Computation and experimental evaluation of Mordell–Tornheim–Witten sum derivatives

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Abstract

In previous work the present authors and others have studied Mordell-Tornheim-Witten sums and their connections with multiple-zeta values. In this note we describe the numerical computation of derivatives at zero of a specialization originating in a preprint by Romik, and the experimental evaluation of these numerical values in terms of well-known constants.

1 Introduction

In previous work the present authors and others have studied Mordell-Tornheim-Witten sums (MTW sums), their connections with multiple-zeta values and various applications [5, 6, 8, 11, 18, 19]. The simplest MTW sum is:

$$W(r,s,t)=\sum_{m,n\geq 1}\frac{1}{m^r n^s (m+n)^t}. \tag{1}$$

Such sums arise in combinatorics, mathematical physics (e.g., Feynmann diagrams and string theory), Lie algebras, number theory and numerous other fields [16, 17, 18, 20]. In special cases these sums have simple evaluations. For example, when $t=0$,

$$W(r,s,0)=\sum_{m,n\geq 1}\frac{1}{m^r n^s} \;=\; \sum_{m\geq 1}\frac{1}{m^r}\sum_{n\geq 1}\frac{1}{n^s} \;=\; \zeta(r)\zeta(s). \tag{2}$$

There are numerous intriguing relationships between these values, including, for example,

$$W(r,0,t)+W(t,0,r)=\zeta(r)\zeta(t)-\zeta(r+t). \tag{3}$$

The n-dimensional MTW sum is defined for integer m_i and positive real r_i as

$$W(r_1,r_2,\ldots,r_n,t)=\sum_{m_1,\ldots,m_n\geq 1}\frac{1}{m_1^{r_1}m_2^{r_2}\cdots m_n^{r_n}(m_1+m_2+\cdots+m_n)^t}. \tag{4}$$

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In [15], Matsumoto proved that (4) can be continued meromorphically to the entire $\mathbb{C}^{r+1}$ space, and that the possible singularities are only on subsets of $\mathbb{C}^{r+1}$ defined by one of the conditions

$$\begin{aligned} s_j + s_{r+1} &= 1 - \ell \quad (1 \le j \le r) \\ s_{j_1} + s_{j_2} + s_{r+1} &= 2 - \ell \quad (1 \le j_1 < j_2 \le r) \\ &\cdots \\ s_{j_1} + s_{j_2} + \cdots + s_{j_{r-1}} + s_{r+1} &= r - 1 - \ell \quad (1 \le j_1 < \cdots < j_{r-1} \le r) \\ s_1 + s_2 + \cdots + s_r + s_{r+1} &= r \end{aligned}$$

for nonnegative integers ℓ. In what follows we will assume this analytic continuation of W.

We will focus on the special case $r_1 = r_2 = \cdots = r_n = t = s$ for real s (analytically continued as above), namely

$$\omega_{n+1}(s) = \sum_{m_1, m_2, \ldots, m_n \ge 1} \frac{1}{(m_1 m_2 \cdots m_n (m_1 + m_2 + \cdots + m_n))^s}, \tag{5}$$

for $n = 2, 3, \cdots$. These generalized sums were studied by Tomkins [19], who conjectured that

$$\omega_{n+1}(0) = \frac{(-1)^n}{n+1}. \tag{6}$$

This was proved by Romik for the case $n = 2$ and in the general case by Borwein and Dilcher in [12].

2 Evaluation using a free parameter

Tomkins analyzed the Witten sums using a free parameter, following the lead of earlier studies [4, 5, 6]. This analysis was based on the following two results that are proved, for instance, in [4] (formulas (46) and (11), respectively):

Lemma 1. *For any complex r that is not a positive integer, and for complex z such that $|\log z| < 2\pi$,*

$$\mathrm{Li}_r(z) = \sum_{n=0}^{\infty} \frac{\zeta(r-n) \log^n(z)}{n!} + \Gamma(1-r)(-\log z)^{r-1}. \tag{7}$$

In the case $z = e^{-x}$ this yields

$$\mathrm{Li}_r(e^{-x}) = \sum_{n=0}^{\infty} \frac{\zeta(r-n)(-x)^m}{m!} + \Gamma(1-r) x^{r-1}. \tag{8}$$

