STOP-OUTS UNDER SERIAL CORRELATION AND "THE TRIPLE PENANCE RULE"

David H. Bailey [α]

Marcos López de Prado [β]

First version: December 2012
This version: October 2014

[α] David H. Bailey is recently retired from the Lawrence Berkeley National Laboratory and is a Research Fellow at University of California, Davis, Department of Computer Science, Davis, CA 95616, USA. E-mail: david@davidhbailey.com. URL: www.davidhbailey.com

[β] Marcos López de Prado is Senior Managing Director at Guggenheim Partners, New York, NY 10017, USA, and Research Affiliate at Lawrence Berkeley National Laboratory, Berkeley, CA 94720, USA. E-mail: lopezdeprado@lbl.gov. URL: www.QuantResearch.info

We are grateful to the Editor-In-Chief and two anonymous referees for their suggestions. We would also like to thank Tony Anagnostakis (Moore Capital), José Blanco (UBS), Peter Carr (Morgan Stanley, NYU), Matthew D. Foreman (University of California, Irvine), Marco Dion (J.P. Morgan), Jeffrey S. Lange (Guggenheim Partners), David Leinweber (CIFT, Lawrence Berkeley National Laboratory), Attilio Meucci (KKR, NYU), Riccardo Rebonato (PIMCO, University of Oxford) and Luis Viceira (HBS).

Supported in part by the Director, Office of Computational and Technology Research, Division of Mathematical, Information, and Computational Sciences of the U.S. Department of Energy, under contract number DE-AC02-05CH11231.

STOP-OUTS UNDER SERIAL CORRELATION AND "THE TRIPLE PENANCE RULE"

ABSTRACT

At what loss should a portfolio manager be stopped-out? What is an acceptable time under water? We demonstrate that, under standard portfolio theory assumptions, the answer to the latter question is strikingly unequivocal: On average, the recovery spans three times the period involved in accumulating the maximum quantile loss for a given confidence level. We denote this principle the "*triple penance rule*".

We provide a theoretical justification to why investment firms typically set less strict stop-out rules to portfolio managers with higher Sharpe ratios, despite the fact that they should be expected to deliver superior performance. We generalize this framework to the case of first-order auto-correlated investment outcomes, and conclude that ignoring the effect of serial correlation leads to a gross underestimation of the downside potential of hedge fund strategies, by as much as 70%. We also estimate that some hedge funds may be firing more than three times the number of skillful portfolio managers, compared to the number that they were willing to accept, as a result of evaluating their performance through traditional metrics, such as the Sharpe ratio.

We believe that our closed-form compact expression for the estimation of downside potential, without having to assume IID cashflows, will open new practical applications in risk management, portfolio optimization and capital allocation. The Python code included confirms the accuracy of our analytical solution.

Keywords: Downside, time under water, stop-out, triple penance, serial correlation, Sharpe ratio.

AMS Classification: 91G10, 91G60, 91G70, 62C, 60E.
JEL Classification: G0, G1, G2, G15, G24, E44.

1.- INTRODUCTION

Multi-manager investment firms are routinely faced with the decision to stop-out a portfolio manager (PM). This is a decision of the utmost importance, intended to protect the well-being of the overall funds. It typically has dramatic consequences, including the removal of the PM involved. Despite of the relevance and recurrence of stop-outs, we are not aware of the existence of a theoretical framework addressing this particular question. Such framework would be very useful in practice, as it would allow the firm to approach the stop-out problem in an objective and transparent manner, thus avoiding the personal conflict that is so often associated with employee dismissals.

The question of when do we possess enough evidence to discontinue (or stop-out) an investment strategy can be approached as a decision problem under uncertainty. This uncertainty arises from the fact that we cannot be sure whether negative outcomes, accumulated over time, are the result of bad performance, or mere bad luck. We can use decision theory to construct at test that, given a significance level α, rejects or not the null hypothesis that a PM's performance is consistent with skill. In this paper we introduce a framework to test for that hypothesis in three alternative formulations: The maximum quantile-loss (*MaxQL*), the quantile-time unde water (*TuW*), and a procedure to translate realized losses into implied time under water (*ITuW*).

Consider a multi-manager firm, whose executives must decide *where* and *when* a particular PM must be stopped-out.[1] Multi-manager firms allocate capital to a PM based on the statistics that characterize his track record. These statistics are then used to generate MaxQL or TuW limits, which reflect the firm's appetite for false positives. Very tight limits are adopted by firms who are willing to fire a truly skillful PM, however unlucky he may have been. In other words, realized MaxQL and TuW exceeding certain limits are taken as sufficient evidence (for a certain confidence level) that the PM is not living up to the expectations which granted him a capital allocation.

We begin our discussion with the standard mean-variance framework introduced by the seminal work of Markowitz [1952, 1956, 1959]. Under the assumption of IID Normal outcomes, we determine the MaxQL and the TuW associated with a particular confidence level. With these results as a backdrop, we generalize our framework to incorporate the possibility of first-order autoregressive (or AR(1)) investment outcomes. Allowing this type of serial conditionality offers a richer analysis than standard mean-variance approaches. Higher-order serial conditionality would lead to different numerical results, however not conceptually different conclusions than those derived from an AR(1) specification. This is because the key feature that leads to substantial MaxQL and TuW is serial-dependence, which is already incorporated by an AR(1) process. To our knowledge, this is the first time that a compact expression is published which provides analytical estimates of quantile-loss potentials under first-order autoregressive cashflows.

A first goal of this paper is to provide an analytical framework to the stop-out problem, allowing for serial-conditionality of outcomes. We show through Monte Carlo

[1] Our analysis can be applied to managers as well as strategies, and so we will refer to one or the other interchangeably.

experiments that the closed-form compact solution[2] presented here is accurate. In providing an analytical estimate of downside potential, even in the presence of serial-conditionality, we open the possibility to integrate our results in optimization problems, rather than having to resort to computationally-expensive numerical methods. A second goal is to provide a theoretical justification as to why PMs with high Sharpe ratios may be given more permissive stop-out limits. A third goal is to formalize the relationship between the two main financial variables involved in a stop-out, MaxQL and TuW, by deriving the ITuW. We believe that ITuW is a more effective way to communicate stop-out limits.

The rest of the study is organized as follows: Section 2 reviews the academic literature on the topic. Section 3 introduces the framework. Section 4 determines the maximum quantile-loss over any horizon. Section 5 determines the time under water for a given confidence level, and the implied time under water for a given loss. Section 6 combines both concepts (maximum quantile-loss and quantile-time under water) into the "triple penance rule." Section 7 presents a numerical example. Section 8 explains why PMs with higher Sharpe ratios tend to receive less strict stop-out limits. Section 9 generalizes our framework to the case of first-order auto-correlated cashflows. Section 10 applies our framework to a long series of Hedge Fund Research (HFR) indices, and evaluates the impact that auto-correlation has on hedge funds' downside potential and firing practices. Section 11 summarizes our conclusions. The mathematical appendices prove the propositions presented throughout the paper. The Python code that numerically validates the accuracy of our analytical results can be found at www.QuantResearch.info/downloads/DD_Appendices.pdf

2.- LITERATURE REVIEW

López de Prado and Peijan [2004] showed that serial correlation is the main feature responsible for large MaxQL and TuW outcomes, even more so than Non-Normality. Non-Normality is a lesser concern because, as long as investment outcomes are independent and identically distributed (IID), the Central Limit Theorem ensures that the cumulative distribution of those investment outcomes converges to a Normal distribution as time passes. That paper allowed for serial-conditionality when modeling MaxQL and TuW, however those values had to be computed through a Monte Carlo.

