Ancient Indian Square Roots

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Introduction

Our modern system of positional decimal notation with zero, together with efficient algorithms for computation, which were discovered in India some time prior to 500 CE, certainly must rank among the most significant achievements of all time. And it was not easy. As Pierre-Simon Laplace (1923) explained, "the difficulty of inventing it will be better appreciated if we consider that it escaped the genius of Archimedes and Apollonius, two of the greatest men of antiquity." (p. 222–223).

The Mayans came close, with a system that featured positional notation with zero. However, in their system successive positions represented the mixed sequence $(1, 20, 360, 7200, 144000, \cdots)$, rather than the purely base-20 sequence $(1, 20, 400, 8000, 160000, \cdots)$, which precluded any possibility that their numerals could be used as part of a highly efficient arithmetic system (Ifrah 2000, p. 311).

What's more, mathematicians in ancient India developed remarkably advanced schemes, at a very early era, for computing square roots. This article summarizes these schemes. It is based on an earlier study by the present authors, to which readers are referred for additional details (2012).

The discovery of positional arithmetic

The original discovery of positional decimal arithmetic is, sadly, unknown. The earliest known physical evidence, using single-character Brahmi numerals (which are the ancestors of our modern digits), is an inscription of the date 346 on a copper plate, which corresponds to 595 CE (Chrisomalis 2010, p. 196). But there are numerous passages of more ancient texts that suggest that both the concept and the practice of positional decimal numeration was known much earlier (Plofker 2009, p. 122).

For example, a fifth century text includes the passage "Just as a line in the hundreds place [means] a hundred, in the tens place ten, and one in the ones place, so one and the same woman is called mother, daughter, and sister [by different people]" (Plofker 2009, p. 46). Similarly, in 499 CE the Indian mathematician Ārayabhaṭa wrote, "The numbers one, ten, hundred, thousand, ten thousand, hundred thousand, million, ten million, hundred million, and billion are from place to place each ten times the preceding" (Clark 1930, p. 21).

These early texts did not use Brahmi numerals, but instead used the Sanskrit words for the digits one through nine and zero, or, when needed to match the meter of the verse, used one of a set of literary words (known as "word-symbols") associated with digits. For example, the medieval Indian manuscript *Sūryasiddhānta* included the verse, "The apsids of the moon in a cosmic cycle are: fire; vacuum; horsemen; vast; serpent; ocean." Here the last six words are word-symbols for 3, 0, 2, 8, 8, 4, respectively (meaning the decimal number 488,203, since the order is reversed) (Ifrah 2000, p. 411).

The most ancient Indian documents are more recent copies, so that we cannot be absolutely certain of their ancient authenticity. But one manuscript whose ancient authenticity cannot be denied is the *Lokavibhāga* ("Parts of the Universe"). This has numerous large numbers in positional decimal notation (using Sanskrit names or word-symbols for the digits) and detailed calculations (Siddhanta-Shastri 1926, p. 70, 79, 131). Near the end of the *Lokavibhāga*, the author provides some astronomical observations that enable modern scholars to determine, in two independent ways, that this text was written on 25 August 458 CE (Julian calendar). The text also mentions that it was written in the 22nd year of the reign of Simhavarman, which also confirms the 458 CE date (Ifrah 2000, p. 417).

One even earlier source of positional word-symbols is the mid-third-century CE text *Yavanajātaka*, whose final verse reads, "There was a wise king named Sphujidhvaja who made this [work] with four thousand [verses] in the Indravajra meter, appearing in the year Visnu; hook-sign; moon." The three word-symbols, "Visnu," "hook-sign" and "moon," mean 1, 9 and 1, signifying year 191 of the Saka era, which corresponds to 270 CE (Plofker 2009, p. 47).

The earliest record of zero may be in the *Chandahsūtra*, dated to the second or third century BCE. Here we see the solution to a mathematical problem relating to the set of all possible meters for multi-syllable verse, which involves the expression of integers using a form of binary notation (Plofker 2009, p. 55). The very earliest origin of the notion of positional decimal notation and arithmetic, however, is still obscure; it may be connected to the ancient Chinese "rod calculus" (Plofker 2009, p. 48).

