6144\mathrm{Li}_4(1/2)$$
$$-5376\log(2)\zeta(3) + 4608\zeta(4) - 256\log(2)^4 + 1536\log(2)^2\zeta(2) + 2960\zeta(5)$$
$$-1280\zeta(2)\zeta(3) - 1600\zeta(6) + 1024\zeta(3)^2 - 2048S_{4,2}(-1) + 1620\zeta(7)$$
$$-768\zeta(2)\zeta(5) - 192\zeta(3)\zeta(4) - 1288\zeta(8) + 1024\zeta(3)\zeta(5) - 1024S_{6,2}(-1)$$
$$+521\zeta(9) - 256\zeta(2)\zeta(7) - 16\zeta(3)\zeta(6) - 64\zeta(4)\zeta(5)) \qquad (130)$$

$$\sum_{k=1}^{\infty}\frac{H(2k)}{(2k)^9(2k+1)} = \frac{-1}{3072}\big(-768\zeta(2) + 2112\zeta(3) - 1536\log(2)^2 - 3072\mathrm{Li}_4(1/2)$$
$$-2688\log(2)\zeta(3) + 2304\zeta(4) - 128\log(2)^4 + 768\log(2)^2\zeta(2) + 1776\zeta(5)$$
$$-768\zeta(2)\zeta(3) - 1200\zeta(6) + 768\zeta(3)^2 - 1536S_{4,2}(-1) + 1620\zeta(7) - 768\zeta(2)\zeta(5)$$
$$-192\zeta(3)\zeta(4) - 1932\zeta(8) + 1536\zeta(3)\zeta(5) - 1536S_{6,2}(-1) + 1563\zeta(9)$$
$$-768\zeta(2)\zeta(7) - 48\zeta(3)\zeta(6) - 192\zeta(4)\zeta(5) - 2691\zeta(10) + 1536\zeta(3)\zeta(7)$$
$$+768\zeta(5)^2 - 1536S_{8,2}(-1)\big) \qquad (131)$$

$$\sum_{k=1}^{\infty}\frac{H(2k)}{(2k)^{10}} = \frac{1}{4096}(2059\zeta(11) - 1024\zeta(2)\zeta(9) - 16\zeta(3)\zeta(8) - 256\zeta(4)\zeta(7)$$
$$-64\zeta(5)\zeta(6)) \qquad (132)$$

Some additional formulas include:

$$\sum_{k=1}^{\infty}\frac{H(k)}{(k)^{10}} = \frac{-\pi^8\zeta(3) - 10\pi^6\zeta(5) - 105\pi^4\zeta(7) - 1575\pi^2\zeta(9) + 56700\zeta(11)}{9450} \qquad (133)$$

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}H(k)}{(k)^{10}} = \frac{-1016\pi^8\zeta(3) - 9920\pi^6\zeta(5) - 94080\pi^4\zeta(7) - 806400\pi^2\zeta(9) + 48332025\zeta(11)}{9676800} \qquad (134)$$

$$\sum_{k=1}^{\infty}\frac{H(k)}{(2k+1)^{10}} = \frac{-1071\pi^8\zeta(3) - 11718\pi^6\zeta(5) - 120015\pi^4\zeta(7) - 1448685\pi^2\zeta(9) + 58032450\zeta(11) - 62\pi^{10}\log(4)}{5806080} \qquad (135)$$

15 Formulas for denominator order 11

$$\sum_{k=1}^{\infty} \frac{H(2k)}{(2k+1)^{11}} = \frac{-1}{8} \left(-9\zeta(12) + 4\zeta(3)\zeta(9) + 4\zeta(5)\zeta(7) + 4S_{10,2}(-1)\right) \tag{136}$$