This is not the first paper to discuss the impact that serial dependence has on performance metrics or quantile-losses. Lo [2002] derived the analytical solution to the asymptotic distribution of the Sharpe ratio under serial correlation, and found that the annual Sharpe ratio for a hedge fund can be overstated by as much as 65% because of the presence of serial correlation in monthly returns. In a very interesting study, Hayes [2006] fits a Markov chain (switching) model to hedge fund returns to estimate their downside potential under that type of serial dependence. Our work differs from Hayes' in a number of ways. We use an autoregressive specification, which allows for Gaussian (continuous)

[2] We use the term closed-form solution in the sense of being exact and analytically derived. With the term compact, we mean that this solution does not involve mathematical operators such as discrete summation, and therefore it is amenable to Analysis.

shocks, while the Markov chain only considers a fixed step (up or down). Empirical evidence supports the autoregressive nature of hedge fund returns' serial dependence. For example, Getmansky et al. [2004] study various sources for hedge fund returns' serial correlation and conclude that exposure to illiquid investments is the most likely. It seems implausible that the returns of illiquid investments will follow the conditioned fixed step implied by a Markov chain. In his study, Hayes argues that a Markov chain asymptotically approximates to an AR(1) model, however he also acknowledges that the speed of convergence is slow, and both models will yield different answers. This is a problem, because we could not rely on a downside risk model which is only accurate after an extended time under water. The conclusion is that it would be desirable to develop a downside risk model with serial correlation following an autoregressive specification. The present paper addresses this gap in the literature.

The academic literature measures downside potential according to at least three distinct approaches: (i) As an extreme value (or drawdown), like in Grossman and Zhou [1993], Magdon-Ismail and Atiya [2004], Carr et al. [2011], Yang and Zhong [2012] or Zhang et al. [2013]; (ii) As a quantile loss (an analogue to VaR), like in López de Prado and Peijan [2004], Mendes and Leal [2005] or Hayes [2006]; and (iii) As the average of a specified percentage of the largest losses over an investment horizon (an analogue to CVaR), like in Chekhlov, Uryasev and Zabarankin [2003, 2005], or Pavlikov, Uryasev and Zabarankin [2012]. We are interested in computing stop-out limits consistent with the testing of a hypothesis, thus the first approach (extreme loss or drawdown) is not useful to our problem. Of the other two approaches, we have opted for the second one because of the prevalence of VaR formulations among practitioners and regulators. Since January 1997, the U.S. Securities and Exchange Commission has required all publicly traded companies to disclose their financial risks. Value-at-Risk (VaR) is one of the three S.E.C. approved formats (Lin et al. [2010]). Practitioners and regulators' preference for quantile-loss risk management methods is also evidenced by the Basel II Accords, published in June 2004 (Chen [2014]). While downside risk can be measured in many different ways, it is important to understand the practical implications of decision-making under the most widely used risk framework.

3.- THE FRAMEWORK

Suppose an investment strategy which yields a sequence of cash inflows $\Delta\pi_\tau$ as a result of a sequence of bets $\tau \in \{1, \dots, \infty\}$, where

$$\Delta\pi_\tau = \mu + \sigma\varepsilon_\tau \quad (1)$$

such that the random shocks are IID distributed $\varepsilon_\tau \sim N(0,1)$. Because these random shocks ε_τ are independent and Normally distributed, so is the random variable $\Delta\pi_\tau$, with $\Delta\pi_\tau \sim N(\mu, \sigma^2)$. This provides for negative as well as positive outcomes, although in the context of stop-outs we are naturally interested in the former rather than the latter. Throughout the remainder of the paper, we will assume that $\mu > 0$. While Bailey and López de Prado [2013] demonstrate that it may be optimal to allocate capital to strategies with $\mu \leq 0$ as long as their correlation to the overall portfolio is sufficiently low, that is not a scenario most practitioners would consider.

Downside periods arise from the accumulation of cash inflows $\Delta\pi_\tau$ over *t* sequential bets (or equivalently, a period of length *t*). If these bets are taken with a certain regular frequency, *t* can also be interpreted in terms of time elapsed. For instance, twelve monthly bets would span a period of one calendar year. Let us define a function π_t that accumulates the cashflows $\Delta\pi_\tau$ over *t* bets.

$$\pi_t = \sum_{\tau=1}^{t} \Delta\pi_\tau \tag{2}$$

where $t \in \{0,1,\dots,\infty\}$ and $\pi_0 = 0$. At the origin of the investment cycle, the cumulative performance is set to zero, because we would like to evaluate the downside potential of π_t following that reset point *t=0*. Because π_t is the aggregation of *t* IID random variables $\Delta\pi_\tau \sim N(\mu, \sigma^2)$, we know that $\pi_t \sim N(\mu t, \sigma^2 t)$.

For a significance level $\alpha < \frac{1}{2}$, we define the quantile function for π_t:

$$Q_{\alpha,t} = \mu t + Z_\alpha \sigma \sqrt{t} \tag{3}$$

where Z_α is the critical value of the Standard Normal distribution associated with a probability α of performing worse than $Q_{\alpha,t}$, i.e. $\alpha = Prob\left[\pi_t \leq Q_{\alpha,t}\right]$. Then, the maximum quantile-loss is defined as:

$$QL_{\alpha,t} = \max\left\{0, -Q_{\alpha,t}\right\} \tag{4}$$

Note that $QL_{\alpha,1}$ coincides with the standard Value-at-Risk (VaR) for that investment at a $(1-\alpha)$ confidence level (see Jorion [2006] for a discussion of VaR), which is a deterministic function of $\{\mu, \sigma, \alpha, t\}$.

4.- MAXIMUM QUANTILE LOSS

VaR is a risk metric limited to a particular horizon, typically one step ahead. We will move beyond VaR by determining the maximum quantile-loss regardless of the number of bets (or time horizon) involved. In words, we would like to answer the question: Up to how much could a particular strategy lose with a given confidence level? Proposition 1 computes analytically the maximum quantile-loss for a given significance level (proved in the Appendix).

DEFINITION 1: Given a significance level α*, we define maximum quantile-loss of an investment as* $MaxQL_\alpha \equiv \max_t\left[QL_{\alpha,t}\right] = \max\left\{0, -\min_t\left[Q_{\alpha,t}\right]\right\}$.

<u>PROPOSITION 1</u>: Assuming IID cashflows $\Delta\pi_\tau \sim N(\mu, \sigma^2)$, and $\mu > 0$, the maximum quantile-loss associated with a significance level $\alpha < \frac{1}{2}$ is

$$MaxQL_\alpha = \frac{(Z_\alpha \sigma)^2}{4\mu} \tag{5}$$

which occurs at the time (or bet)

$$t_\alpha^* = \left(\frac{Z_\alpha \sigma}{2\mu}\right)^2 \tag{6}$$

5.- QUANTILE TIME UNDER WATER

PMs are also routinely stopped-out if they do not recover from a loss after a period of time. In this section we determine that period for a given significance level.

<u>DEFINITION 2</u>: Given a significance level α, we define quantile-time under water of an investment as the minimum time TuW_α, with $TuW_\alpha > 0$, such that $QL_{\alpha,TuW_\alpha} = 0$.

Proposition 2 computes analytically the quantile-time under water for a given significance level (proved in the Appendix).

<u>PROPOSITION 2</u>: Assuming IID cashflows $\Delta\pi_\tau \sim N(\mu, \sigma^2)$, and $\mu > 0$, the quantile-time under water associated with a significance level $\alpha < \frac{1}{2}$ is

$$TuW_\alpha = \left(\frac{Z_\alpha \sigma}{\mu}\right)^2 \tag{7}$$

Suppose that a PM experiences a performance $\tilde{\pi}_t < 0$ after *t* observations. For how long may we not be able to receive performance fee due to $\tilde{\pi}_t$? $\tilde{\pi}_t$ is a realized performance, which is consistent with a quantile loss for some confidence level $\tilde{\alpha}$, $-Q_{\tilde{\alpha},t}$. It would be useful to translate that loss $\tilde{\pi}_t$ in terms of time under water, because that would allow us to express that monetary loss as a cost of opportunity (lost performance fee). Proposition 3 computes what is the time under water implied by $\tilde{\pi}_t$ (proved in the Appendix).

