Nonnormality of Stoneham constants

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Abstract

This paper examines "Stoneham constants," namely real numbers of the form $\alpha_{b,c} = \sum_{n\geq 1} 1/(c^n b^{c^n})$, for coprime integers $b \geq 2$ and $c \geq 2$. These are of interest because, according to previous studies, $\alpha_{b,c}$ is known to be b-normal, meaning that every m-long string of base-b digits appears in the base-b expansion of the constant with precisely the limiting frequency b^{-m}. So, for example, the constant $\alpha_{2,3} = \sum_{n\geq 1} 1/(3^n 2^{3^n})$ is 2-normal. More recently it was established that $\alpha_{b,c}$ is *not* bc-normal, so, for example, $\alpha_{2,3}$ is provably *not* 6-normal. In this paper, we extend these findings by showing that $\alpha_{b,c}$ is *not* B-normal, where $B = b^p c^q r$, for integers b and c as above, $p, q, r \geq 1$, neither b nor c divide r, and the condition $D = c^{q/p} r^{1/p} / b^{c-1} < 1$ is satisfied. It is not known whether or not this is a complete catalog of bases to which $\alpha_{b,c}$ is nonnormal. We also show that the sum of two B-nonnormal Stoneham constants as defined above, subject to some restrictions, is B-nonnormal.

1 Introduction

The question of whether (and why) the digits of well-known constants of mathematics are statistically random in some sense has fascinated mathematicians from the dawn

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of history. Indeed, one prime motivation in computing and analyzing digits of π is to explore the age-old question of whether and why these digits appear "random." The first computation on ENIAC in 1949 of π to 2037 decimal places was proposed by John von Neumann so as to shed some light on the distribution of π (and of e) [10, pg. 277–281]. Since then, numerous computer-based statistical checks of the digits of π, for instance, so far have failed to disclose any deviation from reasonable statistical norms.

Analyses of the digits of π and related constants are discussed in greater length in [4], and by using graphical tools in [2]. We should mention that using the graphical tools described in [2], at least one of the results proved in this paper, namely Theorem 2, is visually quite compelling.

In the following, we say a real constant α is b-normal if, given the positive integer $b \geq 2$, every m-long string of base-b digits appears in the base-b expansion of α with precisely the expected limiting frequency $1/b^m$. It is a well-established albeit counter-intuitive fact that given an integer $b \geq 2$, almost all real numbers, in the measure theory sense, are b-normal. What's more, almost all real numbers are b-normal simultaneously for all positive integer bases (a property known as "absolutely normal").

Nonetheless, it has been frustratingly difficult to exhibit explicit examples of normal numbers, even of numbers that are normal just to a single given base b. The first constant to be proven 10-normal is the Champernowne number, namely the constant $0.12345678910111213141516\ldots$, produced by concatenating the decimal representation of all positive integers in order. Some additional results of this sort were established in the 1940s by Copeland and Erdös [16].

The situation with regards to other, more "natural" constants of mathematics remains singularly grim. Normality proofs are not available for any well-known constant such as $\pi, e, \log 2, \sqrt{2}$. We do not even know, say, that a 1 appears 1/2 of the time, in the limit, in the binary expansion of $\sqrt{2}$ (although it certainly appears to, from extensive empirical analysis). For that matter, it is widely believed that *every* irrational algebraic number (i.e., every irrational root of an algebraic polynomial with integer coefficients) is b-normal to all positive integer bases b, but there is no proof, not for any specific algebraic number to any specific base.

Recently the present authors, together with Richard Crandall and Carl Pomerance, proved the following: If a real y has algebraic degree $D > 1$, then the number $\#(|y|, N)$ of 1-bits in the binary expansion of $|y|$ through bit position N satisfies $\#(|y|, N) > CN^{1/D}$, for a positive number C (depending on y) and all sufficiently large N [5]. A related result has been obtained by Hajime Kaneko of Kyoto University in Japan [18]. However, these results falls far short of establishing b-normality for any irrational algebraic in any base b, even in the single-digit sense.

It is known that whenever α is b-normal, then so is $r\alpha$ and $r+\alpha$ for any nonzero positive rational r [11, pg. 165–166]. It is also easy to see that if there is a positive integer n such that integers $a \geq 2$ and $b \geq 2$ satisfy $a = b^n$, then any real constant that is a-normal is also b-normal. Recently Hertling proved an interesting converse: If there is no such n, then there are an uncountable number of counterexamples, namely constants that are a-normal but not b-normal [17]. Moving in the other direction, Greg Martin has succeeded in constructing an absolutely nonnormal number, namely one which fails to be b-normal for any integer base $b \geq 2$ [19].

2 A recent normality result

In 2002, one of the present authors (Bailey) and Richard Crandall showed that given a real number r in $[0,1)$, with r_k denoting the k-th binary digit of r, the real number

$$\alpha_{2,3}(r) := \sum_{k=1}^{\infty} \frac{1}{3^k 2^{3^k + r_k}} \tag{1}$$

is 2-normal. It can be seen that if $r \neq s$, then $\alpha_{2,3}(r) \neq \alpha_{2,3}(s)$, so that these constants are all distinct. Since r can range over the unit interval, this class of constants is uncountable. So, for example, the constant $\alpha_{2,3} = \alpha_{2,3}(0) = \sum_{k\geq 1} 1/(3^k 2^{3^k}) = 0.0418836808315030\ldots$ is provably 2-normal (this special case was proven by Stoneham in 1973 [20]). A similar result applies if 2 and 3 in formula (1) are replaced by any pair of coprime integers (b, c) with $b \geq 2$ and $c \geq 2$ [6].

