Large Poisson polynomials: Computation, results and analysis

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Abstract

Several studies over the past few years have explored the intriguing phenomenon of algebraic numbers arising from a simple two-dimensional instance of the Poisson potential function of mathematical physics:

$$\phi_2(x,y) = \frac{1}{\pi^2} \sum_{m,n \text{ odd}} \frac{\cos(m\pi x)\cos(n\pi y)}{m^2+n^2}.$$

In particular, one earlier study empirically discovered and then proved the remarkable fact that when x and y are *rational numbers*, then $\phi_2(x,y) = 1/\pi \cdot \log(\beta(x,y))$, where $\beta(x,y)$ is an *algebraic number* of some degree m.

This study presents experimental results using new arbitrary precision software and a new four-level multipair PSLQ algorithm to recover the minimal polynomials satisfied by $\alpha = (\beta(x,y))^8$, given specific rationals (x,y). These computations cover not just the special cases $x = y = 1/s$, or $x = 1/s$, $y = q/s$, for integers $s \le 36$, as in previous studies, but the much larger set $x = p/s$, $y = q/s$, for all $1 \le p \le q < s/2$, $\gcd(p,q,s) = 1$, $10 \le s \le 36$, and also for $s = 38, 40, 42$ and $s = 50$ (a total of 2206 cases). With this much larger catalogue of computational results in hand, we were able to note: (a) a tentative generalization of Kimberley's formula for the degrees; (b) the tentative fact that for a given s, all the cases $x = y = p/s$, with $1 \le p < s/2$ and $\gcd(p,s) = 1$, share the same minimal polynomial; and (c) the tentative fact that whenever s is even, the minimal polynomials are palindromic.

1 Earlier work on Poisson polynomials

Lattice sums related to the Poisson potential function naturally arise in studies of gravitational and electrostatic potentials, and have been studied in the mathematical physics community for some time [1, 12, 13]. More recently, researchers have identified applications of the Poisson potential function for practical image processing [5]. These developments have underscored the need to better understand the mathematical theory underlying this function.

In two earlier papers [5, 6], Jonathan Borwein (deceased 2016), Richard Crandall (deceased 2012) and others analyzed a simple two-dimensional instance of the Poisson potential function:

$$\phi_2(x,y) = \frac{1}{\pi^2} \sum_{m,n \text{ odd}} \frac{\cos(m\pi x)\cos(n\pi y)}{m^2+n^2}. \tag{1}$$

These researchers empirically discovered and then proved the intriguing fact that when x and y are *rational numbers*, then

$$\phi_2(x, y) = \frac{1}{\pi} \log(\beta(x, y)), \tag{2}$$

where $\beta(x, y)$ is an *algebraic number*, namely the root of a degree-m minimal polynomial with integer coefficients, for some m. The minimal polynomials corresponding to these algebraic numbers have simplest form in terms of $\alpha = (\beta(x, y))^8$, so in the following the α notation will be understood.

This result can be explored computationally as follows: Given rationals x, y and an integer m, compute $\alpha = \exp(8\pi\phi_2(x, y))$ to high precision, generate the $(m + 1)$-long vector $(1, \alpha, \alpha^2, \cdots, \alpha^m)$, and then apply an integer relation algorithm to discover the coefficients of the polynomial of degree m, if it exists, satisfied by α. This process will be described in more detail below in Section 3.

Some initial results from one earlier study [5] are shown in Table 1. These results immediately raised questions, such as given a specific pair of rationals, such as $x = y = 1/s$ for some integer s, what is the degree? Also, note that when s is an even integer, the minimal polynomial corresponding to the case $x = y = 1/s$ is always palindromic (i.e., coefficient $a_k = a_{m-k}$, where m is the degree). For instance, when $s = 8$, note that the coefficients of the corresponding polynomial $1 - 88\alpha + 92\alpha^2 - 872\alpha^3 + 1990\alpha^4 - 872\alpha^5 + 92\alpha^6 - 88\alpha^7 + \alpha^8$ read the same backward and forward. Does this extend to larger even integers s?

s	Minimal polynomial for $\alpha = (\beta(x, y))^8$, where $x = y = 1/s$
5	$1 + 52\alpha - 26\alpha^2 - 12\alpha^3 + \alpha^4$
6	$1 - 28\alpha + 6\alpha^2 - 28\alpha^3 + \alpha^4$
7	$-1 - 196\alpha + 1302\alpha^2 - 14756\alpha^3 + 15673\alpha^4 + 42168\alpha^5$
	$-111916\alpha^6 + 82264\alpha^7 - 35231\alpha^8 + 19852\alpha^9$
	$-2954\alpha^{10} - 308\alpha^{11} + 7\alpha^{12}$
8	$1 - 88\alpha + 92\alpha^2 - 872\alpha^3 + 1990\alpha^4 - 872\alpha^5 + 92\alpha^6$
	$-88\alpha^7 + \alpha^8$
9	$-1 - 534\alpha + 10923\alpha^2 - 342864\alpha^3 + 2304684\alpha^4$
	$-7820712\alpha^5 + 13729068\alpha^6 - 22321584\alpha^7 + 39775986\alpha^8$
	$-44431044\alpha^9 + 19899882\alpha^{10} + 3546576\alpha^{11}$
	$-8458020\alpha^{12} + 4009176\alpha^{13} - 273348\alpha^{14} + 121392\alpha^{15}$
	$-11385\alpha^{16} - 342\alpha^{17} + 3\alpha^{18}$
10	$1 - 216\alpha + 860\alpha^2 - 744\alpha^3 + 454\alpha^4 - 744\alpha^5 + 860\alpha^6$
	$-216\alpha^7 + \alpha^8$

Table 1: Sample of polynomials produced in an earlier study [5].

In 2015 Jason Kimberley of the University of Newcastle, Australia observed, based on these and a few other results, that the degree $m(s)$ of the minimal polynomial associated with the special case $x = y = 1/s$ appears to be given by the following number-theoretic rule [6]: Set $m(2) = 1/2$. Otherwise for primes p congruent to 1 modulo 4, set $m(p) = (p - 1)^2/4$, and for primes p congruent to 3 modulo 4, set $m(p) = (p^2 - 1)/4$. Then for any other positive integer s whose prime factorization is $s = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}$,

$$m(s) = 4^{r-1} \prod_{i=1}^{r} p_i^{2(e_i - 1)} \, m(p_i). \tag{3}$$

Does Kimberley's formula hold for larger integers s? Can it be proven?

In one earlier study [6], computations confirmed that Kimberley's formula holds for all integers s up to 40, and also for $s = 42, 44, 46, 48, 50, 52, 60$ and 64. These computations employed up to 64,000-digit precision, and produced polynomials with degrees up to 512 and integer coefficients up to 10^{229}. By doing Google searches on the coefficients of the resulting polynomials, the authors found a connection to a 2010 paper by Savin and Quarfoot [15].

These investigations ultimately led to a proof, given in [6], that Kimberley's formula (3) is valid, at least in the stated special case $x = y = 1/s$, and also a proof of the fact that when s is even, the minimal polynomial is palindromic, again in the stated special case $x = y = 1/s$. In [3], computational results were extended to cover the cases $x = 1/s$ and $y = q/s$, for all $1 \leq q < s/2$ and all $10 \leq s \leq 36$ and $s = 38$.

However, these earlier studies have invariably left more questions than answers. Most notably, the range of cases is still very limited. What about the full set of rationals $x = p/s, y = q/s$, with $1 \leq p \leq q < s/2$ and $\gcd(p, q, s) = 1$, for moderately large s, which is a vastly larger set than previously studied? Does Kimberley's formula or some modification of this rule still hold for the degrees? Does the palindromic property still hold when s is an even integer? Are there regularities and patterns among the minimal polynomials? Can any of these regularities be proved?

Along this line, the earlier study [5] mentioned the closely related function

$$\psi_2(x, y) = \frac{1}{\pi^2} \sum_{m,n \text{ even}} \frac{\cos(\pi m x)\cos(\pi n y)}{m^2 + n^2}. \tag{4}$$

This differs from $\phi_2(x, y)$ by replacing *odd* with *even*. As with $\phi_2(x, y)$, the authors of [5] found that when x and y are rational, then $\psi_2(x, y) = 1/\pi \cdot \log(\beta(x, y))$ for algebraic $\beta(x, y)$, and presented a handful of specific results. As the present study was being concluded, the author discovered formulas and computational techniques, based on results in [5], to find minimal polynomials for $\psi_2(x, y)$, and has obtained a set of intriguing initial results. However, the computations and analysis here are significantly more challenging than with $\phi_2(x, y)$, and require much higher numeric precision — up to 100,000 or more digits in some cases. Details will be provided in a separate report.

2 Finding minimal polynomials using an integer relation algorithm

Given an n-long input vector $x = (x_i, 1 \leq i \leq n)$ of real numbers, typically given as high-precision floating-point values, an integer relation algorithm attempts to find a nontrivial n-long vector of integers (a_i) such that

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = 0, \tag{5}$$

to within the tolerance of the numeric precision being used.

In the application discussed in this paper, where one is given a high-precision floating-point value α that is suspected to be an algebraic number of degree m, the procedure is to compute the $(m+1)$-long vector $x = (1, \alpha, \alpha^2, \cdots, \alpha^m)$ and then apply an integer relation algorithm to this vector. If an integer relation (a_i) is found for x that holds to the level of precision being used, then the resulting vector of integers may be the coefficients of an integer polynomial of degree m satisfied by α, subject to further verification.

As an illustration, suppose one suspects that the real constant α, whose numerical value to 40 digits is $2.119591269829175131329848334934687110628 0\ldots$, is an algebraic number of degree eight. After computing the vector $(1, \alpha, \alpha^2, \cdots, \alpha^8)$ and applying the multipair PSLQ integer relation algorithm, the relation $(1, -216, 860, -744, 454, -744, 860, -216, 1)$ is produced, so that α appears to satisfy the polynomial $1 - 216\alpha + 860\alpha^2 - 744\alpha^3 + 454\alpha^4 - 744\alpha^5 + 860\alpha^6 - 216\alpha^7 + \alpha^8 = 0$. This is the fourth of the polynomials listed in Table 1.

