A short history of π formulas

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Abstract

This note presents a short history mathematical formulas involving the mathematical constant π, and how they have been used in mathematical research through the ages.

1 Background

The mathematical constant we know as $\pi = 3.141592653589793\ldots$ is undeniably the most famous and arguably the most important mathematical constant. Mathematicians since the days of Euclid and Archimedes have computed its numerical value in a search for answers to questions such as: (a) is π a rational number? (i.e., is π the ratio of two whole numbers?) (b) does π satisfy some algebraic equation with integer coefficients? (c) is π simply related to other constants of mathematics?

Some of these questions were eventually resolved: In the late 1700s, Lambert and Legendre proved (by means of a continued fraction argument) that π is irrational, which explains why its digits never repeat. Then in 1882 Lindemann proved that π is transcendental, which means that π is not the root of any algebraic equation with integer coefficients. Lindemann's proof also settled, in the negative, the ancient Greek question of whether a circle could be "squared," or, in other words, whether one could construct, using ruler and compass, a square with the same area as a given circle. This cannot be done, because (as it is easy to show using high school algebra) any length constructible by rule and compass procedures is given by a finite combination of add, subtract, multiply, divide and square root operations, which thus is algebraic.

But many other questions remain. Notable of these is the persistent question of whether π is "normal," or, in other words, whether the digits of π, in either decimal or binary or some other base, are statistically flat in the sense that any finite-length string, such as "345," eventually appears with the correct limiting frequency (exactly 1/1000 in the case of a three-digit decimal string). Sadly, there are *no* significant results in this area: mathematicians do not even know for sure that a "7" appears 1/10 of the time in the decimal expansion of π, or whether a "1" appears 1/2 of the time in the binary expansion of π (although based on numerous statistical analysis, the answers appear to be "yes" in each case). If the conjecture of normality could be rigorously proved, either in decimal or binary, this would establish that the digits of π are a provably good source of pseudorandom numbers.

Thus there is continuing interest in computing digits of π, for both theoretical and practical reasons. Along this line, computations of π have for many years been to test the integrity of computer equipment — results using two different formulas should perfectly agree, except for few digits at the end due to round-off error, or else a hardware or software error has occurred. For example, in 1986 the Borwein quartic iteration for π disclosed some subtle hardware errors in one of the first Cray-2 supercomputers.

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2 History

Here is a brief history of formulas used for π. For additional information, see [1] and [2, Chap. 3].

1. Archimedes' scheme, used by Archimedes ($\sim$250 BC to compute three digits; a variation was used by the Chinese mathematician Tsu Chung-Chih to compute seven digits in the fifth century, and, evidently, also by the Indian mathematician Aryabhata in the fifth century to compute five digits):

$$\begin{aligned} a_0 &= 2\sqrt{3}, \quad b_0 = 3, \text{ then iterate:} \\ a_{n+1} &= \frac{2a_n b_n}{a_n + b_n}, \quad b_{n+1} \;=\; \sqrt{a_{n+1} b_n} \end{aligned} \tag{1}$$

   Both a_n and b_n converge to π, with the error decreasing by a factor of approximately four each iteration.

2. Newton's formula (1666; used by Newton to compute 15 digits):

$$\pi \;=\; 2 + \frac{3\sqrt{3}}{4} - 24\left(\frac{1}{5 \cdot 2^5} + \frac{1}{7 \cdot 2^9} + \frac{1}{9 \cdot 2^{12}} + \frac{5}{11 \cdot 2^{17}} + \cdots\right) \tag{2}$$

3. Euler's formula (1738):

$$\begin{aligned} \frac{\pi}{4} &= \left(\frac{1}{2} - \frac{1}{3 \cdot 2^3} + \frac{1}{5 \cdot 2^5} - \frac{1}{7 \cdot 2^7} + \cdots\right) \\ &\quad + \left(\frac{1}{3} - \frac{1}{3 \cdot 3^3} + \frac{1}{5 \cdot 3^5} - \frac{1}{7 \cdot 3^7} + \cdots\right) \end{aligned} \tag{3}$$

4. Machin's formula (1706; used by Shanks in 1873 to compute 707 digits, although only 527 were correct):

$$\begin{aligned} \frac{\pi}{4} &= 4\left(\frac{1}{5} - \frac{1}{3 \cdot 5^3} + \frac{1}{5 \cdot 5^5} - \frac{1}{7 \cdot 5^7} + \cdots\right) \\ &\quad + \left(\frac{1}{239} - \frac{1}{3 \cdot 239^3} + \frac{1}{5 \cdot 239^5} - \frac{1}{7 \cdot 239^7} + \cdots\right) \end{aligned} \tag{4}$$