More recently, Bailey and Michal Misieurwicz were able to establish 2-normality of $\alpha_{2,3}$ by a simpler argument, by utilizing a "hot spot" lemma proven using ergodic theory methods [7]. In [3], this proof was extended to the more general case $\alpha_{b,c}$, although the result itself was established in the 2002 Bailey-Crandall paper. We reprise this proof below, preceded by a statement of the "hot spot lemma" from [7].

Let $A(\alpha, y, n, m)$ denote the count of occurrences where the m-long binary string y is found to start at position p in the base-b expansion of α, where $1 \leq p \leq n$.

Lemma 1 ("Hot Spot" Lemma): *If x is not b-normal, then there is some $y \in [0,1)$ with the property*

$$\liminf_{m\to\infty} \limsup_{n\to\infty} \frac{b^m A(x, y, n, m)}{n} = \infty. \tag{2}$$

Conversely, if for all $y \in [0, 1)$,

$$\liminf_{m\to\infty} \limsup_{n\to\infty} \frac{b^m A(x, y, n, m)}{n} \quad < \quad \infty, \tag{3}$$

then x *is* b*-normal.*

Note that Lemma 1 implies that if a real constant α is not b-normal, then there must exist some interval $[r_1, s_1)$ with the property that successive shifts of the base-b expansion of α visit $[r_1, s_1)$ ten times more frequently, in the limit, relative to its length $s_1 - r_1$; there must be another interval $[r_2, s_2)$ that is visited 100 times more often relative to its length; there must be a third interval $[r_3, s_3)$ that is visited 1,000 times more often relative to its length; etc. Furthermore, there exists at least one real number y (a "hot spot") such that sufficiently small neighborhoods of y are visited too often by an arbitrarily large factor, relative to the lengths of these neighborhoods. On the other hand, if it can be established that no subinterval of the unit interval is visited 1,000 times (for instance) more often in the limit relative to its length, then this suffices to prove that the constant in question is b-normal.

Theorem 1 *For every coprime pair of integers* (b, c) *with* $b \geq 2$ *and* $c \geq 2$*, the constant* $\alpha_{b,c} = \sum_{m\geq 1} 1/(c^m b^{c^m})$ *is* b*-normal.*

Proof: We can write the the fraction immediately following position n in the base-b expansion of $\alpha_{b,c}$ as:

$$b^n \alpha_{b,c} \bmod 1 \;=\; \left(\sum_{m=1}^{\infty} \frac{b^{n-c^m} \bmod c^m}{c^m}\right) \bmod 1 \tag{4}$$

$$\;=\; \left(\sum_{m=1}^{\lfloor \log_c n \rfloor} \frac{b^{n-c^m} \bmod c^m}{c^m}\right) \bmod 1 + \sum_{m=\lfloor \log_c n \rfloor + 1}^{\infty} \frac{b^{n-c^m}}{c^m}. \tag{5}$$

Note that the first expression can be generated by means of the recursion $z_0 = 0$ and, for $n \geq 1$, $z_n = (b z_{n-1} + r_n) \bmod 1$, where $r_n = 1/n$ if $n = c^k$ for some integer k, and zero otherwise. For example, consider the case $b = 3$ and $c = 4$. The first few members of the z sequence are given as follows:

$$
\begin{aligned}
&0,\ 0,\ 0,\\
&\frac{1}{4},\ \frac{3}{4}, \quad \text{(repeated 6 times)}\\
&\frac{5}{16},\ \frac{15}{16},\ \frac{13}{16},\ \frac{7}{16}, \quad \text{(repeated 12 times)},\\
&\frac{21}{64},\ \frac{63}{64},\ \frac{61}{64},\ \frac{55}{64},\ \frac{37}{64},\ \frac{47}{64},\ \frac{13}{64},\ \frac{39}{64},\ \frac{53}{64},\ \frac{31}{64},\ \frac{29}{64},\ \frac{23}{64},\ \frac{5}{64},\ \frac{15}{64},\ \frac{45}{64},\ \frac{7}{64},\\
&\text{(repeated 12 times), etc.}
\end{aligned} \tag{6}
$$

Note here that the fraction $1/2$ is omitted in the first set, the fractions $1/8, 3/8, 5/8, 7/8$ are omitted in the second set, and the fractions with 32 in the denominators are omitted in the third set. Nonetheless, this critical property holds, both in this particular case and in general, so long as $b \geq 2$ and $c \geq 2$ are coprime [6]: if $n < c^{p+1}$ then z_n is a multiple of $1/c^p$, and furthermore the set $(z_k, 1 \leq k \leq n)$ contains at most t repetitions of any particular value, where the integer t depends only on the choice of b and c. For the case $(b, c) = (2, 3)$, the repetition factor $t = 3$. For the case $(3, 4)$, $t = 12$.

These fractions (z_k) constitute an accurate set of approximations to the sequence $b^n \alpha_{b,c} \bmod 1$ of shifted fractions of $\alpha_{b,c}$, since (z_k) generates the first term of (5). In fact, by examining (5) it can be readily seen that for all (b, c) as above and all $n \geq c$,

$$|b^n \alpha_{b,c} \bmod 1 - z_n| < \frac{1}{9n} \tag{7}$$

(and in most cases is much smaller than this).

