Jonathan Borwein: Renaissance Mathematician

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The death of Jonathan M. Borwein in August 2016, at the age of 65, in London, Ontario, Canada (where he had been visiting from the University of Newcastle, Australia), came as a huge shock to many in the mathematical community.

In the wake of his untimely death, several colleagues made an effort to collect Borwein's many published papers, books, reports and talks, as well as a number of articles written by others (such as book reviews) about Prof. Borwein and his work. One such catalog (`https://www.jonborwein.org`) lists over 2000 items. Even if one focuses only on his published books and refereed articles, there are over 500 items. One paper on optimization theory [**8**] has been cited over 1300 times.

In examining Borwein's writings, what is most striking is their breadth. In an era when many in the academic mathematical community focus ever more tightly on a single specialty, Borwein did notable research in a wide range of fields, ranging from experimental mathematics (in which he can rightly be regarded as a pioneer and leading exponent) and optimization to biomedical imaging, mathematical finance, and computer science. Yet Borwein was equally at home writing for the general public. He published numerous articles in venues such as the *Conversation*, the *Huffington Post* and various blogs that were very much targeted to a wide range of readers. In short, Borwein can certainly be credited as both a master mathematician and a master communicator — truly a Renaissance mathematician.

In this introduction to an issue of the MONTHLY devoted in part to Jonathan Borwein, we briefly mention some of the work and methods for which he was most noted.

1. PI. Jonathan Borwein is perhaps best known for deriving, with his brother Peter, quadratically and higher order convergent algorithms for π, including p-th order convergent algorithms for any prime p, and similar algorithms for certain other fundamental constants and functions [**12, 13, 15**]. One of their algorithms is the following: Set $a_0 = 6 - 4\sqrt{2}$ and $y_0 = \sqrt{2} - 1$. Then iterate, for $k \geq 0$,

$$y_{k+1} = \frac{1 - (1 - y_k^4)^{1/4}}{1 + (1 - y_k^4)^{1/4}},$$
$$a_{k+1} = a_k(1 + y_{k+1})^4 - 2^{2k+3} y_{k+1}(1 + y_{k+1} + y_{k+1}^2). \quad (1)$$

Then $1/a_k$ converges *quartically* to π: each iteration approximately *quadruples* the number of correct digits (presuming, of course, that each iteration is performed using a level of numeric precision at least as great as desired for the final result). This algorithm, together with a quadratically convergent algorithm due to Brent and Salamin, were employed in several large computations of π by Kanada and others.

Another of Jon and Peter's algorithms is the following: Set $a_0 = 1/3,\ r_0 = (\sqrt{3} - 1)/2,\ s_0 = (1 - r_0^3)^{1/3}$. Then iterate, for $k \geq 0$,

$$t_{k+1} = 1 + 2r_k, \quad u_{k+1} = (9r_k(1 + r_k + r_k^2))^{1/3},$$
$$v_{k+1} = t_{k+1}^2 + t_{k+1}u_{k+1} + u_{k+1}^2, \quad w_{k+1} = 27(1 + s_k + s_k^2)/v_{k+1},$$
$$a_{k+1} = w_{k+1}a_k + 3^{2k-1}(1 - w_{k+1}),$$
$$s_{k+1} = \frac{(1 - r_k)^3}{(t_{k+1} + 2u_{k+1})v_{k+1}}, \quad r_{k+1} = (1 - s_{k+1}^3)^{1/3}. \tag{2}$$

Then $1/a_k$ converges *nonically* to π: each iteration approximately *nine times* the number of correct digits (presuming again a level of numeric precision at least as great as desired for the final result).

In 1997 Peter Borwein coauthored a paper presenting the following formula for π:

$$\pi = \sum_{k=0}^{\infty} \frac{1}{16^k} \left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6} \right). \tag{3}$$

This formula has the remarkable property that it permits one to directly calculate hexadecimal digits (and thus binary digits) of π beginning at an arbitrary starting point, without needing to calculate any of the preceding digits, by means of a simple algorithm that can easily be programmed on a computer [**5**].