<u>PROPOSITION 3</u>: Given a realized performance $\tilde{\pi}_t < 0$ and assuming $\mu > 0$, the implied time under water is

$$ITuW_{\tilde{\pi}_t} = \frac{{\tilde{\pi}_t}^2}{\mu^2 t} - 2\frac{\tilde{\pi}_t}{\mu} + t \tag{8}$$

Proposition 3 is useful because, given that $\tilde{\pi}_t$ has occurred, $MaxTuW_{\tilde{\pi}_t}$ has become a realistic scenario of time under water. If, for example, $\tilde{\pi}_t$ is so negative that $ITuW_{\tilde{\pi}_t} >$

TuW_α, the firm has a strong argument to stop-out the strategy, even if $\tilde{\pi}_t > -MaxQL_\alpha$. Eq. (8) makes another key point: It not only matters how much money a PM has lost, but critically, for how long.

As we will see in Section 7, this *Implied Time under Water (ITuW)* is a better way of communicating stop-outs than giving a $MaxQL_\alpha$ limit, because it allows us to enforce limits at all times, even before hitting the maximum admissible loss or exhausting the time under water limit. Moreover, note that the calculation of $TuW_{\tilde{\pi}_t}$ only requires three input variables ($\tilde{\pi}_t$, t, μ), where the first two are directly observable. Unlike in the case of TuW_α, there is no need to input Z_α or σ.

6.- THE TRIPLE PENANCE RULE

The concepts of maximum quantile-loss ($MaxQL_\alpha$) and quantile-time under water (TuW_α) are closely related. This is formalized in the following theorem (proved in the Appendix).

THEOREM 1 (or "triple penance rule"): Under standard portfolio theory assumptions, a strategy's maximum quantile-loss $MaxQL_\alpha$ *for a significance level* α *occurs after* t^*_α *observations. Then, the strategy is expected to remain under water for an additional* $3t^*_\alpha$ *after the maximum* quantile-loss*, with a confidence* $(1-\alpha)$.

If we define $Penance = \frac{TuW_\alpha}{t^*_\alpha} - 1$, then the "triple penance rule" tells us that, assuming independent $\Delta\pi_\tau$ identically distributed as Normal (which is the standard portfolio theory assumption), $Penance = 3$, regardless of the Sharpe ratio of the strategy. In other words, it takes three time longer to recover from the maximum quantile-loss than the time it took to produce it, for a given significance level $\alpha < \frac{1}{2}$. This rule has important practical implications with regards to how long it will take for a PM to recover from a fresh new bottom. Figure 1 provides a graphical representation.

[FIGURE 1 HERE]

Should $\Delta\pi_\tau$ exhibit positive serial correlation, $MaxQL_\alpha$, t^*_α and TuW_α will tend to be substantially greater than in the case of $\Delta\pi_\tau$ IID Normal, however $Penance$ will tend to be smaller than 3. We will discuss this case in Sections 9 and 10.

7.- NUMERICAL EXAMPLE

Consider two portfolio managers: PM1 and PM2. PM1 is expected to make US$10m over a year, with an annual standard deviation of also US$10m and a monthly trading frequency. For simplicity, we will assume a risk-free rate of zero.[3] This implies an annualized Sharpe ratio of 1 (see Sharpe [1975, 1994] for a formal definition). On the

[3] We do this to simplify calculations, without loss of generality. Alternatively, the reader could think of these performance numbers as net of the risk-free rate.

other hand, PM2 will run the same risk budget as PM1, however he is expected to deliver a Sharpe ratio of 1.5. Table 1 summarizes this problem.

[TABLE 1 HERE]

For monthly bets, and a 95% confidence level, we would stop-out PM1 if he hits a cumulative loss of US$6,763,858.64, or stays under water for more than 2.706 years (about 33 months). Because PM2 is supposed to deliver a greater risk-adjusted performance (due to his Sharpe ratio of 1.5), he must trade under tighter constraints: We would stop-out PM2 if his losses exceed US$4,509,239.09, or if he remains under water longer than 1.2 years (about 15 months). Figure 2 plots the quantile-loss for a 95% confidence level, as a function of years passed, assuming monthly bets. In both cases, the time under water that follows the maximum quantile-loss is precisely 3 times the number of observations that occur up to the bottom performance. This is consistent with the "triple penance" rule.

[FIGURE 2 HERE]

Beyond stopping out in terms of maximum quantile-loss and time under water, this framework provides a basis for reassessing investments that perform worse than the quantile lines plotted in Figure 2 at any particular point in time. For example, suppose that PM1 has a cumulative loss of US$5,000,000 after being 2 years under water. Even though the loss is below the maximum quantile-loss, this scenario augurs a time under water of 3.125 years for the same confidence level implied by this observation (applying Proposition 3). That exceeds the pre-established limit of 2.706 years under water, and the firm may decide to stop-out PM1. Therefore, an effective way to communicate a downside limit is to translate a realized loss in terms of the implied time under water: Should $ITuW_{\tilde{\pi}_t} > TuW_{\alpha}$, the strategy or PM will be stopped-out, regardless of the actual $\tilde{\pi}_t$, because the cost of opportunity (lost performance fee) is just too high for the firm.

8.- WHY DO BETTER MANAGERS GET LESS STRICT STOP-OUT LIMITS?

The previous numerical example gave a tighter stop-out limit to the PM with greater Sharpe ratio. And yet, the experienced reader is likely aware that hedge funds typically give greater stop-out limits to PMs with higher Sharpe ratios. To understand the reason for this apparent paradox, we need to incorporate into our model a bit of hedge fund business reality which is currently absent from our formal framework.

Hedge funds fund their operations through the collected management fee, and pay bonuses from the performance fee. Good PMs are likely to leave if they do not perceive a bonus within a certain timeframe. Hedge funds want to minimize the probability of defections among good PMs, who may abandon the firm leaving a loss behind. Thus, firms are willing to give more permissive stop-out limits to higher Sharpe ratio PMs.

Let us see how this argument fits in our framework. We can rewrite Eq. (6) as

$$t_{\alpha}^{*} = T\left(\frac{Z_{\alpha}}{2SR}\right)^{2} \tag{9}$$

where T is the total number of independent bets implemented in a year, and SR is the annualized Sharpe ratio. Combining Eq. (9) in Eq. (7), we obtain that

$$\frac{TuW_{\alpha}}{T} = \left(\frac{Z_{\alpha}}{SR}\right)^{2} \tag{10}$$

$\frac{TuW_{\alpha}}{T}$ is the quantile-time under water, for a confidence level $(1-\alpha)$, expressed in years. However, it is known that many firms fix stop-out limits to a constant value for all PMs, $\frac{TuW_{\alpha}}{T} = \bar{K}$. They do so because they fear that stopping out a star portfolio manager on the basis of time reduces her chances to recover, and may trigger her resignation. This sets a floor to the value that $\bar{K}$ may adopt. Values of $\bar{K}$ are also capped, because firms are aware that a good PM will leave if she does not receive frequent bonuses. What is effect of setting a constant time stop-out for all PMs regardless of their Sharpe ratio? Under these circumstances, Eq. (10) leads to

$$\bar{K} = \left(\frac{Z_{\alpha}}{SR}\right)^{2} \tag{11}$$

Eq. (11) evidences the existence of a trade-off between greater Sharpe ratio and greater tolerance to downside potential. For double SR, a double Z_{α} is admissible, which allows for a substantially lower value of α (recall that $\alpha < \frac{1}{2}$, and thus $Z_{\alpha} < 0$). More precisely, the significance level admissible, subject to an exogenously set target $\bar{K}$, is

$$\bar{\alpha} = Z\left[-SR\sqrt{\bar{K}}\right] \tag{12}$$

Once again, the negative sign appears due to the fact that $\alpha < \frac{1}{2}$. Eq. (12) gives us a nice expression, which incorporates the business reality we discussed at the beginning of this section. It explicitly tells us that a hedge fund is more permissive with PMs with high Sharpe ratios, despite the fact that we should expect those same PMs to perform better and hence operate with lower downside potential.