To establish that $\alpha_{b,c}$ is b-normal via Lemma 1, we seek an upper bound for $b^m A(\alpha_{b,c}, y, n, m)/n$, good for all $y \in [0, 1)$ and all $m \geq 1$. A base-b sequence y out to some length m, translated to a subset of the real unit interval, can be written as $[r, s)$, where $r = 0.y_1 y_2 y_3 \ldots y_m$, and s is the next largest base-b fraction of length m, so that $s - r = b^{-m}$. Observe that the count $A(\alpha_{b,c}, y, n, m)$ is equal to the number of those j between 0 and $n - 1$ for which $b^j \alpha_{b,c} \bmod 1 \in [r, s)$. Also observe, in view of (7), that if $b^j \alpha \bmod 1 \in [r, s)$, then $z_j \in [r - 1/(9j), s + 1/(9j))$.

Let n be any integer greater than b^{2m}, and let c^p denote the largest power of c less than or equal to n, so that $c^p \leq n < c^{p+1}$. Now note that for $j \geq b^m$, we have $[r - 1/(9j), s + 1/(9j)) \subset [r - b^{-m-1}, s + b^{-m-1})$. Since the length of this latter interval is no greater than $2b^{-m}$, the number of multiples of $1/c^p$ that it contains cannot exceed $\lfloor 2c^p b^{-m} \rfloor + 1$. Thus there can be at most t times this many j's less than n for which $z_j \in [r - b^{-m-1}, s + b^{-m-1})$. Therefore we can write

$$\begin{aligned}
\frac{b^m A(\alpha_{b,c}, y, n, m)}{n} &= \frac{b^m \#_{0 \le j < n}\big(b^j \alpha_{b,c} \bmod 1 \in [r, s)\big)}{n} \\
&\le \frac{b^m \big[b^m + \#_{b^m \le j < n}\big(z_j \in [r - b^{-m-1}, s + b^{-m-1})\big)\big]}{n} \\
&\le \frac{b^m \big[b^m + t(2c^p b^{-m} + 1)\big]}{n} < 2t + 2, \qquad (8)
\end{aligned}$$

where t is the fixed repetition factor for (b, c), mentioned above. For a fixed pair of integers (b, c), we have shown that for all $y \in [0, 1)$ and all $m > 0$,

$$\limsup_{n \to \infty} \frac{b^m A(\alpha_{b,c}, y, n, m)}{n} \le 2t + 2, \qquad (9)$$

so by Lemma 1, $\alpha_{b,c}$ is b-normal. **QED**

This result was first proven by Bailey and Crandall in 2002 [6]. The proof above, which utilizes the hot spot lemma, appeared in [3].

3 A general nonnormality result

By Theorem 1, the Stoneham constant $\alpha_{2,3} = \sum_{k \ge 0} 1/(3^k 2^{3^k})$ is 2-normal. Almost as interesting is the fact that $\alpha_{2,3}$ is *not* 6-normal. This was first demonstrated in [3]. Here we briefly sketch why this is so, then present a rigorous proof for general Stoneham constants.

First note that the digits immediately following position n in the base-6 expansion of $\alpha_{2,3}$ can be obtained by computing $6^n \alpha_{2,3} \bmod 1$, which can be written as

$$6^n \alpha_{2,3} \bmod 1 \;=\; \left(\sum_{m=1}^{\lfloor \log_3 n \rfloor} 3^{n-m} 2^{n-3^m} \right) \bmod 1 + \sum_{m=\lfloor \log_3 n \rfloor + 1}^{\infty} 3^{n-m} 2^{n-3^m}. \qquad (10)$$

Note that the first portion of this expression is *zero*, since all terms of the summation are integers. That leaves the second expression.

Consider the case when $n = 3^m$, where $m \ge 1$ is an integer, and examine just the first term of the second summation. We see that this expression is

$$3^{3^m - (m+1)} 2^{3^m - 3^{m+1}} \;=\; 3^{3^m - m - 1} 2^{-2 \cdot 3^m} \;=\; (3/4)^{3^m} / 3^{m+1}. \qquad (11)$$

0.
0130140430003334251130502130000001243555045432233011500243525320551352
3435410104300000000000000005141130054040555455303144250433435101241345
2351125142125134505503545015053522052044434045215150510241155250042513 0
0511244540010441311500324203032130000000000000000000000000000000000000
0000014212034311121452013525445342113412240220525301054204423552411055
4150155204350414555400310145303033532002534340401301240104453254343502
1420204324150255551010040433000455441145010313314511510144514123443342
3412400551313335045423530553151153501533452435450250055521453054234342
1530350125024205404135451231323245353031534552304115020154242121145201
5422225343403404505301233255344404431033324453321414150142334545424124
3203125340050134150245514404300000000000000000000000000000000000000000
0000000000000000000000000000000000000000000000000000000000000000000000
0000000000313350542444431111055534141052014540213412313001424333133115
...

Table 1: Base-6 expansion of $\alpha_{2,3}$.

We can generously bound the sum of all terms of the second summation by 1.00001 times this amount, for all $m \geq 1$, and by many times closer to unity for all $m \geq 2$, etc. Thus we have

$$6^{3^m} \alpha_{2,3} \bmod 1 \quad \approx \quad \frac{\left(\frac{3}{4}\right)^{3^m}}{3^{m+1}}, \tag{12}$$

and this approximation is as accurate as one wishes (in ratio) for all sufficiently large m.