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{H(2k)}{(2k)(2k+1)^{10}} = \frac{-1}{12288} \big(&-3072\zeta(2) + 5376\zeta(3) - 6144\log(2)^2 - 12288\mathrm{Li}_4(1/2)\\
&-10752\log(2)\zeta(3) + 13056\zeta(4) - 512\log(2)^4 + 3072\log(2)^2\zeta(2) + 17856\zeta(5)\\
&-9216\zeta(2)\zeta(3) + 4608\zeta(6) - 3072\zeta(3)^2 - 6144S_{4,2}(-1) + 30480\zeta(7)\\
&-9216\zeta(2)\zeta(5) - 11520\zeta(3)\zeta(4) + 7680\zeta(8) - 6144\zeta(3)\zeta(5) - 6144S_{6,2}(-1)\\
&+42924\zeta(9) - 9216\zeta(2)\zeta(7) - 12096\zeta(3)\zeta(6) - 11520\zeta(4)\zeta(5) + 10752\zeta(10)\\
&-6144\zeta(3)\zeta(7) - 3072\zeta(5)^2 - 6144S_{8,2}(-1) + 55269\zeta(11) - 9216\zeta(2)\zeta(9)\\
&-12240\zeta(3)\zeta(8) - 11520\zeta(4)\zeta(7) - 12096\zeta(5)\zeta(6)\big)\end{aligned} \tag{137}$$

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{H(2k)}{(2k)^2(2k+1)^{9}} = \frac{-1}{1536} \big(&3456\zeta(2) - 6432\zeta(3) + 6912\log(2)^2 + 10752\mathrm{Li}_4(1/2)\\
&+9408\log(2)\zeta(3) - 11424\zeta(4) + 448\log(2)^4 - 2688\log(2)^2\zeta(2) - 13392\zeta(5)\\
&+6912\zeta(2)\zeta(3) - 2880\zeta(6) + 1920\zeta(3)^2 + 3840S_{4,2}(-1) - 15240\zeta(7)\\
&+4608\zeta(2)\zeta(5) + 5760\zeta(3)\zeta(4) - 2880\zeta(8) + 2304\zeta(3)\zeta(5) + 2304S_{6,2}(-1)\\
&-10731\zeta(9) + 2304\zeta(2)\zeta(7) + 3024\zeta(3)\zeta(6) + 2880\zeta(4)\zeta(5) - 1344\zeta(10)\\
&+768\zeta(3)\zeta(7) + 384\zeta(5)^2 + 768S_{8,2}(-1)\big)\end{aligned} \tag{138}$$

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{H(2k)}{(2k)^3(2k+1)^{8}} = \frac{-1}{3072} \big(&-27648\zeta(2) + 54528\zeta(3) - 55296\log(2)^2 - 67584\mathrm{Li}_4(1/2)\\
&-59136\log(2)\zeta(3) + 70848\zeta(4) - 2816\log(2)^4 + 16896\log(2)^2\zeta(2) + 66960\zeta(5)\\
&-34560\zeta(2)\zeta(3) + 11520\zeta(6) - 7680\zeta(3)^2 - 15360S_{4,2}(-1) + 45720\zeta(7)\\
&-13824\zeta(2)\zeta(5) - 17280\zeta(3)\zeta(4) + 5760\zeta(8) - 4608\zeta(3)\zeta(5) - 4608S_{6,2}(-1)\\
&+10731\zeta(9) - 2304\zeta(2)\zeta(7) - 3024\zeta(3)\zeta(6) - 2880\zeta(4)\zeta(5)\big)\end{aligned} \tag{139}$$

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{H(2k)}{(2k)^4(2k+1)^{7}} = \frac{-1}{64} \big(&1344\zeta(2) - 2800\zeta(3) + 2688\log(2)^2 + 2688\mathrm{Li}_4(1/2)\\
&+2352\log(2)\zeta(3) - 2716\zeta(4) + 112\log(2)^4 - 672\log(2)^2\zeta(2) - 1897\zeta(5)\\
&+976\zeta(2)\zeta(3) - 240\zeta(6) + 160\zeta(3)^2 + 320S_{4,2}(-1) - 635\zeta(7) + 192\zeta(2)\zeta(5)\\
&+240\zeta(3)\zeta(4) - 40\zeta(8) + 32\zeta(3)\zeta(5) + 32S_{6,2}(-1)\big)\end{aligned} \tag{140}$$