One issue that immediately arose in the wake of Peter's paper was the question of whether there exist similar formulas of this type that permit direct calculation of arbitrary decimal or other base digits of π. In 2004, Jon Borwein, together with William Galway and Jon's father David Borwein, obtained an important result in this area: they proved that the only formulas for π of this form are binary formulas (although this does not rule out the existence of some rather different formula that leads to a different computational algorithm for this purpose) [**11**].

One reason for Jon's great interest in π was, by his own admission, because π is one topic that everyone, from K–12 schoolchildren to research mathematicians alike, could understand and be excited about. A sample of MONTHLY articles he authored or co-authored on π that were at least in part targeted to a larger audience include [**3, 15, 16**].

2. RAMANUJAN CONTINUED FRACTIONS. One fascinating example of the computational/experimental methodology that Jonathan Borwein tirelessly championed can be seen in two papers coauthored with the late Richard Crandall on "Ramanujan continued fractions" [**17, 18**] (the first paper was also coauthored by Greg Fee). Given $a, b, \eta > 0$, define

$$R_\eta(a,b) = \cfrac{a}{\eta + \cfrac{b^2}{\eta + \cfrac{4a^2}{\eta + \cfrac{9b^2}{\eta + \ddots}}}}. \tag{4}$$

Ramanujan discovered the intriguing fact that

$$\frac{R_\eta(a,b) + R_\eta(b,a)}{2} = R_\eta\left(\frac{a+b}{2}, \sqrt{ab}\right). \tag{5}$$

While investigating this identity for possible inclusion in an upcoming book, Borwein, after some effort, found that $R_1\left(1,1\right) = 0.693\ldots$, suggesting that this might be $\log 2$. Investigating further, he was able to show that for $0 < b < a$,

$$\mathcal{R}_1(a,b) = \frac{\pi}{2}\sum_{n\in\mathrm{Z}} \frac{aK(k)}{K^2(k)+a^2n^2\pi^2}\,\mathrm{sech}\left(n\pi\frac{K(k')}{K(k)}\right), \tag{6}$$

where $k = b/a = \theta_2^2/\theta_3^2,\ k' = \sqrt{1-k^2},\ K$ is a complete elliptic integral of the first kind, and θ_2, θ_3 are Jacobian theta functions (see (19) below) [**14**]. Writing the previous equation as a Riemann sum yields

$$\mathcal{R}(a) = \mathcal{R}_1(a,a) \;=\; \int_0^\infty \frac{\mathrm{sech}(\pi x/(2a))}{1+x^2}\,\mathrm{d}x \;=\; 2a\sum_{k=1}^\infty \frac{(-1)^{k+1}}{1+(2k-1)a}, \tag{7}$$

where the final equality follows from the Cauchy–Lindelof Theorem. This sum may also be written as $\mathcal{R}(a) = \frac{2a}{1+a}F\left(\frac{1}{2a}+\frac{1}{2},1;\frac{1}{2a}+\frac{3}{2};-1\right)$, from which *Maple* or *Mathematica* may be used to compute

$$\mathcal{R}(2) = 0.97499098879872209671990033452 9\ldots. \tag{8}$$

What is this constant? Borwein discovered that if one divides this by $\sqrt{2}$, then, by using the *Inverse Symbolic Calculator-2* (an online tool that Jon helped develop), he found that the quotient is, very likely, $\pi/2 - \log(1+\sqrt{2})$. In other words,

$$\mathcal{R}(2) = \sqrt{2}\left[\pi/2 - \log(1+\sqrt{2})\right]. \tag{9}$$

Based on these two specific experimental evaluations, Borwein, together with Crandall and Fee, were led to conjecture and then rigorously prove the general formula

$$\mathcal{R}(a) = 2\int_0^1 \frac{t^{1/a}}{1+t^2}\,\mathrm{d}t, \tag{10}$$

from which the specific results for $\mathcal{R}(1)$ and $\mathcal{R}(2)$, mentioned above, follow immediately. As Gauss once observed, it is often valuable to "know" the result before attempting to prove it. See [**17, 18**] for additional results and details.

