PALINDROME DEGREE REDUCTION

NITYA MANI

1. Problem Overview

Given some polynomial $f(x) = a_{2n}x^{2n} + ... + a_1x + a_0$ that is palindromic so $f(x) = a_0x^{2n} + a_1x^{2n-1} + ... + a_1x + a_0$, we wish to construct a polynomial $h(y)$ with degree n, so that $f(x) = x^n h(x + 1/x)$. This is possible, since given a palindromic polynomial, if α is the root of such a polynomial, then so is $\frac{1}{\alpha}$

2. Finding $h(y)$

We begin by defining $g(x) = f(x)/x^n = a_0(x^n + \frac{1}{x^n}) + a_1(x^{n-1} + \frac{1}{x^{n-1}}) + ... + a_{n-1}(x + \frac{1}{x}) + a_n$. Then, we wish to use this definition of $g(x)$ to find some polynomial h in $y = x + \frac{1}{x}$. This can be done by noting that for any $k \in \mathbb{Z}^+$ we can write $x^k + \frac{1}{x^k}$ as a polynomial in $x + \frac{1}{x}$ in the following manner: Define $y = x + 1/x$, and let $x_k = x^k + \frac{1}{x^k}$. We can see that x_k is polynomial in y by noting the recurrence $x_k = yx_{k-1} + x_{k-2}$, which allows us to find any arbitrary x_k recursively.

This is true inductively as $x_{k-1}y = (x^{k-1} + \frac{1}{x^{k-1}})(x + \frac{1}{x}) = x^k + \frac{1}{x^k} + x^{k-2} + \frac{1}{x^{k-2}} = x_k + x_{k-2} \implies x_k = yx_{k-1} - x_{k-2}$. This gives a recursive relation, but an ugly closed form solution for finding $x_k = x^k + \frac{1}{x^k}$. However, we notice that x_k can be easily determined by replacing $y = x + \frac{1}{x}$ with $2\cos(\theta)$ and dividing the result by 2 which gives the kth degree Chebyshev polynomial of the first kind. Thus, we can transform $g(x)$ to $h(y)$ by taking $g(x) = a_0(x^n + \frac{1}{x^n}) + a_1(x^{n-1} + \frac{1}{x^{n-1}}) + ... + a_{n-1}(x + \frac{1}{x}) + a_n = a_0x_n + a_1x_{n-1} + .. + a_{n-1}y + a_n$ and replacing each $x_k = x^k + \frac{1}{x^k}$ with its polynomial expression in $y = x + \frac{1}{x}$ obtained from the recurrence. For example, $x_2 = x^2 + \frac{1}{x^2} = y^2 - 2$ and $x_3 = x^3 + \frac{1}{x^3} = y^3 - 3y$.

3. Purpose

This enables computation of results for polynomials of even degree ($2n$) polynomials that are known to be palindromic, by computing results with a polynomial $h(y)$ of half the degree (n), obtained through the above method. The original polynomial $f(x)$ can then be obtained as $f(x) = x^n h(x + \frac{1}{x})$, which will double the speed of discovery or enable even polynomials for up to double the current possible degree to be computationally determined.

Stanford University, Department of Mathematics, Stanford, CA 94305
E-mail address: nityam@stanford.edu

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