5. Ferguson's formula (1945; used by Reitwiesner in 1949 to compute 2037 digits):

$$\begin{aligned} \frac{\pi}{4} &= 3\left(\frac{1}{4} - \frac{1}{3 \cdot 4^3} + \frac{1}{5 \cdot 4^5} - \frac{1}{7 \cdot 4^7} + \cdots\right) \\ &\quad + \left(\frac{1}{20} - \frac{1}{3 \cdot 20^3} + \frac{1}{5 \cdot 20^5} - \frac{1}{7 \cdot 20^7} + \cdots\right) \\ &\quad + \left(\frac{1}{1985} - \frac{1}{3 \cdot 1985^3} + \frac{1}{5 \cdot 1985^5} - \frac{1}{7 \cdot 1985^7} + \cdots\right) \end{aligned} \tag{5}$$

6. Ramanujan's series (1914; used by Gosper in 1988 to compute 17 million digits):

$$\frac{1}{\pi} \;=\; \frac{2\sqrt{2}}{9801} \sum_{k=0}^{\infty} \frac{(4k)!(1103 + 26390k)}{(k!)^4 396^{4k}} \tag{6}$$

7. Brent-Salamin scheme (1976; used by various researchers, most recently by Takahashi in January 2009 to compute 2.65 trillion digits):

$$\begin{aligned} a_0 &= 1, \; b_0 = 1/\sqrt{2}, \; s_0 = 1/2, \text{ then iterate:} \\ a_k &= \frac{a_{k-1} + b_{k-1}}{2}, \quad b_k \;=\; \sqrt{a_{k-1} b_{k-1}}, \\ c_k &= a_k^2 - b_k^2, \quad s_k \;=\; s_{k-1} - 2^k c_k, \quad p_k \;=\; 2a_k^2 / s_k \end{aligned} \tag{7}$$

   Then p_k converges to π, with each iteration approximately *doubling* the number of correct digits.

8. Chudnovskys' series (1987; used by various researchers including by Bellard in December 2009 to compute 2.7 trillion digits, and by "houkouonchi" in October 2014 to compute 13.3 trillion digits) [3].

$$\frac{1}{\pi} = 12 \sum_{k=0}^{\infty} \frac{(-1)^k \, (6k)!(13591409 + 545140134k)}{(3k)! \, (k!)^3 \, 640320^{3k+3/2}} \tag{8}$$

9. Borwein's quartic scheme (1985; used by various researchers, most recently by Takahashi in January 2009 to compute 2.65 trillion digits):

$$\begin{aligned} a_0 &= 6 - 4\sqrt{2}, \; y_0 = \sqrt{2} - 1, \text{ then iterate:} \\ y_{k+1} &= \frac{1 - (1 - y_k^4)^{1/4}}{1 + (1 - y_k^4)^{1/4}}, \\ a_{k+1} &= a_k(1 + y_{k+1})^4 - 2^{2k+3} y_{k+1}(1 + y_{k+1} + y_{k+1}^2) \end{aligned} \tag{9}$$

   Then a_k converges to π, with each iteration approximately *quadrupling* the number of correct digits.

10. The Bailey-Borwein-Plouffe (BBP) formula (1996; used by various researchers, most recently by Zse in July 2010 to calculate a segment of 252 binary digits starting at the *two quadrillionth* (2×10^{15}-th) binary position; a variant of the BBP formula was used by Bellard in 2009 as part of his computation in December 2009 to 2.7 trillion decimal digits, and by "houkouonchi" in October 2014 to compute 13.3 trillion decimal digits) [3].

$$\pi = \sum_{k=0}^{\infty} \frac{1}{16^k} \left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6} \right) \tag{10}$$

   The BBP formula is most often used to compute a segment of binary digits beginning at an arbitrary starting point, without needing to compute the digits that came before, by means of a surprisingly simple algorithm that does not require multiple-precision arithmetic or other exotic computational techniques. How this is done is described in [2, Chap. 3].

There are of course many other formulas for π. Over seventy such formulas are presented, with references and some historical background, in [1].

References

[1] David H. Bailey, "A collection of mathematical formulas involving pi," 8 Nov. 2016, available at `http://www.davidhbailey.com/dhbpapers/pi-formulas.pdf`.

[2] Jonathan M. Borwein and David H. Bailey, *Mathematics by Experiment: Plausible Reasoning in the 21st Century*, AK Peters, Natick, MA, 2008.

[3] Alexander J. Yee, "Y-cruncher: a multi-threaded pi program," 30 Oct. 2016, `http://www.numberworld.org/y-cruncher/`.