Following our example from the previous section, $\bar{\alpha}$ is the answer to the question: *What significance level is consistent with setting a constant time stop-out* $\bar{K}$*?* Following the example presented in Section 7, for $\bar{K} = 1$ (one year), PM1 would be stopped-out at a significance level $\bar{\alpha}_1 = 0.1587$, and PM2 would be stopped-out at a significance level $\bar{\alpha}_2 = 0.0668$. As Figure 3 shows, now we are imposing stricter stop-out limits on PM1 than on PM2, the opposite of what we saw in Section 7. This is how in fact hedge funds typically operate, constrained as they are by the reality of having to protect themselves against defections.

[FIGURE 3 HERE]

Similarly, for an exogenously set $\overline{K} = 2$ (two years), PM1 would be stopped-out at a significance level $\bar{\alpha}_1 = 0.0787$, and PM2 would be stopped-out at a significance level $\bar{\alpha}_2 = 0.0169$. Figure 4 shows that in this case the hedge fund sets even more permissive stop-out limits on PM2 relative to PM1 than we saw in Figure 3.

[FIGURE 4 HERE]

This greater permissiveness towards PMs with higher Sharpe ratios is contrary to what standard portfolio theory would have predicted, and yet our framework shows that hedge funds operating in this way act rationally, in an attempt to minimize the risk of defection among their most talented portfolio managers.

9.- STOP-OUT LIMITS UNDER FIRST-ORDER AUTO-CORRELATED CASHFLOWS

Suppose an investment strategy which yields a sequence of cash inflows $\Delta\pi_\tau$ as a result of a sequence of bets $\tau \in \{1, \dots, \infty\}$, where

$$\Delta\pi_\tau = (1-\varphi)\mu + \varphi\Delta\pi_{\tau-1} + \sigma\varepsilon_\tau \tag{13}$$

such that the random shocks are IID distributed $\varepsilon_\tau \sim N(0,1)$. Eq. (13) is initialized by a seed value $\Delta\pi_0$, which is not necessarily null. These random shocks ε_τ follow an independent and identically distributed Gaussian process, however $\Delta\pi_\tau$ is neither an independent nor an identically distributed process. This is due to the parameter φ, which incorporates a first-order serial-correlation effect of auto-regressive form. Appendix 5 shows that a necessary and sufficient condition for $\Delta\pi_\tau$ to be stationary is that $\varphi \in (-1,1)$. In that case, the above process has an asymptotic expectation $\lim_{\tau\to\infty} E_0[\Delta\pi_\tau] = \mu$, and an asymptotic variance $\lim_{\tau\to\infty} V_0[\Delta\pi_\tau] = \frac{\sigma^2}{1-\varphi^2}$, where the zero subscript at E_0 and V_0 denote expectations formed at the origin ($\tau = 0$). If these bets are taken with a certain regular frequency, t can also be interpreted in terms of time elapsed. For instance, twelve monthly bets would span a period of one year.

Like in Section 3, let us define a function π_t that accumulates the cash inflows $\Delta\pi_\tau$ over time.

$$\pi_t = \sum_{\tau=1}^{t} \Delta\pi_\tau \tag{14}$$

where $t \in \{0,1, \dots, \infty\}$ and $\pi_0 = 0$ at the onset of the investment cycle, when the stop-out limits are set up. The following Proposition is proved in the Appendix.

<u>PROPOSITION 4</u>: Under the stationarity condition $\varphi \in (-1,1)$, the distribution of a cumulative function π_t of a first-order auto-correlated random variable $\Delta\pi_\tau$ follows a Normal distribution with parameters:

$$\pi_t \sim N\left(\frac{\varphi^{t+1}-\varphi}{\varphi-1}(\Delta\pi_0-\mu) + \mu t, \frac{\sigma^2}{(\varphi-1)^2}\left(\frac{\varphi^{2(t+1)}-1}{\varphi^2-1} - 2\frac{\varphi^{t+1}-1}{\varphi-1} + t + 1\right)\right) \quad (15)$$

For a significance level $\alpha < \frac{1}{2}$, we can estimate the quantile-loss function as the lower band for π_t after *t* bets:

$$Q_{\alpha,t} = \frac{\varphi^{t+1}-\varphi}{\varphi-1}(\Delta\pi_0-\mu) + \mu t + Z_\alpha \frac{\sigma}{|\varphi-1|}\left(\frac{\varphi^{2(t+1)}-1}{\varphi^2-1} - 2\frac{\varphi^{t+1}-1}{\varphi-1} + t + 1\right)^{1/2} \quad (16)$$

where Z_α is the critical value of the Standard Normal distribution associated with a probability α of performing worse than $Q_{\alpha,t}$. Like before, the quantile-loss function is finally obtained as $QL_{\alpha,t} = max\{0, -Q_{\alpha,t}\}$.

We are not aware of previously published analytical estimates of quantile-losses under first-order auto-correlated cashflows. Proposition 4 is particularly useful in practice, because it gives us that closed-form solution presented in Eq. (16), and allows us to enunciate Proposition 5 (see the Appendix for a proof).

<u>PROPOSITION 5</u>: For $\mu > 0$, $Q_{\alpha,t}$ is unimodal, a global minimum exists ($MinQ_\alpha$) and $MaxQL_\alpha = \max\{0, -MinQ_\alpha\}$ can be computed.

Appendices 9 and 10 present algorithms to determine the maximum quantile-loss and time under water in this more general framework. These procedures can be easily integrated in optimization problems, such as portfolio optimization subject to quantile-loss or time under water constraints under serial conditionality. This is relevant, because many times researchers are compelled to adopt the ubiquitous IID assumption solely for computational reasons, contrary to empirical evidence that would have advised them to apply an expression like Eq. (16).

10.- DOWNSIDE POTENTIAL IN THE HEDGE FUND INDUSTRY

We are ready to put into practice the theory introduced in the earlier sections. We have downloaded from Bloomberg a long series of monthly Net Asset Values (NAVs) for Hedge Fund Research Indices (HFR), and selected those series that go from January 1st

1990 to January 1st 2013. This gives us 265 data points for each of the indices listed in Table 2.

[TABLE 2 HERE]

NAVs do not follow a stationary process (see Meucci [2005] for a comprehensive discussion of this subject). In order to apply our framework, we first need to perform a logarithmic transformation on the NAVs. The maximum likelihood estimator of φ is $\hat{\varphi} = Cov_0[\Delta\pi_\tau, \Delta\pi_{\tau-1}](Cov_0[\Delta\pi_{\tau-1}, \Delta\pi_{\tau-1}])^{-1}$, where $\Delta\pi_\tau$ is the series of first order differences on the log-NAVs at observation τ, and Cov_0 is the covariance operator. The zero subscript denotes expectations formed at the origin ($\tau = 0$). Following Appendix 5, we estimate μ as $\hat{\mu} = \mu_\infty$, and σ as $\hat{\sigma} = \sigma_\infty\sqrt{(1-\varphi^2)}$, where $\mu_\infty = \lim_{\tau\to\infty} E_0[\Delta\pi_\tau] = \mu$ is the asymptotic expected value and $\sigma_\infty^2 = \lim_{\tau\to\infty} V_0[\Delta\pi_\tau] = \frac{\sigma^2}{1-\varphi^2}$ is the asymptotic variance. μ_∞ and σ_∞^2 can be approximated by the large-sample estimates of mean and variance of cashflows. Once we have estimated the triplet $(\hat{\mu}, \hat{\sigma}, \hat{\varphi})$, we can compute the $MaxQL_\alpha$ and TuW_α using the code in Appendix 11.[4] We are assuming $\alpha = 0.05$ and $\Delta\pi_0 = 0$, but different scenarios can be simulated by changing the appropriate parameters in the code. Table 3 reports the results.