3. CURIOUS INTEGRALS. Jonathan Borwein was born into a distinguished family: his father, David Borwein, was a well-known mathematician at the University of Western Ontario; his mother was a professor of anatomy; and, as mentioned above, Jon's brother Peter coauthored numerous distinguished papers with Jon, many of them on π. Jon also coauthored several papers with his father David (one was mentioned above [**11**]), and, in at least one case [**10**], jointly with both David and Peter.

One intriguing study coauthored by Jonathan Borwein and David Borwein [**9**] arose when a colleague of Jon's, using a computer algebra program, brought to Jon's attention the perplexing fact that

$$I_0 = \int_0^\infty \mathrm{sinc}\,(x)\,\mathrm{d}x \;=\; \frac{\pi}{2},$$
$$I_1 = \int_0^\infty \mathrm{sinc}\,(x)\,\mathrm{sinc}\,(x/3)\,\mathrm{d}x \;=\; \frac{\pi}{2},$$

$$\vdots$$

$$I_6 = \int_0^\infty \operatorname{sinc}(x)\operatorname{sinc}(x/3)\cdots\operatorname{sinc}(x/13)\,\mathrm{d}x = \frac{\pi}{2},$$

$$I_7 = \int_0^\infty \operatorname{sinc}(x)\operatorname{sinc}(x/3)\cdots\operatorname{sinc}(x/15)\,\mathrm{d}x$$
$$= \frac{467807924713440738696537864469\pi}{935615849440640907310521750000} \approx 0.499999999992647\pi. \quad (11)$$

Jon's first reaction was that this was a "bug" in the computer algebra system, but after further analysis, using other computational tools, he verified that this counterintuitive result did hold. Why did the pattern abruptly change beginning at I_7?

Working together with his father, Jon found the reason why, which we briefly sketch here. First define $T_n = \int_0^\infty \prod_{k=0}^n \operatorname{sinc}(a_k x)\,\mathrm{d}x$, for positive reals (a_k) with $a_0 = 1$. Since $\operatorname{sinc}(a_k x) = F[f_k](x)$, where $f_k(x) = 1/(2a_k)\cdot\chi(-a_k, a_k)$ and F denotes the Fourier transform, it follows by the convolution theorem that $T_n = F(f_0 \star f_1 \star \cdots \star f_n)$, and by the Fourier inversion theorem,

$$\int_{-\infty}^\infty T_n(x) = 2\pi(f_0 \star f_1 \star \cdots \star f_n)(0) = \pi \int_{-1}^1 (f_1 \star f_2 \star \cdots \star f_n)(u)\,\mathrm{d}u. \quad (12)$$

Since the support of f_k is $[-a_k, a_k]$, the support of $(f_1 \star f_2 \star \cdots \star f_n)$ is $[-s, s]$, where $s = \sum_{k=1}^n a_k$. Now note that if $s \leq 1$, then

$$\int_{-1}^1 (f_1 \star f_2 \star \cdots \star f_n)(u)\,\mathrm{d}u = F(f_1 \star f_2 \star \cdots \star f_n)(0)$$
$$= \prod_{k=1}^n \operatorname{sinc}(a_k 0) = 1, \quad (13)$$

so that $T_n = \pi/2$. On the other hand, if $s > 1$, then the interval $[-1, 1]$ is strictly inside the support of $(f_1 \star f_2 \star \cdots \star f_n)$, and one can conclude that

$$\int_{-1}^1 (f_1 \star f_2 \star \cdots \star f_n)(x)\,\mathrm{d}x < 1, \quad (14)$$

so that $T_n < \pi/2$.