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{H(2k)}{(2k)^5(2k+1)^{6}} = \frac{1}{768} \big(&24192\zeta(2) - 53088\zeta(3) + 48384\log(2)^2 + 43008\mathrm{Li}_4(1/2)\\
&+37632\log(2)\zeta(3) - 40656\zeta(4) + 1792\log(2)^4 - 10752\log(2)^2\zeta(2) - 19404\zeta(5)\\
&+9792\zeta(2)\zeta(3) - 1140\zeta(6) + 768\zeta(3)^2 + 2304S_{4,2}(-1) - 1905\zeta(7) + 576\zeta(2)\zeta(5)\\
&+720\zeta(3)\zeta(4)\big)\end{aligned} \tag{141}$$

$$\sum_{k=1}^{\infty}\frac{H(2k)}{(2k)^6(2k+1)^5} = \frac{1}{768}\big(-24192\zeta(2)+55776\zeta(3)-48384\log(2)^2-43008\mathrm{Li}_4(1/2)$$
$$-37632\log(2)\zeta(3)+37296\zeta(4)-1792\log(2)^4+10752\log(2)^2\zeta(2)+13356\zeta(5)$$
$$-6336\zeta(2)\zeta(3)-1212\zeta(6)+768\zeta(3)^2-2304S_{4,2}(-1)+405\zeta(7)-192\zeta(2)\zeta(5)$$
$$-48\zeta(3)\zeta(4)\big) \tag{142}$$

$$\sum_{k=1}^{\infty}\frac{H(2k)}{(2k)^7(2k+1)^4} = \frac{-1}{256}\big(-5376\zeta(2)+12992\zeta(3)-10752\log(2)^2-10752\mathrm{Li}_4(1/2)$$
$$-9408\log(2)\zeta(3)+8624\zeta(4)-448\log(2)^4+2688\log(2)^2\zeta(2)+3332\zeta(5)$$
$$-1472\zeta(2)\zeta(3)-1000\zeta(6)+640\zeta(3)^2-1280S_{4,2}(-1)+540\zeta(7)-256\zeta(2)\zeta(5)$$
$$-64\zeta(3)\zeta(4)-161\zeta(8)+128\zeta(3)\zeta(5)-128S_{6,2}(-1)\big) \tag{143}$$

$$\sum_{k=1}^{\infty}\frac{H(2k)}{(2k)^8(2k+1)^3} = \frac{-1}{3072}\big(27648\zeta(2)-69888\zeta(3)+55296\log(2)^2+67584\mathrm{Li}_4(1/2)$$
$$+59136\log(2)\zeta(3)-51648\zeta(4)+2816\log(2)^4-16896\log(2)^2\zeta(2)-26640\zeta(5)$$
$$+11520\zeta(2)\zeta(3)+12000\zeta(6)-7680\zeta(3)^2+15360S_{4,2}(-1)-9720\zeta(7)$$
$$+4608\zeta(2)\zeta(5)+1152\zeta(3)\zeta(4)+5796\zeta(8)-4608\zeta(3)\zeta(5)+4608S_{6,2}(-1)$$
$$-1563\zeta(9)+768\zeta(2)\zeta(7)+48\zeta(3)\zeta(6)+192\zeta(4)\zeta(5)\big) \tag{144}$$

$$\sum_{k=1}^{\infty}\frac{H(2k)}{(2k)^9(2k+1)^2} = \frac{-1}{3072}\big(-6912\zeta(2)+18240\zeta(3)-13824\log(2)^2-21504\mathrm{Li}_4(1/2)$$
$$-18816\log(2)\zeta(3)+16128\zeta(4)-896\log(2)^4+5376\log(2)^2\zeta(2)+10656\zeta(5)$$
$$-4608\zeta(2)\zeta(3)-6000\zeta(6)+3840\zeta(3)^2-7680S_{4,2}(-1)+6480\zeta(7)$$
$$-3072\zeta(2)\zeta(5)-768\zeta(3)\zeta(4)-5796\zeta(8)+4608\zeta(3)\zeta(5)-4608S_{6,2}(-1)$$
$$+3126\zeta(9)-1536\zeta(2)\zeta(7)-96\zeta(3)\zeta(6)-384\zeta(4)\zeta(5)-2691\zeta(10)$$
$$+1536\zeta(3)\zeta(7)+768\zeta(5)^2-1536S_{8,2}(-1)\big) \tag{145}$$

