Jonathan Borwein: Mathematician Extraordinaire

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1 Introduction

Many of us were shocked when our dear colleague Jonathan M. Borwein of the University of Newcastle, Australia, died in August 2016. After his passing, one immediate priority was to gather together as many of his works as possible. Accordingly, David H. Bailey and Nelson H. F. Beebe of the University of Utah began collecting as many of Borwein's published papers, books, reports and talks as possible, together with book reviews and articles written by others about Jon and his work. The current catalog [7] lists nearly 2000 items. Even if one focuses only on formal, published, peer-reviewed articles, there are over 500 such items. These works are heavily cited—the Google citation tracker finds over 22,000 citations.

What is most striking about this catalogue is the breadth of topics. One bane of modern academic research in general, and of the field of mathematics in particular, is that most researchers today focus on a single specialized niche, seldom attempting to branch out into other specialties and disciplines or to forge potentially fruitful collaborations with researchers in other fields. In contrast, Borwein not only learned about numerous different specialities, but in fact did significant research in a wide range of fields, including experimental mathematics, optimization, convex analysis, applied mathematics, computer science, scientific visualization, biomedical imaging and mathematical finance.

It is hard to think of a single mathematician of the modern era who has published notable research in so many different arenas.

2 A Portrait of the Man as a Mathematician

Jonathan was a polymath, by its very definition, and a true renaissance scholar. His knowledge-base was as expansive as it was detailed. Let us take a moment to paint a fuller picture of this man and his engagement with mathematics, technology and the world around him. Born in St Andrews, Scotland on 20 May 1951, Jon attended the Madras College in Fife as a child, and went to university at 15, becoming a rather young professor at 23.

Always steeped in the mathematical world, thanks to his father, David Borwein, Jon started doing AMS Math Monthly problems with his father from an early age. The only time Jon didn't have a math book under his arm, a pen leaking ink in his pocket desperate to be etched across a sheet of crisp white paper—and later an iPad to work on—was the three months he travelled along the route of Xenophon's *Anabasis* through Turkey and Greece in the summer of 1973.

Jon won a 1971 Rhodes Scholarship and settled into life at Oxford. His classes allowed him to rub shoulders with the likes of Michael Atiyah, Professor of Geometry, who actually attended a class on mathematical Linguistics with Jon. Atiyah often sat unheeding and would then ask the professor dumb basic questions about the lecture. At a much later time at a conference Atiyah laughed and said that Jon had certainly gotten more from the class than he did. Such interactions reinforced the humanity of the great professors and in many ways became a template for his interactions with students, accepting of differences and ready to teach all.

Jon was a member of "Professors for Peace for the Middle East," a Canadian organization with very dedicated people who went on to be influential in Canadian society. In 1967 a group went to Israel to discuss the problems with both Palestinians and Israelis, from the Mufti of Ghaza to Teddy Kollek, the Mayor of Jerusalem. They arrived in May, days after the election that heralded the start of the tenure of Menachem Begin. The authorities they had arranged to meet were so rattled by the result of the election that they actually reported the truth about many things 'Middle East', while filling boxes to depart their offices. In some cases the group was able to access the newcomers. This organization still exists despite a name change and continues to press for a peaceful resolution.

In 1985 while on sabbatical, Jon went to Cambridge, England and then Limoges, France. While in Cambridge Jon received a redirected Christmas card from Yasumasa Kanada, a professor in the Department of Information Science at the University of Tokyo in Tokyo, Japan. He looked at the return address and was astonished to see that Kanada was in Cambridge as well. There was a meeting and this led to fruitful discussions about computer modeling algorithms and π. This friendship lasted until Kanada's retirement in 2015. When asked why a computer scientist would come to such a place as Cambridge, Kanada replied that he was looking to imbibe the theoretical underpinnings and classical view of his work. This was something he could not get at home. Next in the sabbatical was a stay in France. Jon and his wife, Judi, spent an idyllic time in a gîte rural on the grounds of a Chateau near Rilhac Rancon. This was a perfect place to ruminate on the mysteries of math. The time spent here lead to a very prolific association with Michel Thera and the French mathematics community, leading eventually to an honorary Doctorate at the University of Limoges.