Lemma 2. *For $t > 0$ and $r_1, r_2, \ldots, r_n > 1$,*

$$\begin{aligned} \Gamma(t) W(r_1, r_2, \ldots, r_n, t) &= \int_0^\infty e^{-(r_1 + r_2 + \cdots + r_n)x} x^{t-1} \, \mathrm{d}t \\ &= \int_0^\infty x^{t-1} \prod_{i=j}^{n} \mathrm{Li}_{r_j}(e^{-x}) \, \mathrm{d}x. \end{aligned} \tag{9}$$

We present here a key result from Tomkins' work [19], with a proof outline:

Theorem 3. *Let $r_1, r_2, \ldots, r_n, t$ be complex variables with $r_i \in \mathbb{N}$ for $1 \le i \le n$. Then for any real $\theta > 0$,*

$$\Gamma(t)W(r_1, r_2, \ldots, r_n, t) = \sum_{m_1, m_2, \ldots m_n \ge 1} \frac{\Gamma(t, (m_1 + m_2 + \cdots + m_n)\theta)}{m_1^{r_1} m_2^{r_2} \cdots m_n^{r_n} (m_1 + m_2 + \cdots + m_n)^t} + \sum_{\{a_1, a_2, \ldots, a_k\} \subset \{1,2,\ldots,n\}} \left(\sum_{u_{a_1}, u_{a_2}, \ldots, u_{a_k} \ge 0} \frac{\theta^w}{w} \prod_{i=1}^{n} \Gamma(1 - r_i) \prod_{j=1}^{k} \frac{(-1)^{u_{a_j}} \zeta(r_{a_j} - u_{a_j})}{u_{a_j}!\, \Gamma(1 - r_{a_j})} \right), \quad (10)$$

where $w = t - (n - k) + \sum_{j=1}^{k} (u_{a_j} - r_{a_j}) + \sum_{i=1}^{n} r_i$.

Proof. From the definition of the gamma function, for any $\theta > 0$,

$$\Gamma(t, (m_1 + m_2 + \cdots + m_n)\theta) = \int_{(m_1+m_2+\cdots+m_n)\theta}^{\infty} y^{t-1} e^{-y}\, \mathrm{d}y, \quad (11)$$

which, after substituting $y = (m_1 + m_2 + \cdots + m_n)x$, yields

$$\begin{aligned} \Gamma(t, (m_1 + m_2 + \cdots + m_n)\theta) &= \\ &\int_{\theta}^{\infty} ((m_1 + m_2 + \cdots + m_n)x)^{t-1} e^{-(m_1+m_2+\cdots+m_n)x} (m_1 + m_2 + \cdots + m_n)\, \mathrm{d}x \\ &= (m_1 + m_2 + \cdots + m_n)^t \int_{0}^{\infty} x^{t-1} e^{-(m_1+m_2+\cdots+m_n)x}\, \mathrm{d}x. \end{aligned} \quad (12)$$

Solving this for the integral yields

$$\int_{0}^{\infty} x^{t-1} e^{-(m_1+m_2+\cdots+m_n)x}\, \mathrm{d}x = \frac{\Gamma(t, (m_1 + m_2 + \cdots m_n)\theta)}{(m_1 + m_2 + \cdots + m_n)^t}. \quad (13)$$

From (9), one can write

$$\begin{aligned} \Gamma(t)W(r_1, r_2, \ldots, r_n, t) &= \sum_{m_1, m_2, \ldots, m_n \ge 1} \frac{1}{m_1^{r_1} m_2^{r_2} \cdots m_n^{r_n}} \int_{0}^{\infty} x^{t-1} e^{-(m_1+m_2+\cdots+m_n)x}\, \mathrm{d}x \\ &= \sum_{m_1, m_2, \ldots, m_n \ge 1} \frac{1}{m_1^{r_1} m_2^{r_2} \cdots m_n^{r_n}} \left(\int_{\theta}^{\infty} x^{t-1} e^{-(m_1+m_2+\cdots+m_n)x}\, \mathrm{d}x + \int_{0}^{\theta} x^{t-1} e^{-(m_1+m_2+\cdots+m_n)x}\, \mathrm{d}x \right) \\ &= \sum_{m_1, m_2, \cdots m_n \ge 1} \frac{\Gamma(t, (m_1 + m_2 + \cdots + m_n)\theta)}{m_1^{r_1} m_2^{r_2} \cdots m_n^{r_n} (m_1 + m_2 + \cdots + m_n)^t} + \int_{0}^{\theta} x^{t-1} \prod_{j=1}^{n} \mathrm{Li}_{r_j}(e^{-x})\, \mathrm{d}x. \end{aligned} \quad (14)$$