It should be noted that even if an integer relation algorithm finds that a given constant appears to numerically satisfy a polynomial of some degree, it is possible that the polynomial is not the *minimal* polynomial. The easiest way to ensure that it is minimal is to employ the polynomial factorization facility of *Maple* or *Mathematica* to verify that the resulting polynomial is irreducible. Alternatively, one may attempt to recover an integer relation with the degree reduced by one, and verify that no numerically significant relation is produced with this smaller degree.

Considerable care needs to be taken in these computations to have a relatively strong confidence that the results are not merely artifacts of approximate floating-point computation. This issue is discussed below in Section 4.

3 The four-level multipair PSLQ algorithm

The multipair PSLQ algorithm [7] is a more efficient and moderately parallelizable variant of PSLQ, a widely used integer relation algorithm. Variants of the LLL algorithm are also used [10]. Full statements of the PSLQ and multipair PSLQ algorithms are presented in the Appendix (Section 8).

In brief, given an n-long input vector x, iterations of PSLQ or multipair PSLQ algorithm generate a sequence of invertible $n \times n$ integer matrices A_k, their inverses B_k and real $n \times (n-1)$ matrices H_k, so that the reduced vector $y = B_k \cdot x$ has steadily smaller entries, until one entry of y is smaller than the specified epsilon, with the relation given in the corresponding row of B_k, or else precision has been exhausted.

Integer relation detection by any algorithm requires very high precision. It can be seen from a combinatorial argument that one must employ at least $n \cdot \max_i \log_{10} |a_i|$ digits, or else there is no chance of finding the underlying relation, since the true relation will be lost in a sea of numerical artifacts. Multipair PSLQ is very efficient with precision, compared with other integer relation algorithms, in the sense that it can typically detect a relation when the numeric precision is only a few percent higher than this minimum bound [7].

One earlier study [6] employed a three-level variable precision implementation of the multipair PSLQ algorithm, based on a scheme sketched in [7]. The more recent study [3] introduced a *four-level* scheme: (a) double precision (15 digits); (b) quad precision (32 digits); (c) medium precision (typically 100 to 1000 digits); and (d) full precision (typically 2,000 to 50,000 digits). With this scheme, almost all iterations of the multipair PSLQ algorithm are performed in ordinary double precision floating-point arithmetic. When an entry of the double precision y vector is smaller than 10^{-14}, or when an entry of the double precision A or B array exceeds 10^{13}, the medium precision arrays are updated from the double precision arrays using matrix multiplication via the formulas

$$y := \hat{B} \cdot y, \; B := \hat{B} \cdot B, \; A := \hat{A} \cdot A, \; H := \hat{A} \cdot H, \tag{6}$$

where the hat notation indicates the double precision arrays.

When an entry of the medium precision y vector is smaller than the medium precision epsilon, or when an entry of the medium precision A or B array exceeds the medium precision maximum, then the full precision arrays are updated from the medium precision arrays using similar formulas. Full details of this process will not be given here (see [7] and [3]), but suffice it to say that considerable care must be taken in this implementation to correctly detect when precision has been exhausted at each level, to reliably process the handoff to higher or lower levels of precision, and to recover from situations where an iteration must be abandoned due to precision overflow.

4 Numerical reliability of these computations

While these computations cannot certify a result in the formal mathematical sense, with some care the results can be quite reliable. The present author has found that the most reliable criterion is to note the size of the drop in $\min_{1 \le i \le n} |y_i|$ when the relation is detected. See, for instance, Figure 1, which is a plot of $\log_{10}(\min |y_i|)$ versus iteration number in a typical multipair PSLQ run. Note the sudden drop at iteration 199, from roughly 10^{-50} to 10^{-250}, a drop of approximately 200 orders of magnitude (the "epsilon" in this run was 10^{-250}, corresponding to 250-digit precision). In the calculations described in Section 6, all listed results exhibited a drop of at least 300 orders of magnitude at detection, and many results exhibited drops of well over 1000 orders of magnitude. In other words, the computed relations hold to hundreds or thousands of digits beyond the precision required to discover them.

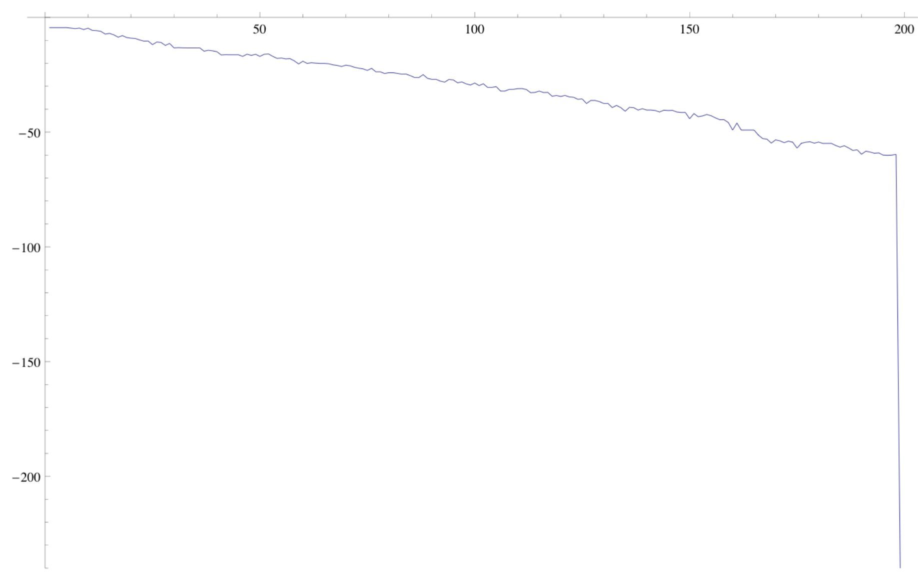

Figure 1: Plot of $\log_{10}(\min |y_i|)$ versus iteration number in a typical multipair PSLQ run. Note the sudden drop at iteration 199 to 10^{-250} (the "epsilon" in this run), a drop of approximately 200 orders of magnitude.

Additionally, the set of coefficients found for a Poisson polynomial is invariably crescent-shaped, with small coefficients at the start (often ± 1), a maximum size in the middle and small again at the end. Table 2, shown in a very small font, presents one representative minimal polynomial, namely the degree-100

-1
$-33300\,\alpha^{1}$
$-8029750\,\alpha^{2}$
$+1542379500\,\alpha^{3}$
$-293348778825\,\alpha^{4}$
$-28080295402080\,\alpha^{5}$
$-6629781350330800\,\alpha^{6}$
$+58722280292148000\,\alpha^{7}$
$-7197243181862115100\,\alpha^{8}$
$+182328184490475806640\,\alpha^{9}$
$+1047464861734017883208\,\alpha^{10}$
$+1806185067754657491760\,\alpha^{11}$
$-279458813247177178256270\,0\,\alpha^{12}$
$+43132729144820226386860000\,\alpha^{13}$
$-256411580898396643830482000\,\alpha^{14}$
$+1396700356872030954211266400\,\alpha^{15}$
$+9088870493426588389401095750\,\alpha^{16}$
$-73359906455331763936496182500\,0\,\alpha^{17}$
$+137041495749203861918095715885\,00\,\alpha^{18}$
$-157819237362090093545769457641320\,\alpha^{19}$
$+1321853578066306850140386747623398\,\alpha^{20}$
$-8347129396578224228862993897237280\,\alpha^{21}$
$+38212880291083166937296196441746800\,\alpha^{22}$
$-104022238521780343721711917072676000\,\alpha^{23}$
$-740320659453459577531079513512959 00\,\alpha^{24}$
$+2695579175102470750839898367021130608\,\alpha^{25}$
$-17580580889577528003991569683501290200\,\alpha^{26}$
$+74952710236773360685253102982788566000\,\alpha^{27}$
$-239080432366992082679933177546734267500\,\alpha^{28}$
$+594175511524312220492120757330324484000\,\alpha^{29}$
$-1200481201725892359379030176076009068400\,\alpha^{30}$
$+2370791216795342465482695956290874196000\,\alpha^{31}$
$-6958365090143220739303965994873839612375\,\alpha^{32}$
$+29340961737653182271816694472006500014100\,\alpha^{33}$
$-118465202788489603844178657796002545834090\,\alpha^{34}$
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$-3729748087865447793480560952748424761349600\,\alpha^{38}$
$+3877725636887659946158475367090512632719680\,\alpha^{39}$
$+846723867393661736729146827990176999234824\,\alpha^{40}$
$-15960428173973401824575411855716955373304480\,\alpha^{41}$
$+4711852072698041211852798100159583408407720\,0\,\alpha^{42}$
$-965603919643933972804021484874384326080040 00\,\alpha^{43}$
$+159442357501276027171661799158377030358819240\,\alpha^{44}$
$-222707090658310894211715578612740014870494016\,\alpha^{45}$
$+268336339651868593336977017479411555696276960\,\alpha^{46}$
$-280387518545606966350071856817231380200513600\,\alpha^{47}$
$+252610498127566038453865516310244367924136500\,\alpha^{48}$
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$+36400498321158716446352704351511072445758400\,\alpha^{53}$
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$-20184727758357355309386195853692979100127600\,\alpha^{58}$
$+10078674543222169818704320601337376637695840\,\alpha^{59}$
$-4216672803412456875435463721793522689401592\,\alpha^{60}$
$+1722566374653535529797841998864588575882560\,\alpha^{61}$
$-940122938034032272236892859372976755103200\,\alpha^{62}$
$+672500761678739947489218957049737685556800\,\alpha^{63}$
$-461584164001204318877039495371276336862015\,\alpha^{64}$
$+265582157437208924810452188771019969018068\,\alpha^{65}$
$-127220613278091048562581392274127224367210\,\alpha^{66}$
$+52420552698515377367407956646216340562900\,\alpha^{67}$
$-19432168360605247012293398573854487548375\,\alpha^{68}$
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$-2160843064282037086551775761827735549808\,\alpha^{70}$
$+617282737944009515075102146542202738080\,\alpha^{71}$
$-145142540389934636970079472194894728300\,\alpha^{72}$
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$-3264527861366792560817221584368055000\,\alpha^{74}$
$+473260992168563618404463109451174000\,\alpha^{75}$
$-178858482702935977606858126791577500\,\alpha^{76}$
$+66028976498454634546379541782684000\,\alpha^{77}$
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$-283939423573690705553213818042394\,\alpha^{80}$
$+27924016290268330781557752871960\,\alpha^{81}$
$-1890264935738308757140819195500\,\alpha^{82}$
$-197646050954205989496871249000\,\alpha^{83}$
$+92058025400867206020845473350\,\alpha^{84}$
$-15898833972285544673912261280\,\alpha^{85}$
$+1776153714893883182147476400\,\alpha^{86}$
$-134561774900933516621588000\,\alpha^{87}$
$+5713502521319481241590900\,\alpha^{88}$
$-280130362788084262599 20\,\alpha^{89}$
$-1074061784968018208453 6\,\alpha^{90}$
$+4164665365649271871 20\,\alpha^{91}$
$-5335454032157718300\,\alpha^{92}$
$-61275002107276000\,\alpha^{93}$
$-662049081315760\,\alpha^{94}$
$+123800803998624\,\alpha^{95}$
$-330633806665\,\alpha^{96}$
$-8699478100\,\alpha^{97}$
$+13986250\,\alpha^{98}$
$+6700\,\alpha^{99}$
$-1\,\alpha^{100}$