$$\sum_{k=1}^{\infty}\frac{H(2k)}{(2k)^{10}(2k+1)} = \frac{-1}{12288}\big(3072\zeta(2)-8448\zeta(3)+6144\log(2)^2+12288\mathrm{Li}_4(1/2)$$
$$+10752\log(2)\zeta(3)-9216\zeta(4)+512\log(2)^4-3072\log(2)^2\zeta(2)-7104\zeta(5)$$
$$+3072\zeta(2)\zeta(3)+4800\zeta(6)-3072\zeta(3)^2+6144S_{4,2}(-1)-6480\zeta(7)$$
$$+3072\zeta(2)\zeta(5)+768\zeta(3)\zeta(4)+7728\zeta(8)-6144\zeta(3)\zeta(5)+6144S_{6,2}(-1)$$
$$-6252\zeta(9)+3072\zeta(2)\zeta(7)+192\zeta(3)\zeta(6)+768\zeta(4)\zeta(5)+10764\zeta(10)$$
$$-6144\zeta(3)\zeta(7)-3072\zeta(5)^2+6144S_{8,2}(-1)-6177\zeta(11)+3072\zeta(2)\zeta(9)$$
$$+48\zeta(3)\zeta(8)+768\zeta(4)\zeta(7)+192\zeta(5)\zeta(6)\big) \tag{146}$$

$$\sum_{k=1}^{\infty}\frac{H(2k)}{(2k)^{11}} = \frac{1}{4096}\big(4609\zeta(12)-2048\zeta(3)\zeta(9)-2048\zeta(5)\zeta(7)+2048S_{10,2}($$
$$-1)\big) \tag{147}$$

16 Formulas of denominator order 12

$$\sum_{k=1}^{\infty}\frac{H(2k)}{(2k+1)^{12}} = \frac{1}{16384}\left(90101\zeta(13)-12288\zeta(2)\zeta(11)-16368\zeta(3)\zeta(10)\right.$$
$$\left.-15360\zeta(4)\zeta(9)-16320\zeta(5)\zeta(8)-16128\zeta(6)\zeta(7)\right) \quad (148)$$

$$\sum_{k=1}^{\infty}\frac{H(2k)}{(2k)(2k+1)^{11}} = \frac{-1}{12288}\left(-3072\zeta(2)+5376\zeta(3)-6144\log(2)^2-12288\mathrm{Li}_4(1/2)\right.$$
$$-10752\log(2)\zeta(3)+13056\zeta(4)-512\log(2)^4+3072\log(2)^2\zeta(2)+17856\zeta(5)$$
$$-9216\zeta(2)\zeta(3)+4608\zeta(6)-3072\zeta(3)^2-6144S_{4,2}(-1)+30480\zeta(7)$$
$$-9216\zeta(2)\zeta(5)-11520\zeta(3)\zeta(4)+7680\zeta(8)-6144\zeta(3)\zeta(5)-6144S_{6,2}(-1)$$
$$+42924\zeta(9)-9216\zeta(2)\zeta(7)-12096\zeta(3)\zeta(6)-11520\zeta(4)\zeta(5)+10752\zeta(10)$$
$$-6144\zeta(3)\zeta(7)-3072\zeta(5)^2-6144S_{8,2}(-1)+55269\zeta(11)-9216\zeta(2)\zeta(9)$$
$$-12240\zeta(3)\zeta(8)-11520\zeta(4)\zeta(7)-12096\zeta(5)\zeta(6)+13824\zeta(12)$$
$$\left.-6144\zeta(3)\zeta(9)-6144\zeta(5)\zeta(7)-6144S_{10,2}(-1)\right) \quad (149)$$

$$\sum_{k=1}^{\infty}\frac{H(2k)}{(2k)^2(2k+1)^{10}} = \frac{-1}{12288}\left(30720\zeta(2)-56832\zeta(3)+61440\log(2)^2\right.$$
$$+98304\mathrm{Li}_4(1/2)+86016\log(2)\zeta(3)-104448\zeta(4)+4096\log(2)^4-24576\log(2)^2\zeta(2)$$
$$-124992\zeta(5)+64512\zeta(2)\zeta(3)-27648\zeta(6)+18432\zeta(3)^2+36864S_{4,2}(-1)$$
$$-152400\zeta(7)+46080\zeta(2)\zeta(5)+57600\zeta(3)\zeta(4)-30720\zeta(8)+24576\zeta(3)\zeta(5)$$
$$+24576S_{6,2}(-1)-128772\zeta(9)+27648\zeta(2)\zeta(7)+36288\zeta(3)\zeta(6)+34560\zeta(4)\zeta(5)$$
$$-21504\zeta(10)+12288\zeta(3)\zeta(7)+6144\zeta(5)^2+12288S_{8,2}(-1)-55269\zeta(11)$$
$$\left.+9216\zeta(2)\zeta(9)+12240\zeta(3)\zeta(8)+11520\zeta(4)\zeta(7)+12096\zeta(5)\zeta(6)\right) \quad (150)$$