[TABLE 3 HERE]

As discussed in Appendix 8, the convergence of the procedure requires $\hat{\mu} > 0$ and $\hat{\varphi} \in [0,1)$, which is consistent with 24 of the 26 cases listed above. As we can deduce from the t-Stat values for $\hat{\varphi}$ reported in Table 3, $\hat{\varphi}$ is statistically significantly in 21 out of 26 cases at a 95% confidence level. Despite of this overwhelming empirical evidence, let us suppose that $\hat{\varphi} = 0$ in all of the above cases. This is of course a misleading assumption, however it is very common for practitioners and academics to assume that returns are independent to avoid dealing with serial correlation. Table 4 reports the corresponding $MaxQL_\alpha$ and TuW_α under that scenario.

[TABLE 4 HERE]

Results in Table 4 can be computed in two different ways: i) Running the code in Appendix 11 for $\hat{\varphi} = 0$ or ii) applying Eqs. (5) and (7). The conclusion is that unrealistically assuming $\hat{\varphi} = 0$ leads to a gross underestimation of the downside potential of hedge fund investments. For example, in the case of *"HFRI RV: Fixed Income-Convertible Arbitrage Index"* (Code "HFRICAI Index"), the maximum quantile-loss under the assumption of independence is only 3.79%, while considering first-order auto-correlation would yield 11.60%. This means that wrongly assuming independence leads to a 67% underestimation compared to taking into account first-order auto-correlation. Quantile-time under water if we assume independence is only 21.28 (monthly) observations, while considering first-order auto-correlation would yield 74.42. This means that wrongly assuming independence leads to a 71% underestimation compared to

[4] Electronic copy available at www.QuantResearch.info/downloads/DD_Appendices.pdf

taking into account first-order auto-correlation. *Penance* measures how long it takes to recover from the maximum quantile-loss, as a multiple of the time it took to reach the bottom. More precisely, $Penance = \frac{TuW_\alpha}{t_\alpha^*} - 1$. Table 4 reports that *Penance* for hedge fund indices ranges between 1.6 and 3. Although positive serial correlation leads to greater downside potential, longer periods to reach the bottom (t_α^*) and longer periods under water, the *Penance* may be substantially smaller. In particular, *Penance* is smaller the higher φ (*Phi*) and the higher the ratio $\frac{\mu}{\sigma}$ (*Mean* divided by *Sigma*). Figure 5 plots *Penance* for hedge fund indices with various $\hat{\varphi}$.

[FIGURE 5 HERE]

These results introduce two interesting implications: First, hedge fund strategies are much riskier than what could be derived from performance metrics that rely on the ubiquitous IID assumption, such as Sharpe ratio, Sortino ratio, Treynor ratio, Information ratio, etc. (see Bailey and López de Prado [2012] for a discussion). This leads to an over-allocation of capital by Markowitz-style approaches to hedge fund strategies. Second, PMs and strategies evaluated by those IID-based metrics are being stopped-out much earlier than it would be appropriate. A good PM running a strategy that delivers auto-correlated cashflows may be unnecessarily stopped-out because the firm assumed IID cashflows. This is a particularly bad decision, because one positive aspect about strategies with auto-correlated cashflows is that their *Penance* is shorter than in the IID case.

We would like to understand whether hedge funds intending to accept a probability α_1 of firing a truly skillful portfolio manager (a "false positive") are effectively taking a different probability α_2 as a result of assuming returns independence. Combining Propositions 1 and 4 we can compute the α_2 associated with $\pi_t = MaxQL_{\alpha_1} = -\frac{\left(Z_{\alpha_1}\sigma\right)^2}{4\mu}$ as

$$\alpha_2 = \Phi\left[\frac{-\frac{\left(Z_{\alpha_1}\sigma\right)^2}{4\mu} - \frac{\varphi^{t_{\alpha_1}^*+1} - \varphi}{\varphi - 1}(\Delta\pi_0 - \mu) + \mu t_{\alpha_1}^*}{\sqrt{\frac{\sigma^2}{(\varphi-1)^2}\left(\frac{\varphi^{2\left(t_{\alpha_1}^*+1\right)} - 1}{\varphi^2 - 1} - 2\frac{\varphi^{t_{\alpha_1}^*+1} - 1}{\varphi - 1} + t_{\alpha_1}^* + 1\right)}}\right] \quad (17)$$

where Φ is the cdf of the standard Normal distribution.[5] Table 5 reports the effective proportion α_2 of truly skillful portfolio managers fired by hedge funds, despite of aiming at a proportion $\alpha_1 = 0.05$. Again, this discrepancy arises because hedge funds assuming returns independence aim at a proportion α_1, however they effectively get a proportion α_2 because that assumption was false in most cases (see t-Stat values for $\hat{\varphi}$ in Table 3).

[5] Incidentally, Eq. (17) can be used to compute $ITuW_{\tilde{\pi}_t}$ (Proposition 3) in the more general framework of first-order serial correlation. In order to do that, we simply have to input this α_2 in the algorithm described in Appendix 10. For the reasons argued in Section 5, this would be a more effective way to communicate stop-out limits.

[TABLE 5 HERE]

For all hedge fund styles, $\alpha_2 > \alpha_1$, which means that they are effectively firing a greater proportion of truly skillful portfolio managers than they originally intended. That proportion of over-firings is $\alpha_2 - \alpha_1$. Most hedge funds evaluate performance through traditional metrics, such as the Sharpe ratio, which assumes returns independence and would lead to the over-firing reported in Table 5. For example, hedge funds similar to those in the *"HFRI RV: Fixed Income-Convertible Arbitrage Index"* (code "HFRICAI Index") may be firing 3.38 times (0.1688 vs. 0.05) the number of truly skillful portfolio managers, compared to the number they were willing to accept under the assumption of returns independence. Skillful managers that are fired by mistake need to be replaced in order to preserve performance, which increases personnel turnover. Our framework explains how the excessive turnover experienced by some hedge funds may be the result of unrealistically expecting their portfolio managers to deliver independent returns.

11.- CONCLUSIONS

Following standard portfolio theory assumptions, we have computed analytically the maximum quantile-loss and quantile-time under water for a certain confidence level. We have shown how these concepts are intimately related through the "triple penance" rule. This rule states that, under standard portfolio theory assumptions, it takes three times longer to recover from the expected maximum quantile-loss than the time it takes to produce it, with the same confidence level. We have introduced a new downside-risk concept called *Penance*, which measures how long it takes to recover from the maximum quantile-loss, as a multiple of the time it took to reach the bottom. We have also demonstrated an effective way to communicate downside limits, via an implied time under water formulation.

According to this framework, for a certain confidence level, we should expect tighter stop-out limits imposed on portfolio managers with higher Sharpe ratios. That is rarely the case in practice. We have provided a theoretical justification to this observation, by recognizing that hedge funds must confront the risk of defection. They know that, should a good portfolio manager not receive a bonus within a certain period of time, he may try his luck at another firm and leave a loss behind. The consequence is that hedge funds assign greater confidence levels to portfolio managers with greater Sharpe ratios, and as a result they are quicker in stopping out portfolio managers with lower Sharpe ratios, just the opposite to what standard portfolio theory would have predicted.