At this same time Jon was reunited with Ephraim J. Borowski, an old friend from Oxford, who joined Jon for a while at the gîte while working on a Collins Dictionary. The two sat on the grass outside, or around the table in the 15th century stone cottage with file cards, filling in definitions. This delightful interlude led to the *Collins Dictionary of Mathematics*. Later, Ephraim, burdened down with file cards, came to work with Jon in Halifax, Nova Scotia. They set up shop in Jon's office, situated in the old Halifax Archives, and had a very strict protocol: nothing was to be changed on the Lisa computer unless it was first listed and dated on the white boards. They never misplaced a letter because of this, but even with five levels of certainty, they almost lost an entire section. Jon had a lifelong interest in utilizing technology and pushing its limits in the pursuit of a plethora of academic interests.

Biographies of Jon, some of which are also obituaries, often glom to certain details about his life, influence, and output. Citing his own writing about "The Best Teacher I Ever Had was ...", these include his childhood experience of teaching another boy a two by two simultaneous equation at the age of six, and how on arrival at Western University, in his second year, he very nearly decided to major in history—which would have been a loss to the mathematical community. These biographies often unevenly focus on the breadth of his contributions, containing them within the lens of the journal or researcher-author penning the biography. It is pertinent to state that he was so accomplished and had expertise in so many fields, that the editors at Springer could not find an appropriate replacement for him as Editor of the SUMAT Series. (Indeed, he was a founding co-editor in chief of Springer-Verlag's SUMAT Series of Springer Undergraduate Mathematics and Technology books.) Apparently, it would have required four editors to cover the fields he knew so intimately.

3 Overview of this Proceedings

It is the intention of this Springer Proceedings to commemorate the breadth of his life and work as explored at the Jonathan M. Borwein Commemorative Conference held on 25–29 September, 2017, at Noah's On The Beach, Newcastle, NSW—one of Jonathan's favourite spots. Associated events included the Sunday, 24th September Satellite meeting on Mathematics and Education; the Tuesday, 26th September Public Lecture given by Keith Devlin, entitled "Finding Fibonacci—The Quest to Rediscover the Forgotten Mathematical Genius Who Changed the World"; and, the Wednesday, 27th September "An Evening of Mathematics, Music and Art" held at the Harold Lobb Concert Hall of the Conservatorium of Music. The conference was devoted to five main areas in which Jonathan made outstanding contributions; these became leading session themes:

1. *Applied Analysis, Optimisation and Convex Functions*, chaired by Regina Burachik and Guoyin Li;
2. *Education*, chaired by Naomi Simone Borwein and Judy-anne Osborn;
3. *Experimental Mathematics and Visualisation*, chaired by David H. Bailey;

4. *Financial Mathematics*, chaired by Qiji (Jim) Zhu; and,
5. *Number Theory, Special Functions and Pi*, chaired by Richard Brent.

The following sections in this preface delve into some of Jon's contributions to these research areas.

4 Optimization

Some of Jon's most significant contributions were in the area of optimization; indeed, papers in the area of optimization and convex analysis are the single most numerous category in the catalog [7].

One notable paper in the optimization arena is [6], which presents what is now known as the Barzilai-Borwein algorithm for large-scale unconstrained optimization. This paper has been cited over 1300 times. There are numerous techniques for this type of problem (unconstrained optimization) in the literature. The standard gradient method, namely to iterate $x_{k+1} = x_k - \alpha_k g_k(x_k)$, where α_k is typically calculated based on a fixed line search procedure, is fairly simple to use, but it makes no use of second order information and sometimes zig-zags rather than converges. Newton's method is to iterate $x_{k+1} = x_k - (F_k(x_k))^{-1} g_k(x_k)$, where $F_k = \nabla^2 f(x_k)$ is the Hessian of the system. It utilizes second-order information and typically converges quite rapidly near the solution, but it requires the expensive computation of the matrix $(F_k(x_k))^{-1}$, and for some applications the scheme requires additional custom modifications to ensure convergence.