$$\sum_{k=1}^{\infty}\frac{H(2k)}{(2k)^3(2k+1)^{9}} = \frac{-1}{3072}\left(-34560\zeta(2)+67392\zeta(3)-69120\log(2)^2-89088\mathrm{Li}_4(1/2)\right.$$
$$-77952\log(2)\zeta(3)+93696\zeta(4)-3712\log(2)^4+22272\log(2)^2\zeta(2)+93744\zeta(5)$$
$$-48384\zeta(2)\zeta(3)+17280\zeta(6)-11520\zeta(3)^2-23040S_{4,2}(-1)+76200\zeta(7)$$
$$-23040\zeta(2)\zeta(5)-28800\zeta(3)\zeta(4)+11520\zeta(8)-9216\zeta(3)\zeta(5)-9216S_{6,2}(-1)$$
$$+32193\zeta(9)-6912\zeta(2)\zeta(7)-9072\zeta(3)\zeta(6)-8640\zeta(4)\zeta(5)+2688\zeta(10)$$
$$\left.-1536\zeta(3)\zeta(7)-768\zeta(5)^2-1536S_{8,2}(-1)\right) \quad (151)$$

$$\sum_{k=1}^{\infty}\frac{H(2k)}{(2k)^4(2k+1)^{8}} = \frac{-1}{3072}\left(92160\zeta(2)-188928\zeta(3)+184320\log(2)^2\right.$$
$$+196608\mathrm{Li}_4(1/2)+172032\log(2)\zeta(3)-201216\zeta(4)+8192\log(2)^4-49152\log(2)^2\zeta(2)$$
$$-158016\zeta(5)+81408\zeta(2)\zeta(3)-23040\zeta(6)+15360\zeta(3)^2+30720S_{4,2}(-1)$$
$$-76200\zeta(7)+23040\zeta(2)\zeta(5)+28800\zeta(3)\zeta(4)-7680\zeta(8)+6144\zeta(3)\zeta(5)$$
$$+6144S_{6,2}(-1)-10731\zeta(9)+2304\zeta(2)\zeta(7)+3024\zeta(3)\zeta(6)$$
$$\left.+2880\zeta(4)\zeta(5)\right) \quad (152)$$

$$\sum_{k=1}^{\infty}\frac{H(2k)}{(2k)^5(2k+1)^7} = \frac{-1}{768}\big(-40320\zeta(2)+86688\zeta(3)-80640\log(2)^2-75264\mathrm{Li}_4(1/2)$$
$$-65856\log(2)\zeta(3)+73248\zeta(4)-3136\log(2)^4+18816\log(2)^2\zeta(2)+42168\zeta(5)$$
$$-21504\zeta(2)\zeta(3)+4020\zeta(6)-2688\zeta(3)^2-6144S_{4,2}(-1)+9525\zeta(7)$$
$$-2880\zeta(2)\zeta(5)-3600\zeta(3)\zeta(4)+480\zeta(8)-384\zeta(3)\zeta(5)-384S_{6,2}(-1)\big) \tag{153}$$

$$\sum_{k=1}^{\infty}\frac{H(2k)}{(2k)^6(2k+1)^6} = \frac{-1}{384}\big(24192\zeta(2)-54432\zeta(3)+48384\log(2)^2+43008\mathrm{Li}_4(1/2)$$
$$+37632\log(2)\zeta(3)-38976\zeta(4)+1792\log(2)^4-10752\log(2)^2\zeta(2)-16380\zeta(5)$$
$$+8064\zeta(2)\zeta(3)+36\zeta(6)+2304S_{4,2}(-1)-1155\zeta(7)+384\zeta(2)\zeta(5)$$
$$+384\zeta(3)\zeta(4)\big) \tag{154}$$

$$\sum_{k=1}^{\infty}\frac{H(2k)}{(2k)^7(2k+1)^5} = \frac{-1}{768}\big(-40320\zeta(2)+94752\zeta(3)-80640\log(2)^2-75264\mathrm{Li}_4(1/2)$$
$$-65856\log(2)\zeta(3)+63168\zeta(4)-3136\log(2)^4+18816\log(2)^2\zeta(2)+23352\zeta(5)$$
$$-10752\zeta(2)\zeta(3)-4212\zeta(6)+2688\zeta(3)^2-6144S_{4,2}(-1)+2025\zeta(7)$$
$$-960\zeta(2)\zeta(5)-240\zeta(3)\zeta(4)-483\zeta(8)+384\zeta(3)\zeta(5)-384S_{6,2}(-1)\big) \tag{155}$$