Note that the integrand of the final term can be expanded as follows (after some rearrangement of terms):

$$\begin{aligned} x^{t-1} \prod_{j=1}^{n} \mathrm{Li}_{r_j}(e^{-x}) &= x^{t-1} \prod_{j=1}^{n} \left(\sum_{u_j=0}^{\infty} \frac{\zeta(r_j - u_j)(-x)^{u_j}}{n!} + \Gamma(1 - r_j) x^{r_j - 1} \right) \\ &= \sum_{\{a_1, a_2, \ldots, a_k\} \subset \{1,2,\ldots,n\}} \left(\sum_{u_{a_1}, u_{a_2}, \ldots, u_{a_k} \ge 0} x^w \prod_{i=1}^{n} \Gamma(1 - r_j) \prod_{j=1}^{k} \frac{(-1)^u_{a_j} \zeta(r_{a_j} - u_{a_j})}{u_{a_j}! \Gamma(1 - r_{a_j})} \right), \end{aligned} \quad (15)$$

where $w = t - 1 + (n-k) + \sum_{j=1}^{k}(u_{a_j} - r_{a_j}) + \sum_{i=1}^{n} r_i$. Integrating this final expression from 0 to θ and substituting above yields the desired result.

In spite of the forbidding notation, specific instances of (10) can be written in a fairly straightforward way. For example, when $n = 4$, one has

$$\omega_4(s) = \frac{1}{\Gamma(s)} \Bigg[\sum_{m,n,p\geq 1} \frac{\Gamma(s,(m+n+p)\theta)}{(mnp(m+n+p))^s} + \sum_{m,n,p\geq 0} \frac{(-1)^{m+n+p}\zeta(s-m)\zeta(s-n)\zeta(s-p)\theta^{m+n+p+s}}{m!n!p!(m+n+p+s)} + 3\Gamma(1-s)\sum_{m,n\geq 0} \frac{(-1)^{m+n}\zeta(s-m)\zeta(s-n)\theta^{m+n+2s-1}}{m!n!(m+n+2s-1)} + 3(\Gamma(1-s))^2 \sum_{p\geq 0} \frac{(-1)^p\zeta(s-p)\theta^{p+3s-2}}{p!(p+3s-2)} + (\Gamma(1-s))^3 \frac{\theta^{4s-3}}{4s-3} \Bigg], \tag{16}$$

where $\theta > 0$ is an arbitrary real parameter, and s is real but not an integer.

Another useful fact is the following, which is a straightforward extension of a result in Tomkins' thesis:

Theorem 4.

$$\sum_{m_1,m_2,\ldots,m_n\geq 1} \Gamma\left(0, \theta \sum_{j=1}^{n} m_j\right) = \int_1^\infty \frac{\mathrm{d}u}{u(e^{\theta u}-1)^n}. \tag{17}$$

Proof. We will illustrate this by proving the special case where $n = 3$, as presented in Tomkins' thesis:

$$\sum_{m,n,p\geq 1} \Gamma\left(0,(m+n+p)\theta\right) = \int_1^\infty \frac{\mathrm{d}u}{u(e^{\theta u}-1)^3}. \tag{18}$$

We start with the identity

$$\Gamma(0,x) = \int_x^\infty \frac{e^{-t}\,\mathrm{d}t}{t}, \tag{19}$$

which, after the substitution $t = (m+n+p)\theta u$ becomes

$$\Gamma(0,(m+n+p)\theta) = \int_1^\infty \frac{e^{-(m+n+p)\theta u}\,\mathrm{d}u}{u}. \tag{20}$$

Thus we can write

$$\begin{aligned} \sum_{m,n,p\geq 1} \Gamma(0,(m+n+p)\theta) &= \int_1^\infty \left(\sum_{m,n,p\geq 1} e^{-(m+n+p)\theta}\right) \frac{\mathrm{d}u}{u} \\ &= \int_1^\infty \left(\sum_{m\geq 1} e^{-m\theta u}\right)\left(\sum_{n\geq 1} e^{-n\theta u}\right)\left(\sum_{p\geq 1} e^{-p\theta u}\right)\frac{\mathrm{d}u}{u} \\ &= \int_1^\infty \left(\sum_{m\geq 1} e^{-m\theta u}\right)^3 \frac{\mathrm{d}u}{u} = \int_1^\infty \frac{1}{(1-e^{-\theta u})^3}\frac{\mathrm{d}u}{u}. \end{aligned} \tag{21}$$

3 Computation of ω_n derivatives

In initial computations of omega derivatives at zero, namely $\omega_d'(0)$ for $d = 3, 4, \ldots$, the authors of [11] found and then proved the intriguing experimental equivalence

$$\omega_3'(0) \;=\; \log(2\pi), \tag{22}$$

based purely on the numerical value of $\omega_3'(0)$ as computed from a more complex evaluation in [18], where the sum arises in counting representations of $SU(3)$.

