An Ergodic Proof that Rational Times Normal is Normal

David H. Bailey and Daniel J. Rudolph

15 February 2002

Lemma 1. (Birkoff ergodic theorem) Let $f(t)$ be an integrable function on a measure space with probability measure μ, and let T be an ergodic transformation (i.e. $T^{-1}A = A$ implies $\mu(A) = 0$ or 1). Then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k x) \;=\; \int f \, d\mu \quad \text{for a.e. } x(\mu)$$

where "for a.e. $x(\mu)$" means for all x except for a set N with $\mu(N) = 0$. This result is proved in [2, pg. 13, 20-29].

Lemma 2. Let μ be a probability measure and T an ergodic transformation on the probability space. Suppose that ν is another measure for which T is ergodic, and further ν is absolutely continuous with respect to μ (i.e., $\nu(A) = 0$ if and only if $\mu(A) = 0$). Then $\mu = \nu$.

Proof. Applying Lemma 1 to $f(t) = I_A(t)$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^n x) \;=\; \int f(t)\, d\mu(t) \;= \mu(A) \quad \text{for a.e. } x(\mu).$$

Since ν is absolutely continuous with respect to μ, the above holds a.e. $x(\nu)$ as well. Now since T preserves the measure ν, we can write, for $n > 0$,

$$\begin{aligned}
\nu(A) \;&=\; \int f(t)\, d\nu(t) \;=\; \frac{1}{n} \sum_{i=0}^{n-1} \int f(T^i x)\, d\nu(x) \\
&=\; \int \frac{1}{n} \sum_{i=0}^{n-1} f(T^i x)\, d\nu(x) \;\to\; \int \mu(A) d\nu \;=\; \mu(A)
\end{aligned}$$

by the dominated convergence theorem.

Lemma 3. The constant α is normal base b if and only if there exists a constant C such that

$$\overline{\lim}_{n\to\infty} \frac{\#_{0\le j\le n-1}(\{b^j \alpha\} \in [\beta, \gamma))}{n} \;\le\; C(\gamma - \beta),$$

for every $0 \le \beta < \gamma < 1$.

Proof. Let μ denote ordinary Lebesgue measure on $[0, 1)$, let $T(x) = \{2x\}$, and let ν be the measure on $[0, 1)$, defined on the interval $[\beta, \gamma)$ to be the LHS of the condition in Lemma 3. It is easily seen that T is ergodic under both μ and ν. The condition in Lemma 3 is easily seen to imply that ν is absolutely continuous with respect to μ. Thus by Lemma 2, $\mu = \nu$, or in other words $\{b^k \alpha\}$ is uniformly distributed in the unit interval, so that α is normal.

Theorem. Whenever α is normal to base b, then so is $r\alpha$ for every nonzero positive rational r.

Proof. First suppose that α is normal, and consider $p\alpha$ for a positive integer p. Then $\{b^j p\alpha\} \in [\beta, \gamma)$ implies that one of the following p mutually exclusive conditions must hold:

$$\begin{array}{lcl} \{b^j\alpha\} & \in & [\beta/p,\ \gamma/p) \\ \{b^j\alpha\} & \in & [\beta/p + 1/p,\ \gamma/p + 1/p) \\ \{b^j\alpha\} & \in & [\beta/p + 2/p,\ \gamma/p + 2/p) \\ \cdots & & \cdots \\ \{b^j\alpha\} & \in & [\beta/p + (p-1)/p,\ \gamma/p + (p-1)/p) \end{array}$$

Since α is normal, the limiting frequency of each of the above is $(\gamma - \beta)/p$. Thus the limiting frequency of $\{b^j p\alpha\} \in [\beta, \gamma)$ is p times this value, or $\gamma - \beta$. This estabishes that $p\alpha$ is normal.

Now suppose that α is normal, and consider α/p for a positive integer p. We can assume that $\gamma - \beta < 1/p$, because otherwise we can take $C = 2p$ in the condition of Lemma 3. Then $\{b^j\alpha/p\} \in [\beta, \gamma)$ implies $\{b^j\alpha\} \in [\{p\beta\}, \{p\gamma\})$, where we understand that in some cases $\{p\beta\} > \{p\gamma\}$, due to "wrapping" around the unit interval, in which case we take this to mean the union of the two intervals $[0, \{p\gamma\})$ and $[\{p\beta\}, 1)$. However, in either case, the total length is $p(\gamma - \beta)$, so that the limiting frequency of $\{b^j\alpha\}$ in this set is $p(\gamma - \beta)$. Thus we can write

$$\lim_{n\to\infty} \frac{\#_{0\le j\le n-1}(\{b^j\alpha/p\} \in [\beta, \gamma))}{n} \quad \le \quad p(\gamma - \beta),$$

where we must use $\le$ since whereas $\{b^j\alpha/p\} \in [\beta, \gamma)$ implies $\{b^j\alpha\} \in [\{p\beta\}, \{p\gamma\})$, the converse is not true. But this is good enough for Lemma 3, which then implies that α/p is normal. See also exercise 8.9 in [1, pg. 77].

References

[1] L. Kuipers and H. Niederreiter, *Uniform Distribution of Sequences*, Wiley-Interscience, New York, 1974.

[2] Patrick Billingsley, *Ergodic Theory and Information*, John Wiley, New York, 1965.