Ārayabhaṭa's square root and cube root

One person who deserves at least some credit for the proliferation of decimal arithmetic calculation is the Indian mathematician Ārayabhaṭa, mentioned above (see Figure 1). He devised ingenious digit-by-digit algorithms for computing square roots and cube roots, as given (tersely) in his 499 CE work *Āryabhaṭīya* (Clark 1930, p. 24–26). These schemes were used, with only minor variations, by Indian mathematicians such as Siddhasena Gani (∼550 CE), Bhāskara I (∼600 CE), Śrīdhara (∼750 CE) and Bhāskara II (∼1150 CE), as well as by numerous later Arabic and European mathematicians (Datta and Singh 1962, I, p. 170–175).

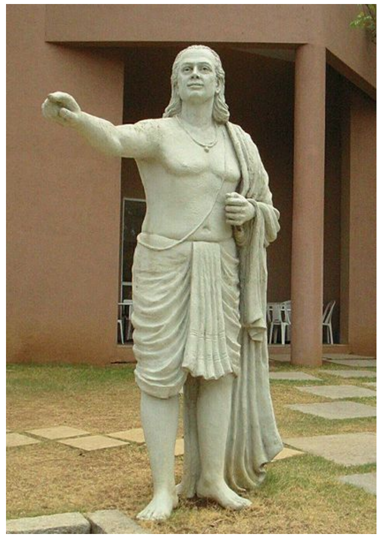

Figure 1: Statue of Ārayabhaṭa on the grounds of IUCAA, Pune, India (no one knows what Ārayabhaṭa actually looked like) [courtesy Wikimedia]

The Bakhshālī manuscript

Another ancient source that clearly exhibits considerable familiarity with decimal arithmetic in general and square roots in particular is the Bakhshālī manuscript. This ancient mathematical treatise was found in 1881 in the village of Bakhshālī, approximately 80 kilometers northeast of Peshawar. Among the topics covered in this document, at least in the fragments that have been recovered, are solutions of systems of linear equations, indeterminate (Diophantine) equations of the second degree, arithmetic progressions of various types, and rational approximations of square roots. The manuscript appears to be a commentary on an even earlier work (Hayashi 1995, p. 86, 148).

Ever since its discovery in 1881, scholars have debated its age. Some, like British scholar G. R. Kaye, assigned the manuscript to the 12th century, in part because he believed that its mathematical content was derivative from Greek sources. Others, such as Rudolf Hoernle, assigned the underlying manuscript to the "3rd or 4th century CE" (Hoernle 1887, p. 9). In the most recent analysis, Takao Hayashi assigned the commentary to the seventh century, with the underlying original not much older (1995, p. 149).

The Bakhshālī square root

One particularly intriguing item in the Bakhshālī manuscript is the following algorithm for computing square roots:

> [1:] *In the case of a non-square [number], subtract the nearest square*

number; divide the remainder by twice [the root of that number]. [2:] *Half the square of that [that is, the fraction just obtained] is divided by the sum of the root and the fraction and subtract [from the sum].* [3:] *[The non-square number is] less [than the square of the approximation] by the square [of the last term].* (Translation is due to Datta (1929), except last sentence is due to Hayashi (1995, p. 431).)

The Bakhshālī square root in modern notation

In modern notation, this algorithm is as follows. To obtain the square root of a number q, start with an approximation x_0 and then calculate, for $n \geq 0$,

$$\begin{aligned} a_n &= \frac{q - x_n^2}{2x_n} && \text{(sentence \#1 above)} \\ x_{n+1} &= x_n + a_n - \frac{a_n^2}{2\,(x_n + a_n)} && \text{(sentence \#2 above)} \\ q &= x_{n+1}^2 - \left[\frac{a_n^2}{2\,(x_n + a_n)}\right]^2. && \text{(sentence \#3) above)} \end{aligned}$$

The last line is merely a check; it is not an essential part of the calculation. In the examples presented in the Bakhshālī manuscript, this algorithm is used to obtain rational approximations to square roots only for integer arguments q, only for integer-valued starting values x_0, and is only applied once in each case (i.e., it is not iterated). But from a modern perspective, the scheme clearly can be repeated, and in fact converges very rapidly to $\sqrt{q}$, as we shall see in the next section.