Table 2: Degree-100 minimal polynomial found for the case $x = y = 1/25$

polynomial found by the program for the case $x = y = 1/25$. It is typical of Poisson polynomials, in that the initial coefficient is -1, then coefficients ascend to a maximum size (here roughly 10^{45}), and then descend back down to -1.

This crescent-shaped pattern, from unity to huge to unity, is strong numerical evidence that the polynomial produced by the computer program is the true minimal polynomial associated with this case, and that all hardware, software and application code performed flawlessly, since otherwise it is *exceedingly unlikely* that the final set of coefficients would have this distinctive and highly improbable pattern. By contrast, in cases where the program fails to find a numerically significant relation, say due to a coding bug, insufficient degree or insufficient precision, the resulting erroneous integer coefficients typically are all roughly the same size, within one or two orders of magnitude. Visually speaking, an erroneous set of coefficients appears as a rectangle rather than a crescent.

5 High-level computational algorithm

It should be emphasized that numerical evaluation of $\phi_2(x, y)$ by the definition formula (1) is utterly impractical — millions of terms would be required to obtain even a few accurate digits. So a key breakthrough in the study of these Poisson polynomials was the discovery, by Jonathan Borwein (deceased 2016), that $\phi_2(x, y)$ can be numerically computed very rapidly using theta functions from the theory of elliptic functions.

In particular, here is the high-level algorithm employed to discover the Poisson polynomials in this study, updated from [6]:

1. Given rationals $x = p/s$ and $y = q/s$, typically satisfying $1 \le p \le q < s/2$ for $s \le 50$, with $\gcd(p, q, s) = 1$, select a conjectured minimal polynomial degree m (say from Kimberley's rule), a medium precision level P_1 digits, a full precision level P_2 digits and other parameters for the run.

2. Calculate $\phi_2(x, y)$ to P_2-digit precision using this formula from [5]:
$$\phi_2(x, y) = \frac{1}{2\pi} \log \left| \frac{\theta_2(z, q)\theta_4(z, q)}{\theta_1(z, q)\theta_3(z, q)} \right|, \tag{7}$$
where $q = e^{-\pi}$ and $z = \pi/2 \cdot (y + ix)$. Compute the four complex theta functions using the following rapidly convergent formulas from [9, p. 52]:
$$\begin{aligned} \theta_1(z, q) &= 2 \sum_{k=1}^{\infty} (-1)^{k-1} q^{(2k-1)^2/4} \sin((2k-1)z), \\ \theta_2(z, q) &= 2 \sum_{k=1}^{\infty} q^{(2k-1)^2/4} \cos((2k-1)z), \\ \theta_3(z, q) &= 1 + 2 \sum_{k=1}^{\infty} q^{k^2} \cos(2kz), \\ \theta_4(z, q) &= 1 + 2 \sum_{k=1}^{\infty} (-1)^k q^{k^2} \cos(2kz). \end{aligned} \tag{8}$$

3. Calculate $\alpha = \exp(8\pi\phi_2(x, y))$ and generate the $(m + 1)$-long vector $x = (1, \alpha, \alpha^2, \cdots, \alpha^m)$ to P_2-digit precision.

4. Apply a three- or four-level multipair PSLQ algorithm to find a numerically significant integer relation for x, if one exists (see Sec. 3 and 8).

5. If a numerically significant relation is not found, try again with a larger degree m or a higher precision P_2. If a tentative relation is found, employ the polynomial factorization facility in *Mathematica* or *Maple* to ensure that the resulting polynomial is irreducible. Alternatively, rerun the problem with the degree m reduced by one, to ensure that no numerically significant relation is found with this smaller degree.

We should add one note to the above algorithm: Recall from Table 1, covering the case $x = y = 1/s$ for $s \leq 10$, that when s is even, the corresponding minimal polynomial is always palindromic, i.e., coefficient $a_k = a_{m-k}$, where m is the degree. In [6] this observation was proved to hold in the special case $x = y = 1/s$. In the more general cases $x = p/s$ and $y = q/s$ discussed in this paper, computational results have confirmed that this palindromic property invariably holds when s is even, although there is no immediately evident route to prove this assertion.

After the initial results of one previous study [6] were obtained, Nitya Mani, then a student at Stanford University, observed [14] that if α satisfies a palindromic polynomial of degree m, then $\alpha + 1/\alpha$ satisfies a polynomial of degree $m/2$, and the degree-m polynomial satisfied by α can then be easily reconstructed from the degree-$m/2$ polynomial satisfied by $\alpha + 1/\alpha$. This fact has been exploited in the present study to *greatly* reduce run time for the cases when s is even.

6 Results and analysis

It can be shown from formula (1) that $\phi_2(a+x, b+y) = \phi_2(x, y)$, for any integers a, b, so there is no need to consider the cases $x = p/s$, $y = q/s$, where either p or q is negative or where either p or q exceeds s. In fact, by symmetry, it follows that one only need examine cases where $1 \leq p \leq q < s/2$. Further, one need not examine any cases where $\gcd(p, q, s) \neq 1$, since these cases are equivalent to cases with smaller p, q and s.

For this study, 2206 individual cases were run, using the algorithms and software described in the previous sections. These cases are: $x = p/s$, $y = q/s$, for $1 \leq p \leq q < s/2$, with $\gcd(p, q, s) = 1$, and for $10 \leq s \leq 36$, and also for $s = 38, 40, 42, 50$.

These runs were performed on an Apple Mac Studio computer with an M1 Max processor, 32 GByte memory and 10 cores. The application program implementing the algorithm described above in Section 5 was coded using a new thread-safe arbitrary precision package with a high-level language interface [2, 3], which achieves nearly the performance of the MPFR package [11], but with a much simpler software compilation process. All code was compiled using gfortran/gcc version 12.1.0.

For each of these cases, the computer runs exhibited a drop of at least 300 orders of magnitude at the iteration of detection, and in many cases exhibited drops of well over 1000 orders of magnitude. Thus the polynomials produced by these calculations hold to hundreds and in many cases thousands of digits beyond the precision required to discover them (see Section 3). *Mathematica* 13.2.0 was employed to confirm that each of these polynomials is irreducible.

Run times varied dramatically in these cases, from less than 0.01 second for the case $x = y = 1/10$, to 153,300 processor core seconds for the case $x = y = 8/31$. Some of the larger runs required up to 32,000-digit arithmetic. The minimal polynomials produced by these runs ranged up to degree 400 for many of the $s = 50$ cases, with coefficients as large as 10^{130}. The output files from these runs, with the full recovered polynomials, are quite voluminous but are available if desired from the author.

The three principal experimental findings of this study are the following:

1. A generalized Kimberley rule. Given $x = p/s$ and $y = q/s$, with $1 \le p \le q < s/2$ and $\gcd(p, q, s) = 1$, let $\phi_2(x, y)$ be defined as in (1), with $\alpha = \exp(8\pi\phi_2(x, y))$. Then the degree of the minimal polynomial of α is given by this rule:

1. If s is even or odd, and $p = q$, then the degree is given by Kimberley's rule (3).
2. Otherwise if s is odd, then the degree is given by Kimberley's rule, except for a few cases where the degree is half Kimberley's rule.
3. If s is even, and both p and q are odd, then the degree is given by Kimberley's rule, except for a few cases where the degree is half Kimberley's rule.
4. If s is even, with one of p or q even and the other odd, then the degree is given by *twice* Kimberley's rule, except for a few cases where the degree is equal to Kimberley's rule.

Table 3 presents a list of the *exceptions* to the generalized rule, or in other words those cases, mentioned above, where the degree of the recovered minimal polynomial was half the rule. At present, it is not immediately clear how this generalized rule may be proved, nor is there any hint as to a pattern followed by the exceptional degree cases.

2. Sharing of minimal polynomials. One particularly intriguing feature of the catalogue of results is that for a given integer s, many of the minimal polynomials corresponding to various $(x, y) = (p/s, q/s)$ cases are identical, even though the α numerical values are distinct. Tables 4 through 12, at the end of this paper, present a complete summary of these data extracted from the computer runs: each row of a given table lists (p, q) pairs, corresponding to $(x, y) = (p/s, q/s)$, whose minimal polynomials are identical.