The Barzilai-Borwein method mimics the gradient method, in that it selects α_k so that $\alpha_k g_k(x_k)$ approximates $(F_k(x_k))^{-1} g_k(x_k)$, but it does not require that one actually compute $(F_k(x_k))^{-1}$. As a result, this scheme often converges nearly as fast as the Newton method, but at significantly lower computational cost. Due to its simplicity and efficiency, variations of this method have been applied in a variety of applications, including sparse optimization, image analysis and signal processing.

5 Experimental Mathematics

Jon is perhaps best known for deriving, with his brother Peter, quadratically and higher order convergent algorithms for π, including p-th order convergent algorithms for any prime p, and similar quadratically convergent algorithms for certain other fundamental constants and functions [8–10]. Here "quadratically convergent" means that each iteration of the algorithm approximately *doubles* the number of correct digits in the result, with a similar definition for higher-order convergence; similarly, p-th order convergent means that the number of correct digits increases approximately by a factor of p with each iteration.

One of their best-known algorithms is the following: Set $a_0 = 6 - 4\sqrt{2}$ and $y_0 = \sqrt{2} - 1$. Then iterate

$$\begin{aligned} y_{k+1} &= \frac{1-(1-y_k^4)^{1/4}}{1+(1-y_k^4)^{1/4}} \\ a_{k+1} &= a_k(1+y_{k+1})^4 - 2^{2k+3} y_{k+1}(1+y_{k+1}+y_{k+1}^2). \end{aligned}$$

Then a_k converge *quartically* to $1/\pi$: each iteration approximately *quadruples* the number of correct digits. This algorithm, together with a quadratically convergent algorithm due to Brent and Salamin, were employed in several large computations of π by Kanada and others.

But an event of more enduring legacy is his advocacy of *experimental mathematics*, in particular his championing of the usage of advanced computing technology to discover new principles and formulas of mathematics, not just verify them with mathematical software.

One of many examples of this methodology in action was his analysis (in conjunction with David H. Bailey and the late Richard Crandall) of the following three classes of integrals that arise in mathematical physics: C_n are connected to quantum field theory, D_n arise in Ising theory, while the E_n integrands are derived from D_n:

$$\begin{aligned} C_n &:= \frac{4}{n!}\int_0^\infty \cdots \int_0^\infty \frac{1}{\left(\sum_{j=1}^n (u_j + 1/u_j)\right)^2} \frac{\mathrm{d}u_1}{u_1} \cdots \frac{\mathrm{d}u_n}{u_n} \\ D_n &:= \frac{4}{n!}\int_0^\infty \cdots \int_0^\infty \frac{\prod_{i<j}\left(\frac{u_i-u_j}{u_i+u_j}\right)^2}{\left(\sum_{j=1}^n (u_j + 1/u_j)\right)^2} \frac{\mathrm{d}u_1}{u_1} \cdots \frac{\mathrm{d}u_n}{u_n} \\ E_n &= 2\int_0^1 \cdots \int_0^1 \left(\prod_{1\le j<k\le n} \frac{u_k-u_j}{u_k+u_j}\right)^2 \mathrm{d}t_2\,\mathrm{d}t_3 \cdots \mathrm{d}t_n, \end{aligned}$$

where in the last line $u_k = t_2 \cdots t_k$ [1].

One early observation was that the C_n integrals can be converted to one-dimensional integrals involving the modified Bessel function $K_0(t)$:

$$C_n = \frac{2^n}{n!}\int_0^\infty t K_0^n(t)\,\mathrm{d}t.$$

It was quickly evident that high-precision numerical values of this sequence, computed using tanh-sinh quadrature, approach a limit. For example:

$$C_{1024} = 0.630473503374386796122040192710878904354587 0787\ldots$$

When the first 50 digits of this constant were copied into the online Inverse Symbolic Calculator-2 (ISC-2) at `https://isc.carma.newcastle.edu.au` (which Jon was instrumental in developing and deploying), the result was:

$$\lim_{n\to\infty} C_n = 2e^{-2\gamma},$$

where γ denotes Euler's constant, a result which was then proved.