Given the very small size of the expression $(3/4)^{3^m}/3^{m+1}$ for even moderate-sized m, it is clear the base-6 expansion will have very long stretches of zeroes beginning at positions $3^m + 1$. For example, by explicitly computing $\alpha_{2,3}$ to high precision, one can produce the counts of consecutive zeroes Z_m that immediately follow position 3^m in the base-6 expansion of $\alpha_{2,3}$—see Tables 1 and 2.

In total, there 14256 zeroes in the first ten segments of zeroes, which, including the last segment, span the first $59049 + 9487 = 68536$ base-6 digits of $\alpha_{2,3}$. In this tabulation we have of course ignored the many zeroes in the large "random" segments of the expansion. Thus the fraction of the first 68536 digits that are zero is at least $14256/68536 = 0.20800747\ldots$, which is significantly more than the expected value $1/6 = 0.166666\ldots$. A careful estimate of the limiting fraction yields the desired nonnormality result.

m	3^m	Z_m
1	3	1
2	9	3
3	27	6
4	81	16
5	243	42
6	729	121
7	2187	356
8	6561	1058
9	19683	3166
10	59049	9487

Table 2: Counts Z_m of consecutive zeroes immediately following position 3^m in the base-6 expansion of $\alpha_{2,3}$.

It is worth pointing out here that in the parlance of Lemma 1, zero is a "hot spot" for the base-6 expansion of $\alpha_{2,3}$. This is because all sufficiently small neighborhoods of zero are visited too often, by an arbitrarily large factor, in a subsequence of the shifted fractions of its base-6 expansion.

The nonnormality of $\alpha_{2,3}$ and some related constants is explored graphically in [2], where the patterns shown above in Table 1 can be seen even more clearly.

Here is a generalization of this result for general Stoneham constants $\alpha_{b,c}$:

Theorem 2 *Given coprime integers $b \geq 2$ and $c \geq 2$, and integers $p, q, r \geq 1$, with neither b nor c dividing r, let $B = b^p c^q r$. Assume that the condition $D = c^{q/p} r^{1/p} / b^{c-1} < 1$ is satisfied. Then the constant $\alpha_{b,c} = \sum_{k \geq 0} 1/(c^k b^{c^k})$ is B-nonnormal.*

Proof. Let $n = \lfloor c^m/p \rfloor$, and let $w = np/c^m$, so that $n = wc^m/p$. Note that for even moderately large m, relative to p, the fraction w is very close to one. Let Q_m be the shifted fraction of $\alpha_{b,c}$ immediately following position n in its base-B expansion.

One can write

$$\begin{aligned} Q_m &= B^n \alpha_{b,c} \bmod 1 \\ &= \left(\sum_{k=0}^{m} b^{pn-c^k} c^{qn-k} r^n\right) \bmod 1 + \sum_{k=m+1}^{\infty} b^{pn-c^k} c^{qn-k} r^n \quad (13) \\ &= \sum_{k=m+1}^{\infty} b^{pn-c^k} c^{qn-k} r^n = \sum_{k=m+1}^{\infty} \frac{c^{qwc^m/p-k} r^{wc^m/p}}{b^{c^k-wc^m}}. \quad (14) \end{aligned}$$

(The first summation in (13) vanishes because all summands are integers.) Thus Q_m is accurately approximated (in ratio) by the first term of the series (14), namely

$$S_1 = \frac{1}{c^{m+1}} \left(\frac{c^{qw/p} r^{w/p}}{b^{c-w}}\right)^{c^m}, \quad (15)$$

and this in turn is very accurately approximated (in ratio) by

$$S_1' = \frac{D^{c^m}}{c^{m+1}}, \quad (16)$$

where $D = c^{q/p} r^{1/p} / b^{c-1}$ as defined in the hypothesis. So for all sufficiently large integers m,

$$S_1'(1 - 1/10) < Q_m < S_1'(1 + 1/10). \quad (17)$$

Given that $D < 1$, as assumed in the hypothesis, it is clear from (16) that Q_m will be very small for even moderate-sized m, and thus the base-B expansion of $\alpha_{b,c}$ will feature long stretches of zeroes beginning immediately after position n, where $n = \lfloor c^m/p \rfloor$. In particular, given $m \geq 1$, let $Z_m = \lfloor \log_B 1/Q_m \rfloor$ be the number of zeroes that immediately follow position $\lfloor c^m/p \rfloor$. Then after noting that $B \geq 6$ (implied by the definition of b, c, p, q, r above), we can rewrite (17) as

$$c^m \log_B(1/D) + (m+1)\log_B c - 2 < Z_m < c^m \log_B(1/D) + (m+1)\log_B c + 2. \quad (18)$$

Now let F_m be the fraction of zeroes up to position $c^m + Z_m$. Clearly

$$F_m > \frac{\sum_{k=1}^{m} Z_k}{c^m + Z_m}, \quad (19)$$

since the numerator only counts zeroes in the long stretches, ignoring many others in the "random" stretches. The summation in the numerator satisfies

$$\begin{aligned}\sum_{k=1}^{m} Z_k &> \frac{c}{c-1}\left(c^m - \frac{1}{c}\right)\log_B(1/D) + \frac{m(m+3)}{2}\log_B c - 2m \\ &> \frac{c^{m+1}}{c-1}\left(1 - \frac{1}{c^{m+1}}\right)\log_B(1/D) - 2m. \end{aligned} \tag{20}$$