In examining these data, one striking regularity is observed: For a given s, all the cases $x = y = p/s$, where $1 \le p < s/2$ and $\gcd(p, s) = 1$, share the same minimal polynomial. For example, when $s = 50$, the minimal polynomials for the cases $(1/50, 1/50), (3/50, 3/50), (7/50, 7/50), (9/50, 9/50), (11/50, 11/50),$ $(13/50, 13/50), (17/50, 17/50), (19/50, 19/50), (21/50, 21/50)$ and $(23/50, 23/50)$ are all identical. Note that this represents a complete set of $(p/50, p/50)$ with $1 \le p < 25$ and $\gcd(p, 50) = 1$.

This sharing feature has not been observed before in studies of Poisson polynomials, and the simplicity of this assertion suggests that it might well be amenable to further theoretical analysis. Doubtless other regularities exist in this large set of data, as yet unrecognized. The reader is invited to search these tables for additional interesting regularities.

3. The palindromic property. Also, as mentioned earlier in Section 5, the palindromic property was observed to hold for *all* minimal polynomials in the study associated with even s.

It should be emphasized again, however, that these three findings are experimental only; the present author has not been able to find formal proofs. But the relative simplicity of these assertions suggests that they may well be amenable to proof or disproof.

7 Conclusions and future research

While these computational results and observations are a useful start, it is clear that a fuller understanding of the structure and behavior of Poisson polynomials will require substantial additional effort, both theoretical and computational.

In particular, recall that the catalogued computations merely cover the cases $x = p/s$, $y = q/s$, for integers $1 \le p, q < s/2$ with $\gcd(p, q, s) = 1$, for $10 \le s \le 36$ and also for $s = 38, 40, 42$ and 50. To obtain further confidence in the three assertions mentioned in the previous section, these limits should be increased, which will require substantial additional computation. In addition, several questions still remain, such as what regularity is exhibited by the exceptional cases noted in Table 3, and, even more intriguingly, why certain sets of cases share the same minimal polynomial, as noted in Tables 4 through 12.

Note also that all of the research results and analyses above are for the simple two-dimensional case, namely $\phi_2(x, y)$. What happens in three or higher dimensions? At present this research is hampered by the lack of rapid and universally applicable computational algorithms similar to that listed above in Section 5 for $\phi_2(x, y)$. Clearly one important next step in this research is to re-examine earlier studies for hints to computational techniques and potential theoretical results applicable to higher dimensions.

In addition, the earlier study [5] mentioned the closely related function

$$\psi_2(x, y) = \frac{1}{\pi^2} \sum_{m,n \text{ even}} \frac{\cos(\pi m x) \cos(\pi n y)}{m^2 + n^2}. \tag{9}$$

This differs from $\phi_2(x, y)$ by replacing *odd* with *even.* As with $\phi_2(x, y)$, when x and y are rational, then $\psi_2(x, y) = 1/\pi \cdot \log(\beta(x, y))$ for algebraic $\beta(x, y)$. As mentioned in Section 1, the present author recently discovered formulas and computational techniques, based on results in [5], to find minimal polynomials for $\psi_2(x, y)$, and has obtained a set of intriguing initial results. However, the computations and analysis here are significantly more challenging than with $\phi_2(x, y)$, and require much higher numeric precision — up to 100,000 or more digits in some cases. Details will be provided in a separate report.

Integer relation computations are clearly central to this research and numerous other experimental mathematics research areas [4]. Thus research is needed in fundamentally more efficient integer relation algorithms. Perhaps, for instance, some variant of the Lenstra-Lenstra-Lovasz (LLL) algorithm [10] might be more efficient than multipair PSLQ for these large problems.

In this study, most of the individual cases were run on separate processor cores, taking advantage of the obvious parallelism over a large number of cases. But for larger problems, it may well be necessary to utilize parallel processing on a single case run. The arbitrary precision package employed in this study [2] is thread-safe, and the multi-pair PSLQ algorithm exhibits moderate parallelism for large problems. Speedups of 12X on a 16-core system have been achieved. While such speedups are welcome, a scheme to efficiently employ many more processors on a single case may be required to extend this research.

Note, by the way, that simply parallelizing a full-precision implementation of an algorithm such as multipair PSLQ, which may achieve large parallel speedups, is not helpful, since this would violate the principle that performance timings in parallel computing must be compared to the most efficient practical serial algorithm; otherwise parallel speedups are illusory [8]. At present, the most efficient practical serial algorithm is a three- or four-level implementation of the multipair PSLQ algorithm, so this or a similarly efficient serial algorithm must be the starting point for any effective parallel version.

8 Appendix: PSLQ and multi-pair PSLQ

Given an input vector $x = (x_j, 1 \leq j \leq n)$ of real numbers, typically given as high-precision floating-point values, the PSLQ and multipair PSLQ integer relation algorithms attempt to find a nontrivial vector of integers (a_j), if one exists, such that

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = 0, \tag{10}$$

to within the numeric precision being employed. The name "PSLQ" derives from its usage of a partial sum of squares vector and an LQ (lower-diagonal-orthogonal) matrix factorization.

The multipair PSLQ algorithm attempts to perform multiple iterations of the standard PSLQ algorithm in a single iteration. It is moderately parallelizable and has the added benefit of running faster, even on a single processor, and of being more efficient with precision.

As noted above in Section 3, integer relation detection by any algorithm requires very high precision. It can be seen from a combinatorial argument that one must employ at least $n \cdot \max_i \log_{10} |a_i|$ digits, or else there is no chance of finding the underlying relation, since the true relation will be lost in a sea of numerical artifacts. PSLQ, and, even more so, multipair PSLQ are very efficient with precision, compared with other integer relation algorithms, in the sense that they can typically detect a relation when the numeric precision is only a few percent higher than this minimum bound [7]. Since computing the input vector x to high precision is often very expensive, and the overall cost of performing an integer relation computation increases very rapidly as precision is increased, efficiency with precision is a very important metric of an integer relation algorithm. More complete details on these algorithms, including details on multilevel precision implementations, are given in [7] and [3].

8.1 The standard PSLQ algorithm

Let x be the n-long input real vector, let nint denote the nearest integer function (for exact half-integer values, define nint to be the integer with greater absolute value) and select $\gamma \geq \sqrt{4/3}$ (we typically select $\gamma = \sqrt{4/3}$, since this is the most efficient with precision).

Initialize:

1. For $j := 1$ to n: for $i := 1$ to n: if $i = j$ then set $A_{ij} := 1$ and $B_{ij} := 1$ else set $A_{ij} := 0$ and $B_{ij} := 0$; endfor; endfor.

2. For $k := 1$ to n: set $s_k := \sqrt{\sum_{j=k}^n x_j^2}$; endfor. Set $t = 1/s_1$. For $k := 1$ to n: set $y_k := tx_k$; $s_k := ts_k$; endfor.

3. Initial H: For $j := 1$ to $n-1$: for $i := 1$ to $j-1$: set $H_{ij} := 0$; endfor; set $H_{jj} := s_{j+1}/s_j$; for $i := j+1$ to n: set $H_{ij} := -y_i y_j/(s_j s_{j+1})$; endfor; endfor.

4. Reduce H: For $i := 2$ to n: for $j := i-1$ to 1 step -1: set $t := \mathrm{nint}(\mathrm{H_{ij}}/\mathrm{H_{jj}})$; and $y_j := y_j + ty_i$; for $k := 1$ to j: set $H_{ik} := H_{ik} - tH_{jk}$; endfor; for $k := 1$ to n: set $A_{ik} := A_{ik} - tA_{jk}$ and $B_{kj} := B_{kj} + tB_{ki}$; endfor; endfor; endfor.

Iteration: Repeat the following steps until precision has been exhausted or a relation has been detected.

1. Select m such that $\gamma^i |H_{ii}|$ is maximal when $i = m$.

2. Exchange the entries of y indexed m and $m+1$, the corresponding rows of A and H, and the corresponding columns of B.

3. Remove corner on H diagonal: If $m \le n-2$ then set $t_0 := \sqrt{H_{mm}^2 + H_{m,m+1}^2}$, $t_1 := H_{mm}/t_0$ and $t_2 := H_{m,m+1}/t_0$; for $i := m$ to n: set $t_3 := H_{im}$, $t_4 := H_{i,m+1}$, $H_{im} := t_1 t_3 + t_2 t_4$ and $H_{i,m+1} := -t_2 t_3 + t_1 t_4$; endfor; endif.

4. Reduce H: For $i := m+1$ to n: for $j := \min(i-1, m+1)$ to 1 step -1: set $t := \mathrm{nint}(\mathrm{H_{ij}}/\mathrm{H_{jj}})$ and $y_j := y_j + ty_i$; for $k := 1$ to j: set $H_{ik} := H_{ik} - tH_{jk}$; endfor; for $k := 1$ to n: set $A_{ik} := A_{ik} - tA_{jk}$ and $B_{kj} := B_{kj} + tB_{ki}$; endfor; endfor; endfor.

5. Norm bound: Compute $M := 1/\max_j |H_{jj}|$. Then there can exist no relation vector whose Euclidean norm is less than M.

6. Termination test: If the largest entry of A or B exceeds the level of numeric precision used, then precision is exhausted. If the smallest entry of the y vector is less than the detection threshold, and the dynamic range between that smallest entry and the largest entry of y is sufficiently large (say at least 30 orders of magnitude), then a relation may have been detected and is given in the corresponding row of B.

8.2 The multipair PSLQ algorithm

Let x be the n-long input real vector, let nint denote the nearest integer function as before and select $\gamma \ge \sqrt{4/3}$ (we typically select $\gamma = \sqrt{4/3}$, since this is the most efficient with precision) and $\beta = 0.4$.