In Tomkins' thesis [19] it was then shown that

$$\omega_4'(0) \;=\; -\log(2\pi) + \zeta'(-2). \tag{23}$$

These results immediately raise the question of whether the higher-degree constants $\omega_d'(0)$ have similarly elegant evaluations.

3.1 Numerical explorations

We decided to explore this question using a methodology we have employed numerous times before in other applications, namely to compute these constants to very high precision (typically 100–1000 digits) and then employ the multipair PSLQ algorithm [9] to attempt to obtain an analytic evaluation.

The principal computational challenge here is to evaluate the higher-degree versions of (10) to high precision. Straightforward evaluation of (10) as it is written is exceedingly expensive, since with each higher degree d, the summations involve one more level of loop nesting, and each higher level of loop nesting typically increases the computational run time by a factor of 10 or more over the previous level. Thus runs with, say, $d = 10$ are literally millions of times more expensive than with $d = 4$.

3.2 Code optimizations

However, after carefully examining these formulas or the equivalent computer code, it is evident that there are significant opportunities for economization in the computational work. For example, the zeta function terms such as $\zeta(s-m)/m!$ can all be precomputed, up to some loop limit N, where N is the largest sum of indices for the particular summation. Similarly, in (10) the expressions

$$\frac{\theta^{4s-3}}{4s-3}, \; \frac{\theta^{p+3s-2}}{p+3s-2}, \; \frac{\theta^{p+2s-1}}{p+2s-1}, \; \frac{\theta^{p+s}}{p+s}$$

can all be precomputed for p up to N, with clear analogues for higher degree.

Still, the deeply nested nature of these summations defeats evaluation for degrees d higher than six or so, particularly given that the computations must be performed to very high precision (we used 400 digits) to obtain a sufficiently accurate result for PSLQ analysis. The solution is to precompute the summations, starting with the next-to-last term of (10) (or its higher dimensional equivalent), and apply this tabulation recursively to compute the second-to-last term, and so forth. If this is done carefully, all terms beginning with the second term on the right-hand side of (10) can be computed very rapidly.

This leaves the first term on the right-hand side of (10). Here we note that the overall objective of this computation is to evaluate the derivative of $w_d(s)$ at $s = 0$. This can be done by selecting a very small argument, such as $s = \epsilon = 10^{-e}$ for $e = P/2$, where P is the working precision level in digits. We then approximate the first right-hand side term of (10) with the integral (18), recognizing that this will be an excellent, albeit not perfect approximation, valid only when s is very small. Note that this requires high-precision numerical quadrature, which can be performed with the tanh-sinh quadrature scheme [10]. We select $e = P/2$ because we found that in computing $w'_d(0)$ by this scheme, that some terms on the right-hand side of (10) and its higher-degree equivalents are of order unity, whereas others are of order $1/\epsilon = 10^e$ or so. In order to avoid catastrophic cancellation when these terms are added or subtracted, it is necessary to perform all computations to twice this level, i.e., with precision $P = 2e$ (or slightly higher). This produces final results accurate to approximately e-digit precision.

3.3 Million-fold speedups

We wish to emphasize the enormous importance of these accelerations. A relatively straightford implemention, or even a moderately tuned implementation, severely limits the degree of sums that can be practically studied.

For example, in an earlier implementation, where we performed some of the above-mentioned optmizations, but not the full-scale recursive precomputation of sum arrays, the run for degree-12 required 57,979 seconds on a 16-core MacPro system, or, in other words, 927,664 total core-seconds of computing.

With our fully-optimized code, this time was reduced to 0.478 seconds on a single-core system, a speedup factor of approximately 1.9 million. But even this is not a fair comparison, because the earlier computation was done using only 100-digit precision arithmetic, whereas the more advanced program employed 400-digit precision, which is roughly 16 times more expensive. Thus the actual speedup factor is closer to 30 million.

The resulting computational algorithm is presented in the form of a *Mathematica* program included in Table 1. For the actual computations reported here, however, we employed an implementation written in Fortran, using the MPFUN-MPFR multiprecision software [2].