Thus given any $\varepsilon > 0$, we can write, for all sufficiently large m,

$$\begin{aligned} F_m &> \frac{\frac{c^{m+1}}{c-1}\left(1 - \frac{1}{c^{m+1}}\right)\log_B(1/D) - 2m}{c^m + c^m \log_B(1/D) + (m+1)\log_B c} \\ &= \frac{\frac{c}{c-1}\left(1 - \frac{1}{c^{m+1}}\right)\log_B(1/D) - \frac{2m}{c^m}}{1 + \log_B(1/D) + \frac{m+1}{c^m}\log_B c} \\ &\geq \frac{\frac{c}{c-1}\log_B(1/D)}{1 + \log_B(1/D)} - \varepsilon \;=\; T - \varepsilon, \end{aligned} \tag{21}$$

where

$$T \;=\; \frac{c}{c-1} \cdot \frac{\log_B(1/D)}{1 + \log_B(1/D)}. \tag{22}$$

To prove our desired result, it suffices to establish that $F_m > T > 1/B$, which means that infinitely often (namely on segments up to position $c^m + Z_m$ for positive integers m) the fraction of zeroes exceeds the "normal" frequency of a zero, namely $1/B$, by the nonzero amount $T - 1/B$. But depending on the particular values of b, c, p, q and r, the condition $T > 1/B$ might not hold. Recall that the calculation above ignores the many zeroes in the "random" portions of the expansion, and thus the estimate T might not be sufficiently accurate to establish nonnormality, at least not in the single-digit frequency sense.

However, a simple modification of the above argument can establish nonnormality in the multi-digit frequency sense. Note that given any integer $M > 1$, then for all m with $Z_m > M$, we will see an M-long string of zeroes beginning immediately after position n, where $n = \lfloor c^m/p \rfloor$ as above. Indeed, the condition that an M-long string of zeroes begins at position t will be fulfilled for $\bar{Z}_m = Z_m - M + 1$ consecutive positions beginning with $t = n + 1 = \lfloor c^m/p \rfloor + 1$. Note that for sufficiently large m, the modified count $\bar{Z}_m$ is nearly as large as Z_m. What's more, when we sum $\bar{Z}_k$ for

$k = 1$ to m, we obtain, as in (20) above,

$$\sum_{k=1}^{m} \bar{Z}_k > \frac{c}{c-1}\left(c^m - \frac{1}{c}\right)\log_B(1/D) + \frac{m(m+3)}{2}\log_B c - (M+1)m$$
$$> \frac{c^{m+1}}{c-1}\left(1 - \frac{1}{c^{m+1}}\right)\log_B(1/D) - (M+1)m. \quad (23)$$

But the small term $(M+1)m$ in this expression disappears when we divide by c^m and take the limit as in (21) above. Thus we obtain exactly the same limiting bound T as we calculated above in (22) for individual zeroes. Note that the natural frequency for an M-long string of zeroes is $1/B^M$. Since $T > 1/B^M$ for all sufficiently large M, we conclude that $\alpha_{b,c}$ is B-nonnormal. **QED**

The following less general result than Theorem 2 first appeared in [3]:

Corollary 1 *Given coprime integers $b \geq 2$ and $c \geq 2$, $\alpha_{b,c}$ is bc-nonnormal.*

Proof: This is a special case of Theorem 2 where $p = q = r = 1$. It follows by checking the condition (see the hypothesis of Theorem 2) that $D = c/b^{c-1} < 1$, or, equivalently, that $\log c < (c-1)\log b$. This condition can be verified as follows. First assume that $b \geq 2$ and $c \geq 3$. In this case, the function $f(c) = \log c - (c-1)\log 2 < 0$, so that $\log c < (c-1)\log 2 \leq (c-1)\log b$. Similarly, when $b \geq 3$ and $c \geq 2$, the function $g(c) = \log c - (c-1)\log 3 < 0$, so that $\log c < (c-1)\log 3 \leq (c-1)\log b$. The remaining case $b = 2$ and $c = 2$ is not allowable, since b and c must be coprime. Thus the key condition $c/b^{c-1} < 1$ in the hypothesis of Theorem 2 is satisfied by all allowable pairs (b, c). Hence $\alpha_{b,c}$ is not bc-normal. **QED**

Example 1 (Normality and nonnormality in various bases) According to Theorem 1, the constant $\alpha_{2,3}$ is normal base 2, and thus is also normal in base $4, 8, 16, 32, \cdots$ (i.e., all powers of two). According to Theorem 2, $\alpha_{2,3}$ is nonnormal base $6, 12, 24, 36, 48, 60, 72, 96, 120, 144, 168, 192, 216, 240, \cdots$. This list can be obtained by checking the condition $3^{q/p}r^{1/p} < 4$ for various candidate bases $B = 2^p 3^q r$, where $p, q, r \geq 1$. Note that while all integers in this list are divisible by 6, not all multiples of 6 are in the list.

However, there are many integer bases not included in either list. For example, it is not known at the present time whether or not $\alpha_{2,3}$ is 3-normal, although it appears to be. For example, statistical analysis of the first 83,736 base-3 digits of $\alpha_{2,3}$ (both single digits and 6-long strings of digits) found no deviations from reasonable statistical norms. But there is no proof of 3-normality. Similar questions remain in the more general case of $\alpha_{b,c}$, where b and c and coprime and at least two.