Initialize:

1. For $j := 1$ to n: for $i := 1$ to n: if $i = j$ then set $A_{ij} := 1$ and $B_{ij} := 1$ else set $A_{ij} := 0$ and $B_{ij} := 0$; endfor; endfor.

2. For $k := 1$ to n: set $s_k := \sqrt{\sum_{j=k}^n x_j^2}$; endfor; set $t = 1/s_1$; for $k := 1$ to n: set $y_k := tx_k$; $s_k := ts_k$; endfor.

3. Initial H: For $j := 1$ to $n-1$: for $i := 1$ to $j-1$: set $H_{ij} := 0$; endfor; set $H_{jj} := s_{j+1}/s_j$; for $i := j+1$ to n: set $H_{ij} := -y_i y_j/(s_j s_{j+1})$; endfor; endfor.

Iteration: Repeat the following steps until precision has been exhausted or a relation has been detected.

1. Sort the entries of the $(n-1)$-long vector $\{\gamma^i |H_{ii}|\}$ in decreasing order, producing the sort indices.

2. Beginning at the sort index m_1 corresponding to the largest $\gamma^i |H_{ii}|$, select pairs of indices (m_i, m_i+1), where m_i is the sort index. If at any step either m_i or m_i+1 has already been selected or is outside the array bound, pass to the next index in the list. Continue until either βn pairs have been selected, or the list is exhausted. Let p denote the number of pairs actually selected in this manner.

3. For $i := 1$ to p, exchange the entries of y indexed m_i and m_i+1, and the corresponding rows of A, B and H; endfor.

4. Remove corners on H diagonal: For $i := 1$ to p: if $m_i \le n-2$ then set $t_0 := \sqrt{H^2_{m_i,m_i} + H^2_{m_i,m_i+1}}$, $t_1 := H_{m_i,m_i}/t_0$ and $t_2 := H_{m_i,m_i+1}/t_0$; for $i := m_i$ to n: set $t_3 := H_{i,m_i}$; $t_4 := H_{i,m_i+1}$; $H_{i,m_i} := t_1 t_3 + t_2 t_4$; and $H_{i,m_i+1} := -t_2 t_3 + t_1 t_4$; endfor; endif; endfor.

5. Reduce H: For $i := 2$ to n: for $j := 1$ to $n-i+1$: set $l := i+j-1$; for $k := j+1$ to $l-1$: set $H_{lj} := H_{lj} - T_{lk} H_{kj}$; endfor; set $T_{lj} := \mathrm{nint}(\mathrm{H_{lj}}/\mathrm{H_{jj}})$ and $H_{lj} := H_{lj} - T_{lj} H_{jj}$; endfor; endfor.

6. Update y: For $j := 1$ to $n-1$: for $i := j+1$ to n: set $y_j := y_j + T_{ij} y_i$; endfor; endfor.

7. Update A and B: For $k := 1$ to n: for $j := 1$ to $n-1$: for $i := j+1$ to n: set $A_{ik} := A_{ik} - T_{ij} A_{jk}$ and $B_{jk} := B_{jk} + T_{ij} B_{ik}$; endfor; endfor; endfor.

8. Norm bound: Compute $M := 1/\max_j |H_{jj}|$. Then there can exist no relation vector whose Euclidean norm is less than M.

9. Termination test: If the largest entry of A or B exceeds the level of numeric precision used, then precision is exhausted. If the smallest entry of the y vector is less than the detection threshold, and the dynamic range between that smallest entry and the largest entry of y is sufficiently large (say at least 30 orders of magnitude), then a relation may have been detected and is given in the corresponding row of B.

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s	Gen. Kimb. degrees	Exceptional cases: (p, q):degree, where $(x, y) = (p/s, q/s)$
10	8/16	(1,3):4
11	30	None
12	16/32	(1,5):8, (2,3):16, (3,4):16
13	36	(1,5):18, (2,3):18, (4,6):18
14	24/48	None
15	32	(1,4): 16, (2,7): 16, (3,5):16, (5,6):16
16	32/64	(1,7):16, (3,5):16
17	64	(1,4):32, (2,8):32, (3,5):32, (6,7):32
18	36/72	None
19	90	None
20	32/64	(1,9):16, (2,5):32, (3,7):16, (4,5):32, (5,6):32, (5,8):32
21	96	(1,8):48, (2,5):48, (3,7):48, (4,10):48, (6,7):48, (7,9):48
22	60/120	None
23	132	None
24	64/128	(1,5):32, (1,7):32, (1,11):32, (3,4):64, (3,8):64, (4,9):64, (5,7):32, (5,11):32, (7,11):32, (8,9):64
25	100	(1,7):50, (2,11):50, (3,4):50, (6,8):50, (9,12):50
26	72/144	(1,5):36, (3,11):36, (7,9):36
27	162	None
28	96/192	(1,13):48, (2,7):96, (3,11):48, (4,7):96, (5,9):48, (6,7):96
29	196	(1,12):98, (2,5):98, (3,7):98, (4,10):98, (6,14):98, (8,9):98, (11,13):98
30	64/128	(1,11):32, (3,5):32, (3,10):64, (5,6):64, (5,9):32, (5,12):64, (7,13):32, (9,10):64
31	240	None
32	128/256	(1,15):64, (3,13):64, (5,11):64, (7,9):64
33	240	(1,10):120, (2,13):120, (3,11):120, (4,7):120, (5,16):120, (6,11):120, (8,14):120, (9,11):120, (11:12):120, (11,15):120
34	128/256	(1,13):64, (3,5):64, (7,11):64, (9,15):64
35	192	(1,6):96, (2,12):96, (3,17):96, (4,11):96, (5,7):96, (5,14):96, (7,10):96, (7,15):96, (8,13):96, (9,16):96, (10,14):96, (14,15):96
36	144/288	(1,17):72, (2,9):144, (4,9):144, (5,13):72, (7,11):72, (8,9):144, (9,10):144, (9,14):144, (9,16):144
38	180/360	None
40	128/256	(1,9):64, (1,11):64, (1,19):64, (3,7):64, (3,13):64, (3,17):64, (4,5):128, (4,15):128, (5,8):128, (5,12):128, (5,16):128, (7,13):64, (7,17):64, (8,15):128, (9,11):64, (11,19):64, (12,15):128, (13,17):64, (15,16):128
42	192/384	(1,13):96, (3,7):96, (3,14):192, (5,19):96, (6,7):192, (7,9):96, (7,12):192, (7,15):96, (7,18):192, (9,14):192, (11,17):96, (14,15):192
50	200/400	(1,7):100, (3,21):100, (9,13):100, (11,23):100, (17,19):100

Table 3: Table of exceptions to the generalized Kimberley degree rule, or in other words, instances where the actual degree is half the generalized Kimberley rule

s	Sets of (p, q) indices sharing the same minimal polynomial; $(x, y) = (p/s, q/s)$
10	(1, 4), (2, 3) (1, 1), (3, 3) (1, 2), (3, 4) (1, 3)
11	(1, 1), (2, 2), (3, 3), (4, 4), (5, 5) (1, 3), (1, 5), (2, 4), (3, 5) (1, 2), (1, 4), (2, 3), (2, 5), (3, 4), (4, 5)
12	(1, 5) (1, 2), (2, 5) (3, 4) (1, 4), (4, 5) (1, 3), (3, 5) (1, 1), (5, 5) (2, 3)
13	(1, 3), (2, 4), (2, 6), (3, 5) (1, 5), (4, 6) (1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6) (1, 2), (1, 4), (1, 6), (2, 5), (3, 4), (3, 6), (4, 5), (5, 6) (2, 3)
14	(1, 4), (2, 3), (5, 6) (1, 2), (3, 6), (4, 5) (1, 6), (2, 5), (3, 4) (1, 3), (1, 5), (3, 5) (1, 1), (3, 3), (5, 5)
15	(1, 3), (1, 7), (2, 4), (2, 6) (1, 6), (2, 3), (2, 5), (4, 5) (3, 5) (1, 1), (2, 2), (4, 4), (7, 7) (1, 2), (3, 4), (4, 7), (6, 7) (1, 4), (2, 7), (5, 6) (1, 5), (3, 7), (4, 6), (5, 7)
16	(1, 6), (2, 3), (2, 5), (6, 7) (1, 1), (3, 3), (5, 5), (7, 7) (1, 7), (3, 5) (1, 2), (2, 7), (3, 6), (5, 6) (1, 4), (3, 4), (4, 5), (4, 7) (1, 3), (1, 5), (3, 7), (5, 7)
17	(1, 3), (2, 4), (2, 6), (3, 7), (4, 8), (5, 7) (1, 2), (1, 6), (1, 8), (2, 5), (3, 6), (3, 8), (4, 5), (4, 7), (5, 6), (7, 8) (2, 8), (3, 5) (1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6), (7, 7), (8, 8) (1, 5), (1, 7), (4, 6), (6, 8) (2, 3), (2, 7), (3, 4), (5, 8) (1, 4), (6, 7)

Table 4: Minimal polynomial groupings for $s = 10$ through 17. Each row lists cases sharing the same minimal polynomial.