$$\sum_{k=1}^{\infty}\frac{H(2k)}{(2k)^8(2k+1)^4} = \frac{-1}{3072}\big(92160\zeta(2)-225792\zeta(3)+184320\log(2)^2$$
$$+196608\mathrm{Li}_4(1/2)+172032\log(2)\zeta(3)-155136\zeta(4)+8192\log(2)^4-49152\log(2)^2\zeta(2)$$
$$-66624\zeta(5)+29184\zeta(2)\zeta(3)+24000\zeta(6)-15360\zeta(3)^2+30720S_{4,2}(-1)-16200\zeta(7)$$
$$+7680\zeta(2)\zeta(5)+1920\zeta(3)\zeta(4)+7728\zeta(8)-6144\zeta(3)\zeta(5)+6144S_{6,2}(-1)$$
$$-1563\zeta(9)+768\zeta(2)\zeta(7)+48\zeta(3)\zeta(6)+192\zeta(4)\zeta(5)\big) \tag{156}$$

$$\sum_{k=1}^{\infty}\frac{H(2k)}{(2k)^9(2k+1)^3} = \frac{1}{3072}\big(34560\zeta(2)-88128\zeta(3)+69120\log(2)^2+89088\mathrm{Li}_4(1/2)$$
$$+77952\log(2)\zeta(3)-67776\zeta(4)+3712\log(2)^4-22272\log(2)^2\zeta(2)-37296\zeta(5)$$
$$+16128\zeta(2)\zeta(3)+18000\zeta(6)-11520\zeta(3)^2+23040S_{4,2}(-1)-16200\zeta(7)$$
$$+7680\zeta(2)\zeta(5)+1920\zeta(3)\zeta(4)+11592\zeta(8)-9216\zeta(3)\zeta(5)+9216S_{6,2}(-1)$$
$$-4689\zeta(9)+2304\zeta(2)\zeta(7)+144\zeta(3)\zeta(6)+576\zeta(4)\zeta(5)+2691\zeta(10)$$
$$-1536\zeta(3)\zeta(7)-768\zeta(5)^2+1536S_{8,2}(-1)\big) \tag{157}$$

$$\sum_{k=1}^{\infty}\frac{H(2k)}{(2k)^{10}(2k+1)^2} = \frac{-1}{12288}\big(30720\zeta(2)-81408\zeta(3)+61440\log(2)^2$$
$$+98304\mathrm{Li}_4(1/2)+86016\log(2)\zeta(3)-73728\zeta(4)+4096\log(2)^4-24576\log(2)^2\zeta(2)$$
$$-49728\zeta(5)+21504\zeta(2)\zeta(3)+28800\zeta(6)-18432\zeta(3)^2+36864S_{4,2}(-1)-32400\zeta(7)$$
$$+15360\zeta(2)\zeta(5)+3840\zeta(3)\zeta(4)+30912\zeta(8)-24576\zeta(3)\zeta(5)+24576S_{6,2}(-1)$$
$$-18756\zeta(9)+9216\zeta(2)\zeta(7)+576\zeta(3)\zeta(6)+2304\zeta(4)\zeta(5)+21528\zeta(10)$$
$$-12288\zeta(3)\zeta(7)-6144\zeta(5)^2+12288S_{8,2}(-1)-6177\zeta(11)+3072\zeta(2)\zeta(9)$$
$$+48\zeta(3)\zeta(8)+768\zeta(4)\zeta(7)+192\zeta(5)\zeta(6)\big) \tag{158}$$