We have complemented our study with a generalization of our framework to deal with the case of first-order auto-correlated cashflows. We derived a closed-form compact expression which estimates the downside potential of a strategy without having to assume IID random shocks. An empirical study of hedge fund indices reveals that ignoring the effect of serial correlation leads to a gross underestimation of the downside potential of hedge fund strategies, by as much as 70%. Although positive auto-correlation leads to greater downside potential, longer periods to reach the bottom and longer periods under water, the *Penance* may be substantially smaller. We find that some hedge funds may be firing more than three times the number of skillful portfolio managers, compared to the number that they were willing to accept, as a result of evaluating their performance through traditional metrics, such as the Sharpe ratio. The excessive turnover experienced by some hedge funds may be the result of unrealistically expecting their portfolio managers to deliver independent returns.

We are aware that researchers have been compelled to adopt the IID assumption in past, in disregard of contradicting empirical evidence, solely for computational reasons. We hope that the expression we have derived in this paper will allow them to take serial correlation into account in risk management, portfolio optimization and capital allocation applications. The Python code included in the Appendix numerically confirms the accuracy of our solution.[6]

[6] Electronic copy available at www.QuantResearch.info/downloads/DD_Appendices.pdf

APPENDICES

A.1.- PROOF TO PROPOSITION 1

The quantile-loss function is defined as $QL_{\alpha,t} = \max\{0, -Q_{\alpha,t}\}$. We can compute its maximum as $MaxQL_\alpha = \max\{0, -MinQ_\alpha\}$, where $MinQ_\alpha = \min_t Q_{\alpha,t}$ is the minimum value over t of the quantile function for a significance level α. Next, we derive the expression for $MinQ_\alpha$. We have seen that, in the case of independent and identically distributed Normal cashflows $\Delta\pi_\tau \sim N(\mu, \sigma^2)$, the quantile function is given by the expression:

$$Q_{\alpha,t} = \mu t + Z_\alpha \sigma \sqrt{t} \quad (18)$$

The maximum quantile-loss is $MaxQL_\alpha = \max\{0, -MinQ_\alpha\}$. Differentiating Eq. (18), the first order necessary condition that determines the globally and unconstrained minimum value for $Q_{\alpha,t}$ is given by:[7]

$$\frac{\partial Q_{\alpha,t}}{\partial t} = \mu + \frac{1}{2\sqrt{t}} Z_\alpha \sigma = 0 \quad (19)$$

$\frac{1}{2\sqrt{t}} Z_\alpha \sigma < 0$ because $\alpha < \frac{1}{2} \Leftrightarrow Z_\alpha < 0$, thus this first order condition requires $\mu > 0$. Solving for *t*, we obtain the number of observations at which the lowest value of the function $Q_{\alpha,t}$ is realized,

$$t_\alpha^* = \left(\frac{Z_\alpha \sigma}{2\mu}\right)^2 \quad (20)$$

The second order sufficient condition is verified in $\frac{\partial^2 Q_{\alpha,t}}{\partial t^2} = -\frac{1}{4} Z_\alpha \sigma t^{-3/2} > 0$. This second derivative of the quantile function with respect to *t* is strictly positive. This means that $Q_{\alpha,t}$ is convex with respect to *t*, which guarantees the existence of a global minimum.

Combining both conditions, we can then evaluate $Q_{\alpha,t}$ at that optimized value t_α^* to obtain its minimum value with a significance level α:

$$MinQ_\alpha = Q_{\alpha,t^*} = \mu \left(\frac{Z_\alpha \sigma}{2\mu}\right)^2 + Z_\alpha \sigma \left|\frac{Z_\alpha \sigma}{2\mu}\right| = -\frac{(Z_\alpha \sigma)^2}{4\mu} \quad (21)$$

As expected, $MinQ_\alpha$ is not a function of *t*. Its negative value appears because, as we saw, $Z_\alpha < 0$ and $\mu > 0$ are sufficient conditions for the global maximum quantile-loss to exist.■

[7] We will treat *t* as a continuous variable in $\mathfrak{R}^+$ for the purpose of differentiation.

A.2.- PROOF TO PROPOSITION 2

The time under water (TuW) is the number of observations (in terms of bets or time), $t > 0$, that elapse until we first observe that $\pi_{t-1} < 0$ and $\pi_t \geq 0$. We can determine its upper boundary for a significance level $\alpha < \frac{1}{2}$ as the value $t > 0$ such that $Q_{\alpha,t} = QL_{\alpha,t} = 0$. This condition is satisfied at

$$\mu t + Z_\alpha \sigma \sqrt{t} = t\left(\mu\sqrt{t} + Z_\alpha \sigma\right) = 0 \tag{22}$$

and because $t > 0 \Longrightarrow \mu\sqrt{t} + Z_\alpha \sigma = 0$, we obtain

$$TuW_\alpha = \left(\frac{Z_\alpha \sigma}{\mu}\right)^2 \tag{23}$$

■

A.3.- PROOF TO PROPOSITION 3

Suppose that a portfolio manager experiences a cumulative performance at time *t* for an amount $\tilde{\pi}_t$, where $\tilde{\pi}_t < \pi_0 = 0$. We can determine the implied significance level, $\tilde{\alpha}$, that is associated with such a performance after *t* independent observations. More precisely, we would like to compute the value of $\tilde{\alpha}$ that verifies

$$\tilde{\pi}_t = Q_{\tilde{\alpha},t} \tag{24}$$

Applying Eq. (18) on Eq. (24), we can solve for $\tilde{\alpha}$ as follows

$$\tilde{\alpha} = \Phi\left[\frac{\tilde{\pi}_t - \mu t}{\sigma\sqrt{t}}\right] \tag{25}$$

where Φ is the cumulative distribution function for the Standard Normal distribution. The symbol Φ represents a function and should not be confounded with a critical value Z_α. For the sake of clarity, note that $\alpha = \Phi[Z_\alpha], \forall \alpha \in (0,1)$.

Eq. (25) tells us that, for any observed performance, we can compute its implied statistical significance, which can be understood as $\text{Prob}[\pi_t \leq \tilde{\pi}_t]$. Hence, $\tilde{\alpha}$ can be interpreted as the ex-ante probability that this strategy would have performed below $\tilde{\pi}_t$ after *t* observations. Inserting Eq. (25) in Eq. (23), we obtain

$$ITuW_{\tilde{\pi}_t} = \frac{{\tilde{\pi}_t}^2}{\mu^2 t} - 2\frac{\tilde{\pi}_t}{\mu} + t \tag{26}$$

Eq. (26) gives us the implied time under water associated with the realized performance $\tilde{\pi}_t$, where $\tilde{\alpha}$ is implied by that same realized performance, $\tilde{\pi}_t$. This proposition allows us to communicate stop-out limits more effectively than a mere $MaxQL_\alpha$ limit, because we do not need to wait until the maximum quantile-loss or time under water is reached. It suffices that $TuW_{\tilde{\pi}_t} > TuW_\alpha$ for any observed $\tilde{\pi}_t$.■

A.4.- PROOF TO THEOREM 1 ("TRIPLE PENANCE RULE")
Comparing Eqs. (20) and (23), we derive the expression

$$t^*_\alpha = \frac{1}{4}TuW_\alpha \tag{27}$$

Eq. (27) is true for any value $\alpha \in \left(0, \frac{1}{2}\right)$. These boundaries for α first appeared in Eq. (19). We call it "triple penance rule" because it tells us that, in the case of independent and identically distributed normal cashflows $\Delta\pi_\tau$, it takes three times longer to recover from the expected maximum quantile-loss than the time it took to produce it, for the same significance level $\alpha < \frac{1}{2}$.■