This result explains precisely why the values of I_n abruptly drop below $\pi/2$ beginning with $n = 7$: This happens because $1/3 + 1/5 + \cdots + 1/13 < 1$, but $1/3 + 1/5 + \cdots + 1/15 > 1$. In their joint paper [**9**], Jon and his father give full details of this result, and also provide a geometric interpretation: the values of the integrals I_n drop below $\pi/2$ precisely when certain polyhedra "bite" into the corresponding hypercube.

In a follow-on paper [**7**] (coauthored with Robert Baillie), Jon and his father further expanded these results, and also proved results for other curious integrals, such as the intriguing integral

$$\int_0^\infty \cos(2x)\prod_{n=1}^\infty \cos(x/n)\,\mathrm{d}x$$

$$= 0.392699081698724154807830422909937860524645434187 23\ldots, \tag{15}$$

which is exceedingly close to $\pi/8$, differing from $\pi/8$ only beginning in the 43rd digit.

4. ALGEBRAIC NUMBERS IN POISSON POTENTIAL FUNCTIONS. One of the most remarkable examples of Borwein's computational/experimental insight arose when Richard Crandall, who was studying lattice sums associated with the Poisson equation, in connection with a technique to sharpen smartphone images, brought to Jon's attention the sums

$$\phi_n(r_1, \ldots, r_n) = \frac{1}{\pi^2} \sum_{m_1,\ldots,m_n \text{ odd}} \frac{e^{i\pi(m_1 r_1 + \cdots + m_n r_n)}}{m_1^2 + \cdots + m_n^2}. \tag{16}$$

After some analysis and numerical experimentation [**1, 2, 4**], Jon and Richard, later assisted by two other colleagues, discovered the intriguing fact that in the simple case when $n = 2$ and x and y are rational numbers,

$$\phi_2(x, y) = \frac{1}{\pi} \log A, \tag{17}$$

where A is an *algebraic number* that depends on the particular values of x and y. However, such analyses were hampered by the extremely slow convergence of the series (16), which frustrated attempts to explore this function computationally.

A key breakthrough here, due to Jon, was to prove that the $\phi_2(x, y)$ function could be written in a form much more conducive to rapid computation, as follows [**1**]:

$$\phi_2(x, y) = \frac{1}{2\pi} \log \left| \frac{\theta_2(z, q)\theta_4(z, q)}{\theta_1(z, q)\theta_3(z, q)} \right|, \tag{18}$$

where $q = e^{-\pi}$ and $z = \frac{\pi}{2}(y + ix)$, and where the four theta functions in turn can be computed using rapidly convergent formulas presented in Jon's book, coauthored with his brother Peter, *Pi and the AGM* [**14**, p. 52]:

$$\begin{aligned}
\theta_1(z, q) &= 2 \sum_{k=1}^{\infty} (-1)^{k-1} q^{(2k-1)^2/4} \sin((2k-1)z), \\
\theta_2(z, q) &= 2 \sum_{k=1}^{\infty} q^{(2k-1)^2/4} \cos((2k-1)z), \\
\theta_3(z, q) &= 1 + 2 \sum_{k=1}^{\infty} q^{k^2} \cos(2kz), \\
\theta_4(z, q) &= 1 + 2 \sum_{k=1}^{\infty} (-1)^k q^{k^2} \cos(2kz).
\end{aligned} \tag{19}$$

In some initial experiments with these formulas, Jon and his colleagues computed, in some cases to thousands of digits, the value of $\alpha = A^8 = \exp(8\pi\phi_2(x, y))$ for various specific rationals x and y. Then they generated the vector $(1, \alpha, \alpha^2, \ldots, \alpha^d)$ as input to a program implementing the multipair PSLQ integer relation algorithm [**6, 19**], which, when successful, returns the likely vector of integer coefficients $(a_0, a_1, a_2, \cdots, a_d)$ of the minimal polynomial satisfied by α. The following table shows some results of these initial computations [**1**]:

s	Minimal polynomial corresponding to $x = y = 1/s$:
5	$1 + 52\alpha - 26\alpha^2 - 12\alpha^3 + \alpha^4$
6	$1 - 28\alpha + 6\alpha^2 - 28\alpha^3 + \alpha^4$
7	$-1 - 196\alpha + 1302\alpha^2 - 14756\alpha^3 + 15673\alpha^4 + 42168\alpha^5 - 111916\alpha^6$
	$+82264\alpha^7 - 35231\alpha^8 + 19852\alpha^9 - 2954\alpha^{10} - 308\alpha^{11} + 7\alpha^{12}$
8	$1 - 88\alpha + 92\alpha^2 - 872\alpha^3 + 1990\alpha^4 - 872\alpha^5 + 92\alpha^6 - 88\alpha^7 + \alpha^8$
9	$-1 - 534\alpha + 10923\alpha^2 - 342864\alpha^3 + 2304684\alpha^4 - 7820712\alpha^5$
	$+13729068\alpha^6 - 22321584\alpha^7 + 39775986\alpha^8 - 44431044\alpha^9$
	$+19899882\alpha^{10} + 3546576\alpha^{11} - 8458020\alpha^{12} + 4009176\alpha^{13} - 273348\alpha^{14}$
	$+121392\alpha^{15} - 11385\alpha^{16} - 342\alpha^{17} + 3\alpha^{18}$
10	$1 - 216\alpha + 860\alpha^2 - 744\alpha^3 + 454\alpha^4 - 744\alpha^5 + 860\alpha^6 - 216\alpha^7 + \alpha^8$

After these initial results were first obtained, Jason Kimberley, who was then a graduate student at the University of Newcastle, Australia, observed that the degree $m(s)$ of the minimal polynomial associated with the case $x = y = 1/s$ appears to be given by the following formula: Set $m(2) = 1/2$. Otherwise for primes p congruent to 1 modulo 4, set $m(p) = (p-1)^2/4$, and for primes p congruent to 3 modulo 4, set $m(p) = (p^2-1)/4$. Then for any other positive integer s whose prime factorization is $s = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}$,

$$m(s) = 4^{r-1} \prod_{i=1}^{r} p_i^{2(e_i-1)}\, m(p_i). \tag{20}$$

This sequence now appears as item A218147 in the *Online Encyclopedia of Integer Sequences*. Does Kimberley's mysterious formula hold for any or all larger s?

Jon and his coauthors subsequently applied some significantly more powerful computational tools, including a new PSLQ program and an implementation on a parallel computer system [**4**]. With this improved facility, they numerically confirmed that Kimberley's formula appeared to hold for all integers s up to 52, except for a handful of cases that were still too expensive to analyze. These computations were very challenging, some requiring up to 64,000-digit precision and hundreds of hours of run time. The computer runs produced minimal polynomials with degrees as large as 512 and with integer coefficients as large as 10^{229}.

By examining the computed results, and, quite literally, doing Google searches on some of the resulting integer coefficients, Jon then discovered connections to a sequence of polynomials defined in a 2010 paper by Savin and Quarfoot of the University of Utah [**20**]. These investigations ultimately led to a proof, by Watson Ladd of the University of California, Berkeley, of Kimberley's formula and also of the fact that when s is even, the corresponding minimal polynomial is palindromic [**4**]. Needless to say, Jon was extremely pleased with this satisfying finale to a problem that initially appeared to be utterly intractable. Sadly, he died before the paper documenting these results was published [**4**].

5. A RENAISSANCE MATHEMATICIAN. Not long before Jon passed, the present author recalls being engaged in a particularly intense collaboration with Jon, marked by frequent Skype video calls, countless exchanged emails and numerous computer runs. He thought he had Prof. Borwein's full attention during this time, but, as he subsequently discovered, Borwein simultaneously was advancing at least three completely different lines of research with other colleagues, and this was in addition to teaching classes, managing students, handling significant editorial responsibilities and writing articles for the public.

Where did he find the time and energy? One way or the other, he was truly a singular genius, and a most effective communicator of the joy of mathematics.

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