Here is one application in the Bakhshālī manuscript (Hayashi 1995, p. 232–233).

Problem 1 *Find an accurate rational approximation to the solution of*

$$3x^2/4 + 3x/4 = 7000 \tag{1}$$

(which arises from the manuscript's analysis of some additive series).

Answer: $x = (\sqrt{336009} - 3)/6$. To calculate an accurate value for $\sqrt{336009}$, start with the approximation $x_0 = 579$. Note that $q = 336009 = 579^2 + 768$. Then calculate as follows (using modern notation):

$$\begin{aligned} a_0 &= \frac{q - x_0^2}{2x_0} = \frac{768}{1158}, \qquad x_0 + a_0 = 579 + \frac{768}{1158}, \\ \frac{a_0^2}{2(x_0 + a_0)} &= \frac{294912}{777307500}. \end{aligned} \tag{2}$$

Thus we obtain the refined root

$$x_1 = x_0 + a_0 - \frac{a_0^2}{2\,(x_0 + a_0)} = 579 + \frac{515225088}{777307500} = \frac{450576267588}{777307500} \tag{3}$$

(note: This is 579.66283303325903841..., which agrees with $\sqrt{336009} = 579.66283303313487498\ldots$ to 12-significant-digit accuracy).

The manuscript then performs a calculation to check that the original quadratic equation is satisfied. It obtains, for the left-hand side of (1),

$$\frac{5075338376274674327193 6}{7250483394675000000}, \tag{4}$$

which, after subtracting the correction

$$\frac{21743271936}{7250483394675000000}, \tag{5}$$

gives,

$$\frac{50753383762725000000000}{7250483394675000000} = 7000. \tag{6}$$

Each of the integers and fractions shown in the above calculation (except the denominator of (5), which is implied) actually appears in the Bakhshālī manuscript, although some of the individual digits are missing at the edges — see Figure 2. The digits are written left-to-right, and fractions are written as one integer directly over another (although there is no division bar). Zeroes are denoted by large dots. Other digits may be recognized by those familiar with ancient Indian languages. □

Convergence of the Bakhshālī square root

Note, in the above example, that starting with the 3-digit approximation 579, one obtains, after a single application of the algorithm, a value for $\sqrt{336009}$ that is correct to 12 significant digits. From a modern perspective, this happens because the Bakhshālī square root algorithm is *quartically convergent* — each iteration approximately quadruples the number of correct digits in the result, provided that sufficiently accurate arithmetic is used (although there is no indication of the algorithm being iterated more than once in the manuscript itself) (Bailey and Borwein, 2012).

An even more ancient square root

There are instances of highly accurate square roots in Indian sources that are even more ancient than the Bakhshālī manuscript. For example, Srinivasiengar noted that the ancient Jain work *Jambūdvīpa-prajñapti* (∼300 BCE), after erroneously assuming that $\pi = \sqrt{10}$, asserts that the "circumference" of a circle of diameter 100,000 *yōjana* is 316227 *yōjana* + 3 *gavyūti* + 128 *dhanu* + $13\,^1/_2$ angula, "and a little over" (1967, p. 21–22). Datta added that this statement is also seen in the *Jībāhigama-sūtra* (∼200 BCE) (1929, p. 43), while Joseph noted that it seen in the *Anuyoga-dvāra-sūtra* (∼0 CE) and the *Triloko-sara* (∼0 CE) (2010, p. 356).

Figure 2: Fragment of Bakhshālī manuscript with a portion of the square root calculation mentioned in Problem 1. For example, the large right-middle section corresponds to the fraction $\frac{50753383762746743271936}{7250483394675000000}$ in Formula (4). Graphic from (Hayashi 1995, p. 574).

According to one commonly used ancient convention these units are: 1 *yōjana* = 14 kilometers (approximately); 4 *gavyūti* = 1 *yōjana*; 2000 *dhanu* = 1 *gavyūti*; and 96 *angula* = 1 *dhanu* (Joseph 2010, p. 356). Converting these units to *yōjana*, we conclude that the "circumference" is $316227.766017578125\ldots$ *yōjana*. This agrees with $100000\sqrt{10} = 316227.766016837933\ldots$ to 12-significant-digit accuracy!