3.4 PSLQ analysis

Once a numerical value for $\omega'_d(0)$ has been computed, our computer program applies the multipair variant of the PSLQ algorithm to find an analytical evaluation [9]. Given an input vector of high-precision floating-point values $x = (x_i, 1 \le i \le n)$, multipair PSLQ attempts to find integers $(a_i, 1 \le i \le n)$ such that $a_0 x_0 + a_1 x_1 + \cdots + a_n x_n = 0$, to within available numeric precision, or else returns a bound on the size of the a_i within which no relation exists.

In this application, we defined $x_0 = \omega'_d(0)$, and then selected a set of candidate constants, based on experience with the cases degree $d = 3$ and $d = 4$, given in (22) and (23), for the other terms x_i. In particular, we tried the following input vectors x in our multipair PSLQ computations, where all terms are computed to at least 400-digit precision:

$$\begin{aligned} x &= (\omega'_d(0),\, \log(2\pi),\, \zeta'(-2),\, \zeta'(-4), \cdots, \zeta'(-d+3)) \quad \text{for odd } d \\ x &= (\omega'_d(0),\, \log(2\pi),\, \zeta'(-2),\, \zeta'(-4), \cdots, \zeta'(-d+2)) \quad \text{for even } d. \end{aligned} \tag{24}$$

4 Numerical results

Using the input vector x as defined above in (24), we succeeded in finding a numerically significant relation, meaning that the relation holds to at least 50 digits beyond the minimal amount of precision required to discover it, for all $d \le 19$. In all cases the resulting relations held to at least 200-digit precision (recall, from Section 3.2, that this scheme produces results accurate to approximately one-half of the working precision, which in this case was 400 digits). Thus, while as yet we have no formal proofs

```
Domega[d_, s_, th_, NN_, DD_] :=
 Module[{i, j, k, m, n, p, S, SI, T1, T2, T3, Z},
  Z = Table[N[Zeta[s - m]/m!, DD], {m, 0, NN}];
  T1 = Table[
    N[th^(j*s + i + 1 - j)/(j*s + i + 1 - j), DD], {i, 0, NN}, {j, 1, d - 1}];
  T2 = Table[N[Z[[i + 1]]*Z[[j + 1]], DD], {i, 0, NN}, {j, 0, NN}];
  T3 = Table[N[0, DD], {i, 0, NN}, {j, 1, d}];
  Do[T3[[j + 1, 1]] =
    N[Sum[T2[[j - n + 1, n + 1]], {n, 0, j}], DD], {j, 0, NN}];
  Do[Do[T3[[j + 1, k]] =
     N[Sum[T3[[j - i + 1, k - 1]]*Z[[i + 1]], {i, 0, j}], DD], {j, 0, NN}], {k, 2, d - 2}];
  SI = N[Integrate[1/(t*(Exp[th/t] - 1)^(d - 1)), {t, 0, 1}], DD];
  Print[SI]; S = Table[0, {j, 0, d}];
  S[[1]] = N[th^(d*s - d + 1)/(d*s - d + 1), DD];
  S[[2]] = N[Sum[(-1)^m*Z[[m + 1]]*T1[[m + 1, d - 1]], {m, 0, NN}], DD];
  S[[3]] = N[Sum[N[Sum[(-1)^m*T1[[m + 1, d - 2]]*T2[[m - n + 1 , n + 1]],
    {n, 0, m}], DD], {m, 0, NN}], DD];
  Do[S[[k]] =
    N[Sum[(-1)^m*T1[[m + 1, d - k + 1]]*T3[[m + 1, k - 2]], {m, 0,
       NN}], DD], {k, 4, d}]; Do[Print[S[[k]]], {k, 1, d}];
  N[1/s*((-1)^d/d + 1/Gamma[s]*(SI + Sum[Binomial[d - 1, i - 1]*Gamma[1 - s]^(d - i)*S[[i]],
    {i, 1, d}])), DD]]
```

Table 1: Mathematica code to evaluate $\omega_d'(0)$, using a specified small value s, a specified value of θ, with NN as the maximum sum of indices in the summations and DD-digit precision. We have found that setting $s = 10^{-DD/2}$, $th = \theta = 3/4$, and $NN = 2/3 \cdot DD$ works well for the problems we have studied.

that these relations are mathematically true, we are quite confident, in an empirical sense, that they are real relations.