s	Sets of (p, q) indices sharing the same minimal polynomial; $(x, y) = (p/s, q/s)$
18	(1, 5), (1, 7), (5, 7)
	(1, 6), (5, 6), (6, 7)
	(1, 2), (4, 7), (5, 8)
	(1, 1), (5, 5), (7, 7)
	(1, 3), (3, 5), (3, 7)
	(2, 3), (3, 4), (3, 8)
	(1, 8), (2, 7), (4, 5)
	(1, 4), (2, 5), (7, 8)
19	(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6), (7, 7), (8, 8), (9, 9)
	(1, 5), (1, 7), (2, 8), (3, 5), (3, 7), (4, 6), (6, 8), (7, 9)
	(1, 3), (1, 9), (2, 4), (2, 6), (3, 9), (4, 8), (5, 7), (5, 9)
	(1, 4), (1, 8), (2, 3), (2, 5), (2, 9), (3, 4), (4, 9), (5, 6), (6, 9), (7, 8)
	(1, 2), (1, 6), (2, 7), (3, 6), (3, 8), (4, 5), (4, 7), (5, 8), (6, 7), (8, 9)
20	(2, 5), (5, 6)
	(1, 3), (1, 7), (3, 9), (7, 9)
	(1, 4), (3, 8), (4, 9), (7, 8)
	(4, 5), (5, 8)
	(1, 8), (3, 4), (4, 7), (8, 9)
	(1, 1), (3, 3), (7, 7), (9, 9)
	(1, 2), (2, 9), (3, 6), (6, 7)
	(1, 5), (3, 5), (5, 7), (5, 9)
	(1, 9), (3, 7)
	(1, 6), (2, 3), (2, 7), (6, 9)
21	(1, 8), (2, 5), (6, 7)
	(1, 1), (2, 2), (4, 4), (5, 5), (8, 8), (10,10)
	(1, 6), (2, 7), (2, 9), (3, 4), (3,10), (4, 7), (7, 8), (7,10)
	(1, 5), (1, 9), (2, 8), (2,10), (3, 5), (4, 6), (6,10), (8,10)
	(1, 2), (1,10), (3, 8), (4, 9), (5, 6), (5, 8), (5,10), (9,10)
	(1, 7), (5, 7), (5, 9), (6, 8)
	(1, 4), (2, 3), (4, 5), (8, 9)
	(1, 3), (2, 4), (2, 6), (4, 8)
	(3, 7), (4,10), (7, 9)
22	(1, 5), (1, 9), (3, 5), (3, 7), (7, 9)
	(1,10), (2, 9), (3, 8), (4, 7), (5, 6)
	(1, 3), (1, 7), (3, 9), (5, 7), (5, 9)
	(1, 4), (2, 5), (3,10), (6, 7), (8, 9)
	(1, 1), (3, 3), (5, 5), (7, 7), (9, 9)
	(1, 8), (2, 3), (4, 5), (6, 9), (7,10)
	(1, 6), (2, 7), (3, 4), (5, 8), (9,10)
	(1, 2), (3, 6), (4, 9), (5,10), (7, 8)

Table 5: Minimal polynomial groupings for $s = 18$ through 22. Each row lists cases sharing the same minimal polynomial.

s	Sets of (p,q) indices sharing the same minimal polynomial; $(x,y)=(p/s,q/s)$
23	(2, 8), (3, 5), (3,11), (5, 7), (6,10)
	(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6), (7, 7), (8, 8), (9, 9), (10,10), (11,11)
	(1, 5), (1, 7), (1, 9), (2,10), (3, 7), (4, 6), (4,10), (5,11), (6, 8), (7,11), (8,10), (9,11)
	(1, 3), (1,11), (2, 4), (2, 6), (3, 9), (4, 8), (5, 9), (7, 9)
	(1, 2), (1, 8), (2, 7), (3, 6), (3,10), (4, 9), (4,11), (5, 6), (5, 8), (5,10), (6,11), (7, 8), (7,10), (10,11)
	(1, 4), (1, 6), (2,11), (4, 7), (8, 9), (9,10)
	(1,10), (2, 3), (2, 5), (2, 9), (3, 4), (3, 8), (4, 5), (6, 7), (6, 9), (8,11)
24	(1, 2), (2,11), (5,10), (7,10)
	(1, 6), (5, 6), (6, 7), (6,11)
	(1, 4), (4, 5), (4, 7), (4,11)
	(2, 3), (2, 9), (3,10), (9,10)
	(1, 7), (5,11)
	(1, 5), (7,11)
	(1, 9), (3, 5), (3,11), (7, 9)
	(1,11), (5, 7)
	(1, 3), (3, 7), (5, 9), (9,11)
	(1,10), (2, 5), (2, 7), (10,11)
	(1, 8), (5, 8), (7, 8), (8,11)
	(3, 4), (4, 9)
	(1, 1), (5, 5), (7, 7), (11,11)
	(3, 8), (8, 9)
25	(1, 3), (2, 4), (2, 6), (3, 9), (3,11), (4, 8), (4,12), (6,12), (7, 9), (7,11)
	(2, 8), (2,12), (3, 5), (3, 7), (4,10), (5, 7), (6,10), (9,11)
	(1, 7), (6, 8)
	(1, 2), (1, 8), (1,12), (2, 9), (3, 6), (4, 7), (6, 7), (8, 9), (8,11), (11,12)
	(1, 1), (2, 2), (3, 3), (4, 4), (6, 6), (7, 7), (8, 8), (9, 9), (11,11), (12,12)
	(2, 3), (2, 7), (3, 8), (3,10), (4, 5), (4,11), (5, 6), (6, 9), (7,10), (7,12)
	(1, 4), (1, 6), (1,10), (2, 5), (3,12), (4, 9), (5, 8), (5,12), (6,11), (7, 8), (9,10), (10,11)
	(1, 5), (1, 9), (1,11), (2,10), (4, 6), (5, 9), (5,11), (8,10), (8,12), (10,12)
	(2,11), (3, 4), (9,12)
26	(1,10), (2, 5), (3, 4), (6,11), (7, 8), (9,12)
	(1,12), (2,11), (3,10), (4, 9), (5, 8), (6, 7)
	(1, 8), (2, 3), (4, 7), (5,12), (6, 9), (10,11)
	(1, 2), (3, 6), (4,11), (5,10), (7,12), (8, 9)
	(1, 6), (2, 9), (3, 8), (4, 5), (7,10), (11,12)
	(1, 4), (2, 7), (3,12), (5, 6), (8,11), (9,10)
	(1, 5), (3,11), (7, 9)
	(1, 7), (1,11), (3, 5), (3, 7), (5, 9), (9,11)
	(1, 3), (1, 9), (3, 9), (5, 7), (5,11), (7,11)
	(1, 1), (3, 3), (5, 5), (7, 7), (9, 9), (11,11)

Table 6: Minimal polynomial groupings for $s = 23$ through 26. Each row lists cases sharing the same minimal polynomial.

s	Sets of (p, q) indices sharing the same minimal polynomial; $(x, y) = (p/s, q/s)$
27	(1, 3), (1,13), (2, 4), (2, 6), (4, 8), (4,12), (5,11), (7,13)
	(1, 7), (2, 8), (2,12), (3, 5), (3,13), (5, 7), (6, 8), (6,10)
	(1, 9), (5, 9), (5,13), (7, 9), (7,11), (8,10), (9,11), (9,13)
	(1, 1), (2, 2), (4, 4), (5, 5), (7, 7), (8, 8), (10,10), (11,11), (13,13)
	(1, 5), (1,11), (2,10), (3, 7), (3,11), (4, 6), (4,10), (8,12), (10,12), (11,13)
	(1,12), (2, 3), (2, 5), (4, 7), (5, 6), (6,13), (7, 8), (8,13)
	(1, 4), (1, 6), (2,13), (3, 4), (4,11), (5, 8), (7,12), (10,11), (10,13), (11,12)
	(1, 2), (3, 8), (3,10), (5,10), (5,12), (6, 7), (6,11), (7,10), (8,11), (12,13)
	(1, 8), (1,10), (2, 7), (2, 9), (2,11), (4, 5), (4, 9), (4,13), (8, 9), (9,10)
28	(1, 1), (3, 3), (5, 5), (9, 9), (11,11), (13,13)
	(1,13), (3,11), (5, 9)
	(1, 4), (3,12), (4,13), (5, 8), (8, 9), (11,12)
	(1,12), (3, 8), (4, 5), (4, 9), (8,11), (12,13)
	(1, 2), (2,13), (3, 6), (5,10), (6,11), (9,10)
	(1, 6), (2, 5), (2, 9), (3,10), (6,13), (10,11)
	(4, 7), (7, 8), (7,12)
	(1,10), (2, 3), (2,11), (5, 6), (6, 9), (10,13)
	(2, 7), (6, 7), (7,10)
	(1, 3), (1, 9), (3, 9), (5,11), (5,13), (11,13)
	(1, 5), (1,11), (3, 5), (3,13), (9,11), (9,13)
	(1, 8), (3, 4), (4,11), (5,12), (8,13), (9,12)
	(1, 7), (3, 7), (5, 7), (7, 9), (7,11), (7,13)
29	(1, 4), (1, 8), (2, 7), (2,13), (3, 4), (3, 8), (3,12), (4,13), (5, 6), (6,13), (7,10), (9,12), (9,14), (10,11), (11,14), (12,13)
	(1, 7), (1,11), (2, 8), (2,14), (3, 5), (4,14), (5, 9), (5,11), (6, 8), (6,10), (7, 9), (10,12)
	(1, 3), (2, 4), (2, 6), (3, 9), (3,13), (4, 8), (4,12), (6,12), (7,11), (9,11)
	(1, 6), (2, 3), (2,11), (3,10), (3,14), (4, 5), (4, 7), (4, 9), (6, 7), (6, 9), (8,11), (11,12)
	(3, 7), (4,10), (6,14), (11,13)
	(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6), (7, 7), (8, 8), (9, 9), (10,10), (11,11), (12,12), (13,13), (14,14)
	(1, 5), (1, 9), (1,13), (2,10), (2,12), (3,11), (4, 6), (5, 7), (5,13), (7,13), (8,10), (8,12), (8,14), (9,13), (10,14), (12,14)
	(1, 2), (1,10), (1,14), (2, 9), (3, 6), (4,11), (5, 8), (5,10), (5,12), (5,14), (6,11), (7, 8), (7,12), (7,14), (8,13), (9,10), (10,13), (13,14)
	(1,12), (2, 5), (8, 9)

Table 7: Minimal polynomial groupings for $s = 27$ through 29. Each row lists cases sharing the same minimal polynomial.