What algorithm did these ancient scholars employ to compute square roots? The present authors conclude, based on a detailed analysis, that the most reasonable conclusion is that the Indian mathematician(s) who published the above did some preliminary computation to obtain the approximation 316227, then used one Heron iteration (i.e., $x_{n+1} = (x_n + q/x_n)/2$), which was known in ancient times in Greece and elsewhere) to compute an approximate fractional value, and then converted the final result to the length units above (2012). Evidently the Bakhshālī formula had not yet been developed.

Note that just to perform one Heron iteration, with starting value 316227, one would need to perform at least the following rather demanding calculation:

$$\begin{aligned} x_1 &= \frac{1}{2}\left(x_0 + \frac{q}{x_0}\right) = \frac{1}{2}\left(316227 + \frac{100000000000}{316227}\right) \\ &= \frac{1}{2}\left(\frac{316227^2 + 100000000000}{316227}\right) = \frac{99999515529 + 100000000000}{2 \cdot 316227} \\ &= \frac{199999515529}{632454} = 316227 + \frac{484471}{632454}, \end{aligned} \tag{7}$$

followed by several additional steps to convert the result to the given units. By any reasonable standard, this is a rather impressive computation for such an ancient vintage (200-300 BCE). Numerous other examples of prodigious computations in various ancient Indian sources are mentioned by Datta and Singh (1962), Joseph (2010), Plofker (2009) and Srinivasiengar (1967). Although some impressive calculations are also seen in ancient Mesopotamia, Greece and China, as far as we are aware there are more of these prodigious calculations in ancient Indian literature than in other ancient sources.

In any event, it is clear that ancient Indian mathematicians, roughly contemporaneous with Greeks such as Euclid and Archimedes, had command of a rather powerful system of arithmetic, possibly some variation of the Chinese "rod calculus," or perhaps even some primitive version of decimal arithmetic.

Controversies

In spite of these discoveries, we should caution that there is a tendency amongst some ethnomathematicians to optimistically ascribe independent or prior discovery to various Indian sources, such as the Vedas (a collection pre-Christian-era texts) and the Kerala school (a group of mathematicians writing from the 14th to 16th centuries). For example, while Kerala mathematicians appear to have found the Gregory series for the arctangent and computed π to 12-digit accuracy, nonetheless they did not formulate any systematic theory of calculus, nor is there any evidence that they transmitted their findings outside the school (Plofker 2009, p. 253).

Conclusion

The discovery of positional decimal arithmetic with zero, together with efficient algorithms for computation, by unknown Indian mathematicians, certainly by 500 CE and probably several centuries earlier, is a mathematical development of the first magnitude. And the schemes they developed for computing square roots are also quite remarkable for this era.

It should be noted that these ancient Indian mathematicians missed some key points. For one thing, the notion of decimal fraction notation eluded them and everyone else until the tenth century, when a rudimentary form was seen in the writings of the Arabic mathematician al-Uqlidisi, and in the twelfth century, when al-Samaw'al illustrated its use in division and root extraction (Joseph

2010, p. 468). Also, as mentioned above, there is no indication that Indian mathematicians iterated algorithms for finding roots.

Aside from historical interest, does any of this matter? As historian Kim Plofker notes, in ancient Indian mathematics, "True perception, reasoning, and authority were expected to harmonize with one another, and each had a part in supporting the truth of mathematics." (Plofker 2009, p. 12). As she neatly puts it, mathematics was not "an epistemologically privileged subject." Similarly, mathematical historian George G. Joseph writes, "A Eurocentric approach to the history of mathematics is intimately connected with the dominant view of mathematics ... as a deductive system." In contrast, as Joseph continues, "[s]ome of the most impressive work in Indian and Chinese mathematics ... involve computations and visual demonstrations that were not formulated with reference to any formal deductive system." (Joseph 2010, p. xiii).

In short, the Greek heritage that underlies much of Western mathematics may have unduly predisposed many of us against experimental approaches that are now facilitated by the availability of powerful computer technology. Thus a renewed exposure to non-Western traditions may lead to new insights and results, and may clarify the age-old issue of the relationship between mathematics as a language of science and technology, and mathematics as a supreme human intellectual discipline.

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