A.5.- PROOF TO PROPOSITION 4
This proposition generalizes the framework discussed in Section 3, by deriving the distribution of a cumulative function of a first-order auto-correlated random variable. Suppose an investment strategy which yields a sequence of cash inflows $\Delta\pi_\tau$ as a result of a sequence of bets $\tau \in \{1, \ldots, \infty\}$, where

$$\Delta\pi_\tau = (1-\varphi)\mu + \varphi\Delta\pi_{\tau-1} + \sigma\varepsilon_\tau \tag{28}$$

such that the random shocks are IID distributed as $\varepsilon_\tau \sim N(0,1)$. Recursively replacing the previous expression *j* times leads to

$$\Delta\pi_\tau = (1-\varphi)\mu\sum_{i=0}^{j-1}\varphi^i + \varphi^j\Delta\pi_{\tau-j} + \sigma\sum_{i=0}^{j-1}\varphi^i\varepsilon_{\tau-i} \tag{29}$$

For a process initialized at $\Delta\pi_0$, we can extend the recursion back to the origin, in which case $j = \tau$, and Eq. (29) becomes

$$\Delta\pi_\tau = (1-\varphi)\mu\sum_{i=0}^{\tau-1}\varphi^i + \varphi^\tau\Delta\pi_0 + \sigma\sum_{i=0}^{\tau-1}\varphi^i\varepsilon_{\tau-i} \tag{30}$$

This expression evidences that $\Delta\pi_\tau$ is a linear function of independent Gaussian random variables, hence $\Delta\pi_\tau$ is also Gaussian (Grinstead and Snell [1997]). From Eq. (30), we can derive its mean and variance

$$\begin{gathered} E_0[\Delta\pi_\tau] = (1-\varphi)\mu\sum_{i=0}^{\tau-1}\varphi^i + \varphi^\tau\Delta\pi_0 \\ V_0[\Delta\pi_\tau] = \sigma^2\sum_{i=0}^{\tau-1}\varphi^{2i} \end{gathered} \tag{31}$$

Eq. (31) shows that a necessary and sufficient condition for $\Delta\pi_\tau$ to be stationary is that $\varphi \in (-1,1)$, in which case the above mean and variance asymptotically converge to $\lim_{\tau\to\infty} E_0[\Delta\pi_\tau] = \mu$ and $\lim_{\tau\to\infty} V_0[\Delta\pi_\tau] = \frac{\sigma^2}{1-\varphi^2}$. From Eq. (31) we obtain that

$$\Delta\pi_\tau \sim N\left((1-\varphi)\mu\sum_{i=0}^{\tau-1}\varphi^i + \varphi^\tau\Delta\pi_0, \sigma^2\sum_{i=0}^{\tau-1}\varphi^{2i}\right) \tag{32}$$

Note that $\Delta\pi_\tau$ follows a Gaussian law, however it is not independent (due to φ) and it is not identically distributed (due to τ). Readers familiar with the time series literature may recognize Eq. (32) (see Hamilton [1990], for example). It is not immediately useful to us in its current form for two reasons. First, we are interested in the distribution of the cumulative process $\pi_t = \sum_{\tau=1}^{t}\Delta\pi_\tau$, because quantile-losses are not defined on $\Delta\pi_\tau$. A VaR approach is typically interested in $\Delta\pi_\tau$, but ours is a downside approach, which requires the modeling of the cumulative process over time. Second, Eq. (32) is not amenable to symbolic optimization because of the presence of the argument variable (t) in the discrete summation operators. Let us turn now our attention to the cumulative process, π_t. From Eq. (30):

$$\pi_t = \sum_{\tau=1}^{t}\Delta\pi_\tau = (1-\varphi)\mu\sum_{\tau=1}^{t}\sum_{i=0}^{\tau-1}\varphi^i + \Delta\pi_0\sum_{\tau=1}^{t}\varphi^\tau + \sigma\sum_{\tau=1}^{t}\sum_{i=0}^{\tau-1}\varphi^i\varepsilon_{\tau-i} \tag{33}$$

The last term can be conveniently operated as

$$\begin{gathered} E_0\left[\sigma\sum_{\tau=1}^{t}\sum_{i=0}^{\tau-1}\varphi^i\varepsilon_{\tau-i}\right] = 0 \\ V_0\left[\sigma\sum_{\tau=1}^{t}\sum_{i=0}^{\tau-1}\varphi^i\varepsilon_{\tau-i}\right] = \sigma^2 V_0\left[\sum_{\tau=1}^{t}\left(\varepsilon_\tau\left(\sum_{i=0}^{t-\tau}\varphi^i\right)\right)\right] \end{gathered} \tag{34}$$

We conclude that π_t is also a linear function of independent Gaussian random variables. This means that π_t is also Gaussian. We can compute the mean and variance of π_t as:

$$\begin{gathered} E_0[\pi_t] = (1-\varphi)\mu\sum_{\tau=1}^{t}\sum_{i=0}^{\tau-1}\varphi^i + \Delta\pi_0\sum_{\tau=1}^{t}\varphi^\tau \\ V_0[\pi_t] = \sigma^2\sum_{\tau=1}^{t}\left(\sum_{i=0}^{t-\tau}\varphi^i\right)^2 \end{gathered} \tag{35}$$

Eq. (35) addresses the first limitation we discussed earlier (these are the moments on π_t, not $\Delta\pi_\tau$), however it is still unsatisfactory with respect to the second feature: In order to analyze the expressions in Eq. (35), we need to compute their compact form, i.e. excluding the summation operators that involve t and τ. Applying the Theorem of Geometric Series on $E_0[\pi_t]$, we obtain:

$$\sum_{i=0}^{\tau-1} \varphi^i = \frac{\varphi^\tau - 1}{\varphi - 1}$$

$$\sum_{\tau=1}^{t}\sum_{i=0}^{\tau-1} \varphi^i = \sum_{\tau=1}^{t} \frac{\varphi^\tau - 1}{\varphi - 1} = \frac{1}{\varphi - 1}\left(-t + \sum_{\tau=1}^{t} \varphi^\tau\right) \tag{36}$$

$$= \frac{1}{\varphi - 1}\left(-t + \frac{\varphi^{t+1} - 1}{\varphi - 1} - 1\right) = \frac{\varphi^{t+1} - 1}{(\varphi - 1)^2} - \frac{t+1}{\varphi - 1}$$

$$\sum_{\tau=1}^{t} \varphi^\tau = \frac{\varphi^{t+1} - 1}{\varphi - 1} - 1$$

Thus,

$$E_0[\pi_t] = \frac{\varphi^{t+1} - \varphi}{\varphi - 1}(\Delta\pi_0 - \mu) + \mu t \tag{37}$$

This new expression for $E_0[\pi_t]$ is easier to deal with from an Analysis perspective. Likewise, we can reduce $V_0[\pi_t]$ by recognizing that

$$\left(\sum_{i=0}^{t-\tau} \varphi^i\right)^2 = \frac{\varphi^{2(t-\tau+1)} - 2\varphi^{t-\tau+1} + 1}{(\varphi - 1)^2}$$

$$\sum_{\tau=1}^{t}\left(\sum_{i=0}^{t-\tau} \varphi^i\right)^2 = \frac{1}{(\varphi - 1)^2}\left(\sum_{\tau=1}^{t} \varphi^{2(t-\tau+1)} - 2\sum_{\tau=1}^{t} \varphi^{t-\tau+1} + t\right) \tag{38}$$

$$= \frac{1}{(\varphi - 1)^2}\left(\frac{\varphi^{2(t+1)} - 1}{\varphi^2 - 1} - 2\frac{\varphi^{t+1} - 1}{\varphi - 1} + t + 1\right)$$