We present these results in two tables: Table 2 contains the raw relations as discovered by multipair PSLQ; in Table 3, the relations have been solved for $\omega'_d(0)$. The hat notation is used in these tables to emphasize that these are experimental results only.

Careful analysis of this table reveals some interesting patterns. For example, note in Table 2 that the first coefficient is $(2n-2)!/2$ when $2n$ is even and $(2n-2)!/(2n)$ when $2n+1$ is odd; also, the final coefficient is always 1 or -1, with sign always opposite that of the first term. Equivalently, note in Table 3 that the coefficient of first term is -1 for even n and 1 for odd n, with the final coefficient $2/(2n-2)!$ for even n and $2n/(2n-2)$ when $2n+1$ is odd.

Another interesting observation in Table 2 is that the absolute value of the sum of the coefficients of the zeta derivative terms is always $(n-3)$ times the absolute value of the first omega coefficient in the relation, or, equivalently, that the absolute value of the sum of the coefficients of the zeta derivative terms in Table 3 is always $(n-3)$. These patterns, which we noticed early in this research, gave us considerable additional confidence that higher-order results that we obtained in subsequent computations are correct.

As mentioned above in (24), we employed only $\log(2\pi)$ and terms of the form $\zeta'(-2n)$ in our search for relations. However, given the fact that

$$\zeta'(-2n) = \frac{(-1)^n(2n)!\zeta(2n+1)}{2(2\pi)^{2n}} = \frac{(-1)^n\zeta(2n+1)|B_{2n}|}{4\zeta(2n)}, \tag{25}$$

(as follows from the reflection formula for ζ [13]), it is clear that we could alternatively discover integer relations by setting

$$\begin{aligned} x &= \big(\omega'_d(0),\, \log(2\pi),\, \zeta(3)/(2\pi),\, \zeta(5)/(2\pi)^2, \cdots, \zeta(d-2)/(2\pi)^{d-3}\big) \quad \text{for odd } d \\ x &= \big(\omega'_d(0),\, \log(2\pi),\, \zeta(3)/(2\pi),\, \zeta(5)/(2\pi)^2, \cdots, \zeta(d-1)/(2\pi)^{d-2}\big) \quad \text{for even } d, \end{aligned} \tag{26}$$

or, as a second alternative option,

$$\begin{aligned} x &= (\omega'_d(0),\, \log(2\pi),\, \zeta(3)/\zeta(2),\, \zeta(5)/\zeta(4), \cdots, \zeta(d-2)/\zeta(d-3)) \quad \text{for odd } d \\ x &= (\omega'_d(0),\, \log(2\pi),\, \zeta(3)/\zeta(2),\, \zeta(5)/\zeta(4), \cdots, \zeta(d-1)/\zeta(d-2)) \quad \text{for even } d. \end{aligned} \tag{27}$$

Indeed, we computationally verified that valid experimental relations are found when we run multipair PSLQ with either of these alternate sets of inputs.

5 Conclusions

We finish with a few observations.

1. It is possible to also use (10) and its extensions to determine the poles of ω_n, while the contour integral method in [18] seem tailored to $n = 3$.
2. The difference in times between our initial and final computations of $\omega'_n(0)$ highlights that for reproducible computational science [7] to flourish, it is crucial to record complete details of the methods used and results for all computer runs performed in the study.
3. As is often the case, our applications of (10) illustrate the happy interplay between the computational and theoretical applications of a method.