$$\sum_{k=1}^{\infty} \frac{H(2k)}{(2k)^{11}(2k+1)} = \frac{-1}{12288} \big(-3072\zeta(2) + 8448\zeta(3) - 6144\log(2)^2 - 12288\mathrm{Li}_4(1/2)$$
$$-10752\log(2)\zeta(3) + 9216\zeta(4) - 512\log(2)^4 + 3072\log(2)^2\zeta(2) + 7104\zeta(5)$$
$$-3072\zeta(2)\zeta(3) - 4800\zeta(6) + 3072\zeta(3)^2 - 6144S_{4,2}(-1) + 6480\zeta(7)$$
$$-3072\zeta(2)\zeta(5) - 768\zeta(3)\zeta(4) - 7728\zeta(8) + 6144\zeta(3)\zeta(5) - 6144S_{6,2}(-1)$$
$$+6252\zeta(9) - 3072\zeta(2)\zeta(7) - 192\zeta(3)\zeta(6) - 768\zeta(4)\zeta(5) - 10764\zeta(10)$$
$$+6144\zeta(3)\zeta(7) + 3072\zeta(5)^2 - 6144S_{8,2}(-1) + 6177\zeta(11) - 3072\zeta(2)\zeta(9)$$
$$-48\zeta(3)\zeta(8) - 768\zeta(4)\zeta(7) - 192\zeta(5)\zeta(6) - 13827\zeta(12) + 6144\zeta(3)\zeta(9)$$
$$+6144\zeta(5)\zeta(7) - 6144S_{10,2}(-1)\big) \tag{159}$$

$$\sum_{k=1}^{\infty} \frac{H(2k)}{(2k)^{12}} = \frac{1}{16384} (8205\zeta(13) - 4096\zeta(2)\zeta(11) - 16\zeta(3)\zeta(10)$$
$$-1024\zeta(4)\zeta(9) - 64\zeta(5)\zeta(8) - 256\zeta(6)\zeta(7)) \tag{160}$$

17 Difference formulae

As a postscript, we list these relations for any $F(k)$ for which the sums below converge:

$$\sum_{k=1}^{\infty} F(k)\left(\frac{1}{(2k)} - \frac{1}{(2k+1)}\right) = \sum_{k=1}^{\infty} \frac{F(k)}{(2k)(2k+1)} \tag{161}$$

$$\sum_{k=1}^{\infty} F(k)\left(\frac{1}{(2k)^2} - \frac{1}{(2k+1)^2}\right) = \sum_{k=1}^{\infty} F(k)\left(\frac{1}{(2k)(2k+1)^2} + \frac{1}{(2k)^2(2k+1)}\right) \tag{162}$$

$$\sum_{k=1}^{\infty} F(k)\left(\frac{1}{(2k)^3} - \frac{1}{(2k+1)^3}\right) = \sum_{k=1}^{\infty} F(k)\left(\frac{3}{(2k)^2(2k+1)^2} + \frac{1}{(2k)^3(2k+1)^3}\right) \tag{163}$$

$$\sum_{k=1}^{\infty} F(k)\left(\frac{1}{(2k)^4} - \frac{1}{(2k+1)^4}\right) = \sum_{k=1}^{\infty} F(k)\left(\frac{4}{(2k)^2(2k+1)^3} + \frac{2}{(2k)^3(2k+1)^3}\right.$$
$$\left. + \frac{1}{(2k)^3(2k+1)^4} + \frac{1}{(2k)^4(2k+1)^3}\right) \tag{164}$$

$$\sum_{k=1}^{\infty} F(k)\left(\frac{1}{(2k)^5} - \frac{1}{(2k+1)^5}\right) = \sum_{k=1}^{\infty} F(k)\left(\frac{5}{(2k)^3(2k+1)^3} + \frac{5}{(2k)^4(2k+1)^4}\right.$$
$$\left. + \frac{1}{(2k)^5(2k+1)^5}\right) \tag{165}$$

$$\sum_{k=1}^{\infty} F(k)\left(\frac{1}{(2k)^6} - \frac{1}{(2k+1)^6}\right) = \sum_{k=1}^{\infty} F(k)\left(\frac{6}{(2k)(2k+1)^6} + \frac{15}{(2k)^2(2k+1)^6}\right.$$
$$\left. + \frac{20}{(2k)^3(2k+1)^6} + \frac{15}{(2k)^4(2k+1)^6} + \frac{6}{(2k)^5(2k+1)^6} + \frac{1}{(2k)^6(2k+1)^6}\right) \tag{166}$$

$$\sum_{k=1}^{\infty} F(k)\left(\frac{1}{(2k)^7}-\frac{1}{(2k+1)^7}\right) = \sum_{k=1}^{\infty} F(k)\left(\frac{7}{(2k)^4(2k+1)^4}+\frac{14}{(2k)^5(2k+1)^5}+\frac{7}{(2k)^6(2k+1)^6}\right.$$
$$\left.+\frac{1}{(2k)^7(2k+1)^7}\right) \tag{167}$$