Subsequently high-precision computations, in conjunction with Ferguson's PSLQ algorithm [11, 5], were applied to find experimental evaluations of numerous other specific instances of these integrals, including:

$$\begin{aligned}
D_3 &= 8 + 4\pi^2/3 - 27\,\mathrm{L}_{-3}(2) \\
D_4 &= 4\pi^2/9 - 1/6 - 7\zeta(3)/2 \\
E_2 &= 6 - 8\log 2 \\
E_3 &= 10 - 2\pi^2 - 8\log 2 + 32\log^2 2 \\
E_4 &= 22 - 82\zeta(3) - 24\log 2 + 176\log^2 2 - 256(\log^3 2)/3 + 16\pi^2\log 2 - 22\pi^2/3 \\
E_5 &= 42 - 1984\,\mathrm{Li}_4(1/2) + 189\pi^4/10 - 74\zeta(3) - 1272\zeta(3)\log 2 + 40\pi^2\log^2 2 \\
&\quad - 62\pi^2/3 + 40(\pi^2\log 2)/3 + 88\log^4 2 + 464\log^2 2 - 40\log 2,
\end{aligned}$$

where $\mathrm{L}_{-3}(2)$ is a Dirichlet L-function constant, $\zeta(x)$ is the Riemann zeta function and $\mathrm{Li}_n(x)$ is the polylogarithm function [1]. The formula for E_5, which was initially found by Borwein (and which he was quite proud of), remained a numerically discovered but open conjecture for several years, but was finally proven in 2014 by Erik Panzer [14]. Resolution of the general case is still open.

6 Number Theory, Special Functions and Pi

Jon's work on number theory, special functions and π is inextricably linked to his work on experimental mathematics. Typically, using his excellent mathematical intuition supported by experimental mathematics tools such as PSLQ [11], Jon would make a conjecture that was almost certainly true, and in many cases could be proved rigorously.

To give just one example, we mention Jon's work, together with collaborators David H. Bailey, Richard Crandall, Karl Dilcher, Armin Straub, James Wan and others, on the so-called *Mordell-Tornheim-Witten zeta function* [2]. This function is a vast generalisation of the Riemann zeta-function, and is defined by

$$\omega(s_1, \ldots, s_{K+1}) := \sum_{m_1,\ldots,m_K>0} \frac{1}{m_1^{s_1}\cdots m_K^{s_K}(m_1+\cdots+m_K)^{s_{K+1}}}$$

with suitable restrictions on the parameters $s_1, \ldots, s_{K+1}$. For integer values of the parameters, there are many interesting identities satisfied by the ω values and their derivatives.

Further examples are described in the the preface to the theme "Number Theory, Special Functions and Pi", and in the various papers contributed to that theme.

7 Mathematical Finance

A notable example of how Jon ventured into arenas quite far afield from his core research in optimization and computational mathematics is his work in mathematical finance. This began in 2013, when David H. Bailey mentioned to Jon some research he had been doing with Marcos Lopez de Prado, a financial mathematician in New York City. Bailey and Lopez de Prado were concerned about the yawning gap between state-of-the-art mathematical techniques that were being successfully applied in leading quantitative investment funds, on one hand, and the mathematically and statistically naive schemes and practices that were often being promoted to the public and even being presented in presumably peer-reviewed journals. It had become clear, based on the preliminary research, that "backtest overfitting", namely the statistical overfitting of historical market data, was rampant in the finance field, and is arguably the principal reason why so many financial strategies and investment fund designs, which look great on paper and in promotional literature, fall flat when actually fielded. David and Marcos were also concerned with the many pseudoscientific techniques and strategies that are mentioned on a daily basis in the financial press.

When they presented some of their findings and thoughts on the topic to Jon, he immediately understood the technical issues, appreciated their gravity and concurred that these issues deserved rigorous treatment. So Bailey, Borwein, Lopez de Prado and Jim Zhu then co-authored a pair of papers with full details. The first paper, entitled "Pseudo-mathematics and financial charlatanism: The effects of backtest overfitting on out-of-sample performance" (a provocative title that Borwein himself proposed), was published as a feature article in the *Notices of the American Mathematical Society* [3], and has been circulated widely in the financial community. The second paper addressed the probability of backtest overfitting in more technical depth [4].