4 Sums of Stoneham constants

We now examine the normality or nonnormality of the sum of two Stoneham constants.

Under the hypothesis $b, c_1, c_2 \geq 2$, with (b, c_1) coprime and (b, c_2) coprime, we know from Theorem 1 that α_{b,c_1} and α_{b,c_2} are each b-normal. But it is not known at the present time whether the sum $\alpha_{b,c_1} + \alpha_{b,c_2}$ is b-normal. However, the sum of two such constants that individually are B-nonnormal, for some base B as given in the hypothesis of Theorem 2, is also B-nonnormal:

Theorem 3 *Let α_{b_1,c_1} and α_{b_2,c_2} be two Stoneham constants satisfying the conditions of Theorem 2 to be B-nonnormal: $b_1 \geq 2$ and $c_1 \geq 2$ are coprime; $B = b_1^{p_1} c_1^{q_1} r_1$ for integers $p_1, q_1, r_1 \geq 1$ with neither b_1 nor c_1 dividing r_1; and $D_1 = c_1^{q_1/p_1} r_1^{1/p_1} / b_1^{c_1 - 1} < 1$ (with similar conditions on b_2, c_2, p_2, q_2, r_2 and D_2). Assume further there are no integers s and t such that $c_1^s = c_2^t$. Then $\alpha_{b_1,c_1} + \alpha_{b_2,c_2}$ is B-nonnormal.*

Proof. Given the hypothesized conditions, the proof of Theorem 2 established that the base-B expansion of α_{b_1,c_1} has long stretches of zeroes beginning at positions $P_{1,m} = \lfloor c_1^m / p_1 \rfloor + 1$ (for positive integers m), extending for length $Z_{1,m} \approx c_1^m \log_B(1/D_1) \approx P_{1,m} p_1 \log_B(1/D_1)$, where $D_1 = c_1^{q_1/p_1} r_1^{1/p_1} / b_1^{c_1-1}$. Similarly, the base-$B$ expansion of α_{b_2,c_2} has long stretches of zeroes beginning at positions $P_{2,n} = \lfloor c_2^n / p_2 \rfloor + 1$ (for positive integers n), extending for length $Z_{2,n} \approx c_2^n \log_B(1/D_2) \approx P_{2,n} p_2 \log_B(1/D_2)$, where $D_2 = c_2^{q_2/p_2} r_2^{1/p_2} / b_2^{c_2-1}$. In each case, the approximation indicated is as accurate in ratio as desired, for all sufficiently large m or n, respectively.

Note that the base-B expansions of the two constants will share a long stretch of zeroes, provided there exists some pair of integers (m, n) such that the corresponding starting points $P_{1,m}$ and $P_{2,n}$ are very close in ratio. In that case, the corresponding strings of zeroes will overlap for a length L that is close in ratio to the shorter of the two lengths. In other words,

$$\begin{aligned} L &\approx \min(Z_{1,m},\ Z_{2,n}) \approx \min(P_{1,m} p_1 \log_B(1/D_1),\ P_{2,n} p_2 \log_B(1/D_2)) \\ &\approx P_{1,m} \min(p_1 \log_B(1/D_1),\ p_2 \log_B(1/D_2)) = P_{1,m} E, \end{aligned} \tag{24}$$

where $E = \min(p_1 \log_B(1/D_1),\ p_2 \log_B(1/D_2))$, and where the approximations shown are as close in ratio as desired for all sufficiently large m and n.

What's more, since the base-B expansions of α_{b_1,c_1} and α_{b_2,c_2} share this section of zeroes, beginning at position $P_{1,m} \approx P_{2,n}$ and continuing for length L, so will the base-B expansion of $\alpha_{b_1,c_1} + \alpha_{b_2,c_2}$.

Now suppose that we can construct a sequence of pairs of integers (m_k, n_k), where the above condition, namely $P_{1,m_k} \approx P_{2,n_k}$ culminating with $L_k \approx P_{1,m_k} E$, is met for each k. At each k, even if we count only the zeroes in the common stretch L_k (ignoring all zeroes in all stretches and all "random" segments that precede it), we obtain, as an estimate of the fraction F_k of zeroes up to position $P_{1,m_k} + L_k$,

$$F_k \;\geq\; \frac{L_k}{P_{1,m_k} + L_k} \;\approx\; \frac{P_{1,m_k} E}{P_{1,m_k} + P_{1,m_k} E} \;=\; \frac{E}{1+E}, \tag{25}$$

where the approximation is as accurate as desired (in absolute terms, not just in ratio) for all sufficiently large k. Recall that $E = \min\left(p_1 \log_B(1/D_1),\ p_2 \log_B(1/D_2)\right) > 0$ by hypothesis, so that that the expression $E/(1+E)$ is independent of k and strictly greater than zero.