s	Sets of (p, q) indices sharing the same minimal polynomial; $(x, y) = (p/s, q/s)$
30	(1,14), (2,13), (4,11), (7, 8)
	(3, 5), (5, 9)
	(1,12), (6, 7), (6,13), (11,12)
	(1,11), (7,13)
	(1, 1), (7, 7), (11,11), (13,13)
	(1, 2), (4,13), (7,14), (8,11)
	(2, 3), (3, 8), (4, 9), (9,14)
	(2, 9), (3, 4), (3,14), (8, 9)
	(1, 5), (5, 7), (5,11), (5,13)
	(1, 9), (3, 7), (3,13), (9,11)
	(1, 6), (6,11), (7,12), (12,13)
	(5, 6), (5,12)
	(1, 4), (2, 7), (8,13), (11,14)
	(1, 3), (3,11), (7, 9), (9,13)
	(1, 8), (2,11), (4, 7), (13,14)
	(1,10), (7,10), (10,11), (10,13)
	(2, 5), (4, 5), (5, 8), (5,14)
	(3,10), (9,10)
	(1, 7), (1,13), (7,11), (11,13)
31	(1, 7), (1, 9), (2,14), (6, 8), (7,13), (8,10), (11,15)
	(1, 3), (1,15), (2, 4), (2, 6), (3, 9), (4, 8), (4,12), (5,13), (5,15), (6,12), (9,11), (9,13)
	(1, 5), (1,11), (2,10), (2,12), (3,11), (3,13), (3,15), (4, 6), (5, 7), (7,11), (7,15), (8,12), (8,14), (10,12), (10,14), (13,15)
	(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6), (7, 7), (8, 8), (9, 9), (10,10), (11,11), (12,12), (13,13), (14,14), (15,15)
	(1,13), (2, 8), (3, 5), (3, 7), (4,10), (4,14), (5, 9), (5,11), (6,10), (6,14), (7, 9), (9,15), (11,13), (12,14)
	(1, 4), (1, 8), (1,12), (2, 5), (2, 7), (2,15), (3, 8), (3,12), (4,15), (6, 7), (6,15), (8,11), (9,10), (10,13), (11,12), (13,14)
	(2,13), (3, 4), (3,10), (4, 5), (5,14), (6,11), (9,12), (12,15)
	(1, 2), (1,10), (2,11), (3, 6), (3,14), (4, 9), (5,10), (5,12), (6,13), (7, 8), (7,10), (7,12), (7,14), (8,13), (8,15), (10,11), (11,14), (14,15)
	(1, 6), (1,14), (2, 3), (2, 9), (4, 7), (4,11), (4,13), (5, 6), (5, 8), (6, 9), (8, 9), (9,14), (10,15), (12,13)
32	(1,14), (2, 7), (2, 9), (3,10), (5, 6), (6,11), (10,13), (14,15)
	(1, 6), (2, 5), (2,11), (3,14), (6,15), (7,10), (9,10), (13,14)
	(1, 1), (3, 3), (5, 5), (7, 7), (9, 9), (11,11), (13,13), (15,15)
	(1,12), (3, 4), (4, 5), (4,11), (4,13), (7,12), (9,12), (12,15)
	(1, 4), (3,12), (4, 7), (4, 9), (4,15), (5,12), (11,12), (12,13)
	(1, 8), (3, 8), (5, 8), (7, 8), (8, 9), (8,11), (8,13), (8,15)
	(1, 2), (2,15), (3, 6), (5,10), (6,13), (7,14), (9,14), (10,11)
	(1, 5), (1,13), (3, 7), (3,15), (5, 7), (9,11), (9,13), (11,15)
	(1,15), (3,13), (5,11), (7, 9)
	(1,10), (2, 3), (2,13), (5,14), (6, 7), (6, 9), (10,15), (11,14)
	(1, 7), (1, 9), (3, 5), (3,11), (5,13), (7,15), (9,15), (11,13)
	(1, 3), (1,11), (3, 9), (5, 9), (5,15), (7,11), (7,13), (13,15)

Table 8: Minimal polynomial groupings for $s = 30, 31, 32$. Each row lists cases sharing the same minimal polynomial.

s	Sets of (p,q) indices sharing the same minimal polynomial; $(x,y)=(p/s,q/s)$
33	(1,12), (2, 9), (2,11), (3, 8), (3,14), (4,11), (4,15), (5, 6), (8,11), (10,11), (11,14), (11,16)
	(1,11), (5,11), (6,16), (7,11), (7,15), (9,13), (10,12), (11,13)
	(3,11), (8,14), (9,11), (11,15)
	(1, 7), (1, 9), (2,14), (3, 7), (4,10), (6, 8), (6,14), (8,10), (12,16), (13,15)
	(1, 4), (1, 6), (1, 8), (3,16), (4, 9), (7,10), (8,15), (12,13), (13,14), (14,15)
	(1,10), (2,13), (4, 7), (5,16), (6,11), (11,12)
	(1, 3), (2, 4), (2, 6), (4, 8), (4,12), (5,15), (7,13), (8,16)
	(2, 3), (2, 7), (4,13), (5, 8), (6, 7), (7, 8), (9,16), (10,15)
	(1,14), (2, 5), (2,15), (3, 4), (4, 5), (5,12), (7,16), (8,13), (9,10), (13,16)
	(1, 2), (1,16), (3,10), (5,10), (5,14), (6,13), (7,12), (7,14), (8, 9), (9,14), (10,13), (15,16)
	(2, 8), (2,12), (2,16), (3, 5), (4,16), (5, 7), (5,13), (6,10), (7, 9), (10,14)
	(1, 5), (1,13), (1,15), (2,10), (3,13), (4, 6), (4,14), (5, 9), (8,12), (10,16), (12,14), (14,16)
	(1, 1), (2, 2), (4, 4), (5, 5), (7, 7), (8, 8), (10,10), (13,13), (14,14), (16,16)
34	(1,14), (2, 5), (3, 8), (4, 7), (6,15), (9,10), (11,16), (12,13)
	(1, 3), (1,11), (3, 9), (5,13), (5,15), (7, 9), (7,13), (11,15)
	(1,10), (2, 7), (3, 4), (5,16), (6,13), (8,11), (9,12), (14,15)
	(1,16), (2,15), (3,14), (4,13), (5,12), (6,11), (7,10), (8, 9)
	(1, 1), (3, 3), (5, 5), (7, 7), (9, 9), (11,11), (13,13), (15,15)
	(1, 4), (2, 9), (3,12), (5,14), (6, 7), (8,15), (10,11), (13,16)
	(1, 2), (3, 6), (4,15), (5,10), (7,14), (8,13), (9,16), (11,12)
	(1, 9), (1,15), (3, 7), (3,11), (5, 7), (5,11), (9,13), (13,15)
	(1,13), (3, 5), (7,11), (9,15)
	(1, 5), (1, 7), (3,13), (3,15), (5, 9), (7,15), (9,11), (11,13)
	(1,12), (2, 3), (4,11), (5, 8), (6, 9), (7,16), (10,15), (13,14)
	(1, 6), (2,11), (3,16), (4, 5), (7, 8), (9,14), (10,13), (12,15)
	(1, 8), (2,13), (3,10), (4, 9), (5, 6), (7,12), (11,14), (15,16)
35	(2,12), (3,17), (5, 7), (7,15), (10,14)
	(1, 7), (1,13), (2,14), (2,16), (3,11), (6, 8), (7, 9), (7,11), (8,14), (11,17), (12,14), (12,16)
	(1, 2), (1,12), (2,11), (3, 6), (3,16), (4,13), (6,17), (8, 9), (8,11), (11,12), (13,16), (16,17)
	(1, 1), (2, 2), (3, 3), (4, 4), (6, 6), (8, 8), (9, 9), (11,11), (12,12), (13,13), (16,16), (17,17)
	(1,16), (2, 3), (2,13), (4, 9), (4,15), (5, 6), (5, 8), (6, 9), (8,17), (9,10), (11,16), (12,17)
	(1, 9), (2, 8), (3, 5), (4,16), (5,11), (5,17), (6,10), (6,16), (8,10), (9,11), (9,15), (13,17)
	(1,15), (3, 7), (4,10), (4,14), (5, 9), (6,14), (7,13), (7,17), (13,15), (14,16)
	(1,14), (2, 5), (2, 7), (3,10), (5,12), (5,16), (6,15), (7, 8), (7,12), (8,15), (9,14), (10,11), (10,17), (11,14)
	(1, 8), (2, 9), (3, 4), (3,14), (4, 7), (4,17), (6, 7), (6,13), (7,16), (9,12), (13,14), (14,17)
	(1, 3), (1,17), (2, 4), (2, 6), (3, 9), (4, 8), (4,12), (6,12), (8,16), (9,13), (9,17), (11,13)
	(1, 5), (1,11), (2,10), (3,13), (3,15), (4, 6), (5,13), (8,12), (10,12), (10,16), (11,15), (15,17)
	(1, 4), (1,10), (2,15), (2,17), (3, 8), (3,12), (4, 5), (6,11), (10,13), (12,13), (12,15), (15,16)
	(1, 6), (4,11), (5,14), (7,10), (8,13), (9,16), (14,15)

Table 9: Minimal polynomial groupings for $s = 33, 34, 35$. Each row lists cases sharing the same minimal polynomial.