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$$
\begin{aligned}
0 &\hat{=} \omega_4'(0) + \log(2\pi) - \zeta'(-2) \\
0 &\hat{=} -\omega_5'(0) + \log(2\pi) - 2\,\zeta'(-2) \\
0 &\hat{=} 12\,\omega_6'(0) + 12\,\log(2\pi) - 35\,\zeta'(-2) - \zeta'(-4) \\
0 &\hat{=} 4\,\omega_7'(0) - 4\,\log(2\pi) + 15\,\zeta'(-2) + \zeta'(-4) \\
0 &\hat{=} -360\,\omega_8'(0) - 360\,\log(2\pi) + 1624\,\zeta'(-2) + 175\,\zeta'(-4) + \zeta'(-6) \\
0 &\hat{=} 90\,\omega_9'(0) - 90\,\log(2\pi) + 469\,\zeta'(-2) + 70\,\zeta'(-4) + \zeta'(-6) \\
0 &\hat{=} -20160\,\omega_{10}'(0) - 20160\,\log(2\pi) + 118124\,\zeta'(-2) + 22449\,\zeta'(-4) \\
&\quad +546\,\zeta'(-6) + \zeta'(-8) \\
0 &\hat{=} -4032\,\omega_{11}'(0) + 4032\,\log(2\pi) - 26060\,\zeta'(-2) - 5985\,\zeta'(-4) - 210\,\zeta'(-6) \\
&\quad -\zeta'(-8) \\
0 &\hat{=} 1814400\,\omega_{12}'(0) + 1814400\,\log(2\pi) - 12753576\,\zeta'(-2) - 3416930\,\zeta'(-4) \\
&\quad -157773\,\zeta'(-6) - 1320\,\zeta'(-8) - \zeta'(-10) \\
0 &\hat{=} -302400\,\omega_{13}'(0) + 302400\,\log(2\pi) - 2286636\,\zeta'(-2) - 696905\,\zeta'(-4) \\
&\quad -39963\,\zeta'(-6) - 495\,\zeta'(-8) - \zeta'(-10) \\
0 &\hat{=} 239500800\,\omega_{14}'(0) + 239500800\,\log(2\pi) - 1931559552\,\zeta'(-2) - 657206836\,\zeta'(-4) \\
&\quad -44990231\,\zeta'(-6) - 749463\,\zeta'(-8) - 2717\,\zeta'(-10) - \zeta'(-12) \\
0 &\hat{=} 34214400\,\omega_{15}'(0) - 34214400\,\log(2\pi) + 292271616\,\zeta'(-2) + 109425316\,\zeta'(-4) \\
&\quad +8691683\,\zeta'(-6) + 183183\,\zeta'(-8) + 1001\,\zeta'(-10) + \zeta'(-12) \\
0 &\hat{=} -43589145600\,\omega_{16}'(0) - 43589145600\,\log(2\pi) + 392156797824\,\zeta'(-2) + 159721605680\,\zeta'(-4) \\
&\quad +14409322928\,\zeta'(-6) + 368411615\,\zeta'(-8) + 2749747\,\zeta'(-10) + 5005\,\zeta'(-12) + \zeta'(-14) \\
0 &\hat{=} 5448643200\,\omega_{17}'(0) - 5448643200\,\log(2\pi) + 51381813456\,\zeta'(-2) + 22556777880\,\zeta'(-4) \\
&\quad +2273360089\,\zeta'(-6) + 68396900\,\zeta'(-8) + 654654\,\zeta'(-10) + 1820\,\zeta'(-12) + \zeta'(-14) \\
0 &\hat{=} 10461394944000\,\omega_{18}'(0) + 10461394944000\,\log(2\pi) - 102992244837120\,\zeta'(-2) \\
&\quad -48366009233424\,\zeta'(-4) - 5374523477960\,\zeta'(-6) - 185953177553\,\zeta'(-8) - 2185031420\,\zeta'(-10) \\
&\quad -8394022\,\zeta'(-12) - 8500\,\zeta'(-14) - \zeta'(-16) \\
0 &\hat{=} 1162377216000\,\omega_{19}'(0) - 1162377216000\,\log(2\pi) + 11905898330880\,\zeta'(-2) \\
&\quad +5943136639504\,\zeta'(-4) + 720447491400\,\zeta'(-6) + 28157550993\,\zeta'(-8) + 393481660\,\zeta'(-10) \\
&\quad +1958502\,\zeta'(-12) + 3060\,\zeta'(-14) + \zeta'(-16)
\end{aligned}
$$