Such a sequence of integer pairs (m_k, n_k) can be constructed as follows: First consider the simpler special case where $p_1 = p_2$. Given $\epsilon > 0$, we require that for all sufficiently large pairs (m_k, n_k),

$$1 - \epsilon \;<\; \frac{P_{1,m_k}}{P_{2,n_k}} \;<\; 1 + \epsilon. \tag{26}$$

But this can equivalently be rewritten in any of the forms

$$\begin{aligned} 1 - \epsilon \;&<\; \frac{c_1^{m_k}}{c_2^{n_k}} \;<\; 1 + \epsilon \\ -\epsilon \;&<\; m_k \log c_1 - n_k \log c_2 \;<\; \epsilon \\ \left| \frac{m_k}{n_k} - \frac{\log c_2}{\log c_1} \right| \;&<\; \frac{\epsilon}{n_k \log c_1}. \end{aligned} \tag{27}$$

This last condition is fulfilled if we specify, for the sequence of pairs (m_k, n_k), the sequence of fractions produced by the infinite continued fraction approximation for $\log c_2 / \log c_1$ (note the continued fraction is infinite, since by assumption there are no integers s and t such that $c_1^s = c_2^t$, which is the same as saying that $\log c_2 / \log c_1$ is not rational). Recall that the error in the continued fraction approximation at each step is less than the square of the reciprocal of the current denominator [13, pg. 373]. Thus we can write,

$$\left| \frac{m_k}{n_k} - \frac{\log c_2}{\log c_1} \right| \;<\; \frac{1}{n_k^2} \;<\; \frac{\epsilon}{n_k \log c_1}, \tag{28}$$

for all sufficiently large k, which satisfies the condition in (27), and thus in (26) also.

Now consider the more general case where p_1 is not necessarily the same as p_2. Given $\epsilon > 0$, we require that for all sufficiently large pairs (m_k, n_k),

$$\begin{aligned} 1 - \epsilon &< \frac{P_{1,m_k}}{P_{2,n_k}} < 1 + \epsilon \\ 1 - \epsilon &< \frac{c_1^{m_k}/p_1}{c_2^{n_k}/p_2} < 1 + \epsilon \\ -\epsilon &< m_k \log c_1 - n_k \log c_2 + (\log p_2 - \log p_1) < \epsilon. \end{aligned} \tag{29}$$

In this case, we can apply a generalization of the continued fraction algorithm presented as Algorithm 0.3 in [12] (see also Lemma 2.5.9 in [1]) to construct the requisite sequence of integer pairs (m_k, n_k). A simple normalization of (29) reduces it to the form required in [12].

In short, for any choice of coprime pairs of integers (b_1, c_1) and (b_2, c_2) satisfying the hypothesis, we can construct an infinite sequence of positions $(P_{1,m_k} + L_k)$ in the base-B expansion of $\alpha_{b_1,c_1} + \alpha_{b_2,c_2}$ such that the fraction F_k of zeroes up to position $P_{1,m_k} + L_k$ exceeds the fixed bound $E/(1 + E)$. If this bound satisfies $E/(1+E) > 1/B$, we are done. If not, a simple extension of the preceding argument to count the number of indices where an M-long strings of zeroes begins, as was done near the end of the proof of Theorem 2, shows that the asymptotic bound $E/(1+E)$ also applies to the frequency of M-long strings of zeroes. Since for all sufficiently large M, the condition $E/(1 + E) > 1/B^M$ is satisfied, this concludes the proof. **QED**

Example 2 (Nonnormality of sums in various bases) Consider the Stoneham constants $\alpha_{2,3}$ and $\alpha_{2,5}$. By Theorem 1, both are 2-normal. Consider base $60 = 2^p \cdot 3^q \cdot r$, where $p = 2, q = 1$ and $r = 5$. By checking the condition $3^{1/2} \cdot 5^{1/2} < 2^2$, we verify that $\alpha_{2,3}$ is 60-nonnormal, according to Theorem 2. In a similar way, write $60 = 2^p \cdot 5^q \cdot r$, where $p = 2, q = 1$ and $r = 3$. Then by checking the condition $5^{1/2} \cdot 3^{1/2} < 2^4$, we verify that $\alpha_{2,5}$ is also 60-nonnormal. Thus, according to Theorem 3, $\alpha_{2,3} + \alpha_{2,5}$ is 60-nonnormal.

5 Alternate proofs

When we first submitted this manuscript for publication, a referee pointed out that Theorems 2 and 3 could both be proven by means of the following lemma, which lemma appears to be potentially quite useful in establishing normality (or nonnormality), independent of the application to the theorems in this paper:

Lemma 2 . *Given the positive integer $b \geq 2$ and α in the unit interval, let $\alpha = 0.a_1a_2a_3\ldots$ be the base-b expansion of α. Assume that there is an increasing sequence of positive integers $(n_k)_{k\geq 1}$ and a real number $0 < \delta < 1$ such that*

$$|b^{n_k}\alpha \bmod 1| < \delta^{n_k}. \tag{30}$$

Then α is not b-normal.

Proof. Let us assume that α is b-normal, so that

$$\lim_{n\to\infty} \frac{\#\{1 \leq n \leq N : a_n = 0\}}{N} = \frac{1}{b}. \tag{31}$$

Assuming the given condition (30), there is a block of $\lfloor \tau n_k \rfloor$ consecutive zeroes starting at position n_k in the base-b expansion of α, where $\tau = -\log_b \delta$. Thus

$$\lim_{k\to\infty} \frac{\#\{1 \leq n \leq n_k + \lfloor \tau n_k \rfloor : a_n = 0\}}{n_k + \lfloor \tau n_k \rfloor} = \lim_{k\to\infty} \frac{\#\{1 \leq n \leq n_k : a_n = 0\} + \lfloor \tau n_k \rfloor}{n_k + \lfloor \tau n_k \rfloor}$$
$$= \frac{1/b + \tau}{1+\tau} \;=\; \frac{1}{b} + \frac{\tau(1-1/b)}{1+\tau} \;>\; \frac{1}{b}, \tag{32}$$

which contradicts the assumption that α is b-normal. **QED**

We now briefly sketch how Theorems 2 and 3 can be proven using Lemma 2:

Alternate proof of Theorem 2: For integers $k \geq 1$, define $n_k = \lfloor c^k/p \rfloor$. Following the first few paragraphs of the earlier proof of Theorem 2, observe that

$$B^{n_k}\alpha_{b,c} \bmod 1 < \frac{D^{n_k}}{c^{n_k+1}} \;<\; D^{n_k}, \tag{33}$$

where

$$D = \frac{c^{q/p}r^{1/p}}{b^{c-1}} \tag{34}$$

as before. By the hypothesis of Theorem 2, $D < 1$. Thus the conclusion follows by Lemma 2, where B takes the place of b, and D takes the place of δ. **QED**

Alternate proof of Theorem 3: Set $m_k = \lfloor c_1^k/p_1 \rfloor$ and $n_k = \lfloor c_2^k/p_2 \rfloor$. According to the earlier proof of Theorem 2, there exist two real numbers δ_1 and δ_2 in $(0,1)$ such that, for every $k \geq 1$,

$$B^{m_k}\alpha_{b_1,c_1} \bmod 1 < \delta_1^{m_k}$$
$$B^{n_k}\alpha_{b_2,c_2} \bmod 1 < \delta_2^{n_k}. \tag{35}$$

Given the assumed fact that there do not exist integers s and t such that $c_1^s = c_2^t$, we note from the earlier proof of Theorem 3, in particular from equation (29), that given any $\epsilon > 0$, there exist two increasing sequences of integers (k_j) and (l_j) such that

$$1 - \epsilon < \frac{c_1^{k_j}/p_1}{c_2^{l_j}/p_2} < 1 + \epsilon \tag{36}$$

for every j. It further can be seen that sequences can be found ensuring $1 - \epsilon < m_{k_j}/n_{l_j} < 1+\epsilon$ for every j, since m_{k_j} is very close (in ratio) to $c_1^{k_j}/p_1$ for all sufficiently large k_j, and n_{l_j} is very close (in ratio) to $c_2^{l_j}/p_2$, for all sufficiently large l_j. Choose ϵ sufficiently small so that $B^\epsilon \delta_2^{1-\epsilon} < \delta_1$. Then

$$\begin{aligned} B^{m_{k_j}}(\alpha_{b_1,c_1} + \alpha_{b_2,c_2}) \bmod 1 &= B^{m_{k_j}} \alpha_{b_1,c_1} \bmod 1 + B^{m_{k_j}} \alpha_{b_2,c_2} \bmod 1 \\ &< \delta_1^{m_{k_j}} + B^{m_{k_j} - n_{l_j}} B^{n_{l_j}} \alpha_{b_2,c_2} < \delta_1^{m_{k_j}} + (B^\epsilon)^{m_k} \delta_2^{n_{l_j}} \\ &< \delta_1^{m_{k_j}} + (B^\epsilon)^{m_{k_j}} \delta_2^{m_{k_j}} \delta_2^{n_{l_j} - m_{k_j}} \\ &< \delta_1^{m_{k_j}} + (B^\epsilon \delta_2^{1-\epsilon})^{m_{k_j}} < 2\delta_1^{m_{k_j}} = (2^{1/m_{k_j}} \delta_1)^{m_{k_j}}. \end{aligned} \tag{37}$$

But since $2^{1/m_{k_j}} \delta_1 < 1$ for all sufficiently large j, Lemma 2 applies, with B in the place of b and $2^{1/m_{k_j}} \delta_1$ in the place of δ, to establish that $\alpha_{b_1,c_1} + \alpha_{b_2,c_2}$ cannot be B-normal. **QED**

6 Conclusion

As mentioned above, under the hypothesis that integers $b \geq 2$, $c_1 \geq 2$ and $c_2 \geq 2$ are coprime, we know from Theorem 1 that α_{b,c_1} and α_{b,c_2} are each b-normal, but it is not known at the present time whether the sum $\alpha_{b,c_1} + \alpha_{b,c_2}$ is b-normal (although from substantial empirical analysis of specific cases, this appears to be true). Such a result, if it could be proven and extended, may yield a construction of an explicit computable constant that is absolutely normal, namely b-normal for all integer bases $b \geq 2$ simultaneously.

One example of an absolutely normal constant, as defined in the previous paragraph, is Chaitin's omega constant. Fix a prefix-free universal Turing machine U: (i.e., if instances $U(p)$ and $U(q)$ each halt, then neither p nor q is a prefix of the other.) Then Chaitin's omega is defined by

$$\Omega = \sum_{\{U(p) \text{ halts}\}} 2^{-|p|},$$

where $|p|$ is the length of the program p in bits. In 1994, Cristian Calude [14] demonstrated that Ω is absolutely normal. Although a scheme is known to explicitly compute the value of an initial segment of Chaitin's constant (for a certain encoding of a Turing machine), fewer than 100 bits are known [15]. Another explicit construction has been given by Becher and Figueira [8]. However, unlike Chaitin's constant, while it is possible in principle to compute digits of the the Becher-Figueira constant, it is nearly impossible in practice. It transpires that Alan Turing visited this same issue many decades ago — as described in [9].

In any event, there is continuing interest in explicitly constructive real numbers that are both absolutely normal and which can be computed to high precision with reasonable effort.

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