s	Sets of (p, q) indices sharing the same minimal polynomial; $(x, y) = (p/s, q/s)$
36	(2, 3), (2,15), (3,10), (3,14), (10,15), (14,15)
	(1, 6), (5, 6), (6, 7), (6,11), (6,13), (6,17)
	(1, 3), (3,11), (3,13), (5,15), (7,15), (15,17)
	(1, 9), (5, 9), (7, 9), (9,11), (9,13), (9,17)
	(1,17), (5,13), (7,11)
	(1,14), (2, 5), (2,13), (7,10), (10,11), (14,17)
	(1,10), (2, 7), (2,11), (5,14), (10,17), (13,14)
	(1, 1), (5, 5), (7, 7), (11,11), (13,13), (17,17)
	(2, 9), (9,10), (9,14)
	(1, 5), (1, 7), (5,11), (7,13), (11,17), (13,17)
	(1, 8), (4, 5), (4,13), (7,16), (8,17), (11,16)
	(4, 9), (8, 9), (9,16)
	(1,11), (1,13), (5, 7), (5,17), (7,17), (11,13)
	(1, 2), (2,17), (5,10), (7,14), (10,13), (11,14)
	(1,15), (3, 5), (3, 7), (3,17), (11,15), (13,15)
	(1,12), (5,12), (7,12), (11,12), (12,13), (12,17)
	(3, 4), (3, 8), (3,16), (4,15), (8,15), (15,16)
	(1,16), (4, 7), (4,11), (5, 8), (8,13), (16,17)
	(1, 4), (4,17), (5,16), (7, 8), (8,11), (13,16)
38	(1, 6), (2,13), (3,18), (4, 7), (5, 8), (9,16), (10,11), (12,17), (14,15)
	(1, 3), (1,13), (3, 9), (5,11), (5,15), (7,15), (7,17), (9,11), (13,17)
	(1,12), (2, 3), (4,13), (5,16), (6, 9), (7, 8), (10,15), (11,18), (14,17)
	(1,10), (2,15), (3, 8), (4,11), (5,12), (6, 7), (9,14), (13,16), (17,18)
	(1,16), (2, 7), (3,10), (4, 5), (6,17), (8, 9), (11,14), (12,15), (13,18)
	(1, 7), (1,11), (3, 5), (3,17), (5,17), (7,11), (9,13), (9,15), (13,15)
	(1,18), (2,17), (3,16), (4,15), (5,14), (6,13), (7,12), (8,11), (9,10)
	(1,14), (2,11), (3, 4), (5, 6), (7,16), (8,13), (9,12), (10,17), (15,18)
	(1, 9), (1,17), (3,11), (3,13), (5, 7), (5, 9), (7,13), (11,15), (15,17)
	(1, 1), (3, 3), (5, 5), (7, 7), (9, 9), (11,11), (13,13), (15,15), (17,17)
	(1, 2), (3, 6), (4,17), (5,10), (7,14), (8,15), (9,18), (11,16), (12,13)
	(1, 5), (1,15), (3, 7), (3,15), (5,13), (7, 9), (9,17), (11,13), (11,17)
	(1, 4), (2, 9), (3,12), (5,18), (6,11), (7,10), (8,17), (13,14), (15,16)
	(1, 8), (2, 5), (3,14), (4, 9), (6,15), (7,18), (10,13), (11,12), (16,17)

Table 10: Minimal polynomial groupings for $s = 36, 38$. Each row lists cases sharing the same minimal polynomial.

s	Sets of (p, q) indices sharing the same minimal polynomial; $(x, y) = (p/s, q/s)$
40	(2, 5), (2,15), (5, 6), (5,14), (5,18), (6,15), (14,15), (15,18)
	(5, 8), (5,16), (8,15), (15,16)
	(1,19), (3,17), (7,13), (9,11)
	(1, 3), (1,13), (3, 9), (7,11), (7,19), (9,13), (11,17), (17,19)
	(1,16), (3, 8), (7, 8), (8,13), (8,17), (9,16), (11,16), (16,19)
	(4, 5), (4,15), (5,12), (12,15)
	(1, 7), (1,17), (3,11), (3,19), (7, 9), (9,17), (11,13), (13,19)
	(1,14), (2, 3), (2,17), (6, 9), (6,11), (7,18), (13,18), (14,19)
	(1,11), (3, 7), (9,19), (13,17)
	(1, 6), (2, 7), (2,13), (3,18), (6,19), (9,14), (11,14), (17,18)
	(1,18), (2, 9), (2,11), (3,14), (6, 7), (6,13), (14,17), (18,19)
	(1,12), (3, 4), (4, 7), (4,13), (4,17), (9,12), (11,12), (12,19)
	(1, 4), (3,12), (4, 9), (4,11), (4,19), (7,12), (12,13), (12,17)
	(1, 1), (3, 3), (7, 7), (9, 9), (11,11), (13,13), (17,17), (19,19)
	(1,15), (3, 5), (5,11), (5,13), (5,19), (7,15), (9,15), (15,17)
	(1, 8), (3,16), (7,16), (8, 9), (8,11), (8,19), (13,16), (16,17)
	(1, 9), (3,13), (7,17), (11,19)
	(1,10), (3,10), (7,10), (9,10), (10,11), (10,13), (10,17), (10,19)
	(1, 5), (3,15), (5, 7), (5, 9), (5,17), (11,15), (13,15), (15,19)
	(1, 2), (2,19), (3, 6), (6,17), (7,14), (9,18), (11,18), (13,14)
42	(1,12), (5,18), (6,11), (6,17), (12,13), (18,19)
	(1, 4), (2,11), (5,20), (8,19), (10,13), (16,17)
	(6, 7), (7,12), (7,18)
	(1, 9), (3, 5), (3,19), (9,13), (11,15), (15,17)
	(2,15), (3, 8), (3,20), (4, 9), (9,10), (15,16)
	(2, 3), (3,16), (4,15), (8, 9), (9,20), (10,15)
	(1, 3), (3,13), (5,15), (9,11), (9,17), (15,19)
	(3,14), (9,14), (14,15)
	(1, 7), (5, 7), (7,11), (7,13), (7,17), (7,19)
	(2, 7), (4, 7), (7, 8), (7,10), (7,16), (7,20)
	(1, 8), (2, 5), (4,11), (10,17), (13,20), (16,19)
	(2, 9), (3, 4), (3,10), (8,15), (9,16), (15,20)
	(1,20), (2,19), (4,17), (5,16), (8,13), (10,11)
	(1,10), (2,17), (4,13), (5, 8), (11,16), (19,20)
	(1,13), (5,19), (11,17)
	(1,18), (5, 6), (6,19), (11,12), (12,17), (13,18)
	(1,16), (2,13), (4, 5), (8,11), (10,19), (17,20)
	(1,11), (1,19), (5,11), (5,13), (13,17), (17,19)
	(1,14), (5,14), (11,14), (13,14), (14,17), (14,19)
	(1, 1), (5, 5), (11,11), (13,13), (17,17), (19,19)
	(1, 5), (1,17), (5,17), (11,13), (11,19), (13,19)
	(1,15), (3,11), (3,17), (5, 9), (9,19), (13,15)
	(3, 7), (7, 9), (7,15)
	(1, 6), (5,12), (6,13), (11,18), (12,19), (17,18)
	(1, 2), (4,19), (5,10), (8,17), (11,20), (13,16)

Table 11: Minimal polynomial groupings for $s = 40, 42$. Each row lists cases sharing the same minimal polynomial.

s	Sets of (p, q) indices sharing the same minimal polynomial; $(x, y) = (p/s, q/s)$
50	(1,20), (3,10), (7,10), (9,20), (10,13), (10,17), (10,23), (11,20), (19,20), (20,21)
	(1,19), (1,21), (3, 7), (3,13), (7,17), (9,11), (9,21), (11,19), (13,23), (17,23)
	(1, 1), (3, 3), (7, 7), (9, 9), (11,11), (13,13), (17,17), (19,19), (21,21), (23,23)
	(1, 7), (3,21), (9,13), (11,23), (17,19)
	(1,13), (1,23), (3,11), (3,19), (7, 9), (7,11), (9,17), (13,19), (17,21), (21,23)
	(1, 2), (3, 6), (4,23), (7,14), (8,21), (9,18), (11,22), (12,19), (13,24), (16,17)
	(1,16), (2, 3), (4,19), (6, 9), (7,12), (8,13), (11,24), (14,21), (17,22), (18,23)
	(1, 9), (1,11), (3,17), (3,23), (7,13), (7,23), (9,19), (11,21), (13,17), (19,21)
	(1,10), (3,20), (7,20), (9,10), (10,11), (10,19), (10,21), (13,20), (17,20), (20,23)
	(1,15), (3, 5), (5, 7), (5,13), (5,17), (5,23), (9,15), (11,15), (15,19), (15,21)
	(1,12), (2,21), (3,14), (4,17), (6,13), (7,16), (8, 9), (11,18), (19,22), (23,24)
	(1, 4), (2,13), (3,12), (6,11), (7,22), (8,23), (9,14), (16,21), (17,18), (19,24)
	(1, 8), (2,19), (3,24), (4,13), (6, 7), (9,22), (11,12), (14,17), (16,23), (18,21)
	(1,22), (2, 9), (3,16), (4, 7), (6,23), (8,11), (12,21), (13,14), (17,24), (18,19)
	(1,24), (2,23), (3,22), (4,21), (6,19), (7,18), (8,17), (9,16), (11,14), (12,13)
	(1, 3), (1,17), (3, 9), (7,19), (7,21), (9,23), (11,13), (11,17), (13,21), (19,23)
	(1,14), (2, 7), (3, 8), (4,11), (6,21), (9,24), (12,17), (13,18), (16,19), (22,23)
	(1, 6), (2,17), (3,18), (4, 9), (7, 8), (11,16), (12,23), (13,22), (14,19), (21,24)
	(1, 5), (3,15), (5, 9), (5,11), (5,19), (5,21), (7,15), (13,15), (15,17), (15,23)
	(1,18), (2,11), (3, 4), (6,17), (7,24), (8,19), (9,12), (13,16), (14,23), (21,22)
	(2, 5), (4,15), (5, 8), (5,12), (5,18), (5,22), (6,15), (14,15), (15,16), (15,24)
	(2,15), (4, 5), (5, 6), (5,14), (5,16), (5,24), (8,15), (12,15), (15,18), (15,22)

Table 12: Minimal polynomial groupings for $s = 50$. Each row lists cases sharing the same minimal polynomial.