Jon also urged Bailey, Lopez de Prado and Zhu to start a blog presenting many of these related issues for an even broader audience. The result was the *Mathematical Investor* blog [12], with the provocative subtitle (also proposed by Jon) "Mathematicians against fraudulent financial and investment advice (MAFFIA)". Its mission was and is to identify and draw attention to abuses of mathematics and statistics in the financial field, and also to call out the financial mathematics community for their silence on these abuses. These abuses include:

1. Failing to disclose the number of models or variations that were used to develop an investment strategy or fund (which failure makes the strategy or fund highly susceptible to backtest overfitting).
2. Making vague predictions that do not permit rigorous testing and falsification.
3. Misusing probability theory, statistics and stochastic calculus.
4. Suggesting in press reports and promotions that investors can achieve above-market returns via unsophisticated chart-watching techniques (e.g., "technical analysis", "Elliott waves", etc.).
5. Using pseudomathematical technical jargon: "stochastic oscillators", "Fibonacci ratios", "cycles", "waves", "golden ratios", "parabolic SAR", "pivot point", "momentum", etc., none of which has any rigorous scientific basis.

As Jon and the other authors of the "Pseudo-mathematics" paper explained, "Our silence is consent, making us accomplices in these abuses" [3].

8 Mathematical education and public communication

Jon's passion for sharing the joy of mathematical research and communicating this joy to the public was central to his career. He personally mentored scores of graduate students, and taught hundreds of others. Many of these students have in turn become notable mathematicians and computer scientists themselves. This alone would be an achievement worthy of acclaim.

Along this line, Jon specifically selected many of his research topics based on their potential for public appeal and inspiring students. This is particularly clear with his interest in π, formulas for π and experimental mathematics in general, which he saw as a powerful vehicle to convey the excitement of modern mathematics to the younger, tech-savvy crowd, and yet basic enough to be comprehensible even to high school and undergraduate students. The depth of Jon's personal engagement with mathematics education, and experimental mathematics as an educational tool, is explored in great detail in Naomi Borwein and Judy-anne Osborn's contribution to this Springer volume.

As mentioned above, Jon was an avid blogger, which again was rooted in his passion for communicating with students and the public at large. David H. Bailey is deeply grateful to have been a part of this effort with Jon. Beginning in 2009, when Bailey and Jon Borwein founded the "Math Drudge" blog [13], he and Bailey co-authored over 200 articles on a wide range of topics, covering virtually every facet of modern mathematics, computing and science. A few of the topics they addressed in these blogs include:

1. The psychology of mathematics.
2. Pseudoscience and anti-science.
3. The fallacies of creationism and intelligent design.
4. The sad state of math and science education.
5. Global warming and global warming denial.
6. The computation of π.
7. New developments in physics and cosmology.
8. New developments in computer science.
9. Fermi's paradox.
10. Artificial intelligence.
11. Computer games vs. humans.
12. Supercomputers.
13. Ancient Indian mathematics.
14. The ancient origins of decimal arithmetic.
15. Moore's law and the future of mathematics.
16. The discovery of the Higgs boson.
17. New ways to visualize the digits of π.
18. DNA and evolution.
19. Pseudoscience from the political left and right.
20. New energy technologies, including LENR and fusion.
21. Fields medalists, Abel Prize recipients and Breakthrough Prize recipients.

It should be emphasized that although Bailey did most of the actual writing, Jon personally proposed, co-wrote and co-edited virtually every one of these blogs. They reflect both his interests and his passions to communicate better with the public. Some of these blogs were subsequently published in venues such as *The Conversation* and *The Huffington Post*. Such expository writing was an extension of Jon's dynamic educational praxis across institutional and popular lines.

9 Summary

Jonathan Borwein's prodigious output in optimization and experimental mathematics is certainly his singular contribution to modern mathematics. But beyond his technical accomplishments, he was a master of mathematical communication (his lectures were always paragons of well-organized and visually appealing mathematics and graphics), mathematical education (part of his interest in π was to bring the joy of mathematical discovery to students), and in promoting science, mathematics and computing to the general public. To this end, he wrote and lectured tirelessly. By one reckoning he presented an average of one lecture per week for decades, and wrote hundreds of articles targeted to the general public. His death is a loss to all those who treasure modern mathematics, science and clear thinking.

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