Table 2: Computationally discovered experimental relations

$$
\begin{aligned}
\omega_4'(0) &\hat{=} -\log(2\pi) + \zeta'(-2) \\
\omega_5'(0) &\hat{=} \log(2\pi) - 2\,\zeta'(-2) \\
\omega_6'(0) &\hat{=} -\log(2\pi) + \frac{35\,\zeta'(-2)}{12} + \frac{\zeta'(-4)}{12} \\
\omega_7'(0) &\hat{=} \log(2\pi) - \frac{15\,\zeta'(-2)}{4} - \frac{\zeta'(-4)}{4} \\
\omega_8'(0) &\hat{=} -\log(2\pi) + \frac{203\,\zeta'(-2)}{45} + \frac{35\,\zeta'(-4)}{72} + \frac{\zeta'(-6)}{360} \\
\omega_9'(0) &\hat{=} \log(2\pi) - \frac{469\,\zeta'(-2)}{90} - \frac{7\,\zeta'(-4)}{9} - \frac{\zeta'(-6)}{90} \\
\omega_{10}'(0) &\hat{=} -\log(2\pi) + \frac{29531\,\zeta'(-2)}{5040} + \frac{1069\,\zeta'(-4)}{960} + \frac{13\,\zeta'(-6)}{480} + \frac{\zeta'(-8)}{20160} \\
\omega_{11}'(0) &\hat{=} \log(2\pi) - \frac{6515\,\zeta'(-2)}{1008} - \frac{95\,\zeta'(-4)}{64} - \frac{5\,\zeta'(-6)}{96} - \frac{\zeta'(-8)}{4032} \\
\omega_{12}'(0) &\hat{=} -\log(2\pi) + \frac{177133\,\zeta'(-2)}{25200} + \frac{341693\,\zeta'(-4)}{181440} + \frac{7513\,\zeta'(-6)}{86400} + \frac{11\,\zeta'(-8)}{15120} + \frac{\zeta'(-10)}{1814400} \\
\omega_{13}'(0) &\hat{=} \log(2\pi) - \frac{190553\,\zeta'(-2)}{25200} - \frac{139381\,\zeta'(-4)}{60480} - \frac{1903\,\zeta'(-6)}{14400} - \frac{11\,\zeta'(-8)}{6720} - \frac{\zeta'(-10)}{302400} \\
\omega_{14}'(0) &\hat{=} -\log(2\pi) + \frac{1676701\,\zeta'(-2)}{207900} + \frac{14936519\,\zeta'(-4)}{5443200} + \frac{4090021\,\zeta'(-6)}{21772800} + \frac{22711\,\zeta'(-8)}{7257600} + \frac{247\,\zeta'(-10)}{21772800} \\
&\quad + \frac{\zeta'(-12)}{239500800} \\
\omega_{15}'(0) &\hat{=} \log(2\pi) - \frac{63427\,\zeta'(-2)}{7425} - \frac{2486939\,\zeta'(-4)}{777600} - \frac{790153\,\zeta'(-6)}{3110400} - \frac{5551\,\zeta'(-8)}{1036800} - \frac{91\,\zeta'(-10)}{3110400} \\
&\quad - \frac{\zeta'(-12)}{34214400} \\
\omega_{16}'(0) &\hat{=} -\log(2\pi) + \frac{30946717\,\zeta'(-2)}{3439800} + \frac{21939781\,\zeta'(-4)}{5987520} + \frac{899683\,\zeta'(-6)}{2721600} + \frac{515261\,\zeta'(-8)}{60963840} + \frac{2747\,\zeta'(-10)}{43545600} \\
&\quad + \frac{\zeta'(-12)}{8709120} + \frac{\zeta'(-14)}{43589145600} \\
\omega_{17}'(0) &\hat{=} \log(2\pi) - \frac{13215487\,\zeta'(-2)}{1401400} - \frac{2065639\,\zeta'(-4)}{498960} - \frac{2271089\,\zeta'(-6)}{5443200} - \frac{4783\,\zeta'(-8)}{381024} - \frac{109\,\zeta'(-10)}{907200} \\
&\quad - \frac{\zeta'(-12)}{2993760} - \frac{\zeta'(-14)}{5448643200} \\
\omega_{18}'(0) &\hat{=} -\log(2\pi) + \frac{993366559\,\zeta'(-2)}{100900800} + \frac{37319451569\,\zeta'(-4)}{8072064000} + \frac{1476517439\,\zeta'(-6)}{2874009600} + \frac{1300371871\,\zeta'(-8)}{73156608000} \\
&\quad + \frac{763997\,\zeta'(-10)}{3657830400} + \frac{46121\,\zeta'(-12)}{57480192000} + \frac{17\,\zeta'(-14)}{20922789888} + \frac{\zeta'(-16)}{10461394944000} \\
\omega_{19}'(0) &\hat{=} \log(2\pi) - \frac{344499373\,\zeta'(-2)}{33633600} - \frac{371446039969\,\zeta'(-4)}{72648576000} - \frac{13195009\,\zeta'(-6)}{21288960} - \frac{7292813\,\zeta'(-8)}{301056000} \\
&\quad - \frac{137581\,\zeta'(-10)}{406425600} - \frac{3587\,\zeta'(-12)}{2128896000} - \frac{17\,\zeta'(-14)}{6457651200} - \frac{\zeta'(-16)}{1162377216000}
\end{aligned}
$$

Table 3: Solved relations

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