Thus,

$$V_0[\pi_t] = \frac{\sigma^2}{(\varphi - 1)^2}\left(\frac{\varphi^{2(t+1)} - 1}{\varphi^2 - 1} - 2\frac{\varphi^{t+1} - 1}{\varphi - 1} + t + 1\right) \tag{39}$$

Finally, we conclude that

$$\pi_t \sim N\left(\frac{\varphi^{t+1}-\varphi}{\varphi-1}(\Delta\pi_0-\mu)+\mu t, \frac{\sigma^2}{(\varphi-1)^2}\left(\frac{\varphi^{2(t+1)}-1}{\varphi^2-1}-2\frac{\varphi^{t+1}-1}{\varphi-1}+t+1\right)\right) \tag{40}$$

Eq. (40) has the two features we were looking for: First, it will allow us to compute the quantile-loss of a strategy with performance π_t. Second, we have been able to express this result in a compact form, amenable to Analysis.■

A.6.- PROOF TO PROPOSITION 5

We can use the analytical result obtained in Proposition 4 to study the behavior of the quantile-loss and maximum quantile-loss functions in the more general case of first-order auto-correlated cashflows. The quantile function in this case is

$$Q_{\alpha,t}=\frac{\varphi^{t+1}-\varphi}{\varphi-1}(\Delta\pi_0-\mu)+\mu t+Z_\alpha\frac{\sigma}{|\varphi-1|}\left(\frac{\varphi^{2(t+1)}-1}{\varphi^2-1}-2\frac{\varphi^{t+1}-1}{\varphi-1}+t+1\right)^{1/2} \tag{41}$$

where Z_α is the critical value of the Standard Normal distribution associated with a probability α of performing worse than $Q_{\alpha,t}$, i.e. $\alpha=\text{Prob}\left[\pi_t\leq Q_{\alpha,t}\right]$. Differentiating $Q_{\alpha,t}$ with respect to *t*:

$$\frac{\partial^i Q_{\alpha,t}}{\partial t^i}=\frac{\partial^i E_0[\pi_t]}{\partial t^i}+Z_\alpha\frac{\partial^i\sqrt{V_0[\pi_t]}}{\partial t^i} \tag{42}$$

for $i=1,2$, where

$$\frac{\partial E_0[\pi_t]}{\partial t}=\frac{Ln[\varphi]\varphi^{t+1}}{\varphi-1}(\Delta\pi_0-\mu)+\mu$$

$$\frac{\partial\sqrt{V_0[\pi_t]}}{\partial t}=\frac{\sigma}{2|\varphi-1|\sqrt{\frac{\varphi^{2(t+1)}-1}{\varphi^2-1}-2\frac{\varphi^{t+1}-1}{\varphi-1}+t+1}}\left(\frac{2Ln[\varphi]\varphi^{2(t+1)}}{\varphi^2-1}-\frac{2Ln[\varphi]\varphi^{t+1}}{\varphi-1}+1\right) \tag{43}$$

Note that these derivatives only have real solutions for $\varphi > 0$.[8] That, combined with the stationarity condition, limits our range of study to $\varphi \in (0,1)$. For that range of first-order autocorrelation, $\frac{2Ln[\varphi]\varphi^{2(t+1)}}{\varphi^2-1} - \frac{2Ln[\varphi]\varphi^{t+1}}{\varphi-1} + 1 > 0$, thus $\frac{\partial\sqrt{V_0[\pi_t]}}{\partial t} > 0$. Risk is monotonically increasing over time, however the rate of increases converges to zero over time, as $\lim_{t\to\infty}\frac{\partial\sqrt{V_0[\pi_t]}}{\partial t} = 0^+$ (a convergence to zero from the right). $\frac{\partial E_0[\pi_t]}{\partial t}$ converges asymptotically to $\lim_{t\to\infty}\frac{\partial E_0[\pi_t]}{\partial t} = \mu$, starting below that level if $\Delta\pi_0 < \mu$, and starting above that level if $\Delta\pi_0 > \mu$. If μ is so large that at *t=1* occurs that $\frac{\partial E_0[\pi_t]}{\partial t} > -Z_\alpha \frac{\partial\sqrt{V_0[\pi_t]}}{\partial t}$, $Q_{\alpha,t}$ will have its minimum at $Q_{\alpha,1}$ and $QL_\alpha = 0$. For a smaller but still positive μ, $Q_{\alpha,t}$ will be initially a decreasing function of *t*, but it will eventually become an increasing function of *t*, once the effect of $\frac{\partial E_0[\pi_t]}{\partial t}$ overcomes the effect of $\frac{\partial\sqrt{V_0[\pi_t]}}{\partial t}$.

$\alpha \in \left(0,\frac{1}{2}\right) \Longrightarrow Z_\alpha < 0$, so in order to guarantee a global minimum we would like to find that $\frac{\partial^2 E_0[\pi_t]}{\partial t^2} \geq 0$ and $\frac{\partial^2\sqrt{V_0[\pi_t]}}{\partial t^2} < 0$. If we differentiate again the first expression in Eq. (43), we get:

$$\frac{\partial^2 E_0[\pi_t]}{\partial t^2} = \frac{(Ln[\varphi])^2\varphi^{t+1}}{\varphi-1}(\Delta\pi_0 - \mu) \tag{44}$$

We can see that $\Delta\pi_0 < \mu \Longrightarrow \frac{\partial^2 E_0[\pi_t]}{\partial t^2} > 0$, and $\Delta\pi_0 > \mu \Longrightarrow \frac{\partial^2 E_0[\pi_t]}{\partial t^2} < 0$, but in any case this component wears out, since $\lim_{t\to\infty}\frac{\partial^2 E_0[\pi_t]}{\partial t^2} = 0$. If we differentiate again the second expression in Eq. (43), after some operations we get:

$$\frac{\partial^2\sqrt{V_0[\pi_t]}}{\partial t^2} = \frac{\sigma(Ln[\varphi])^2\left(\frac{2\varphi^{2(t+1)}}{\varphi^2-1} - \frac{\varphi^{t+1}}{\varphi-1}\right)}{|\varphi-1|\sqrt{\frac{\varphi^{2(t+1)}-1}{\varphi^2-1} - 2\frac{\varphi^{t+1}-1}{\varphi-1} + t + 1}} - \frac{\sigma\left(\frac{2\varphi^{2(t+1)}Ln[\varphi]}{\varphi^2-1} - \frac{2\varphi^{t+1}Ln[\varphi]}{\varphi-1} + 1\right)^2}{4|\varphi-1|\left(\frac{\varphi^{2(t+1)}-1}{\varphi^2-1} - 2\frac{\varphi^{t+1}-1}{\varphi-1} + t + 1\right)^{3/2}} \tag{45}$$

[8] The case $\varphi = 0$ would be easy to handle, by simplifying φ in $Q_{\alpha,t}$ before differentiating. However, we do not need to concern ourselves with an analysis of that case, since that was already addressed by Propositions 1 and 2. The case $-1 < \varphi < 0$ introduces solutions in the complex domain, which would require a separate treatment. Numerical experiments show that even in this case the quantile function is unimodal and is endowed with a global minimum.

Because $\frac{2\varphi^{2(t+1)}}{\varphi^2-1} - \frac{\varphi^{t+1}}{\varphi-1} > 0$ within the range $\varphi \in (0,1)$, $\frac{\partial^2 \sqrt{V_0[\pi_t]}}{\partial t^2}$ can be positive for a small t. So $Q_{\alpha,t}$ could be initially concave, either because of $\frac{\partial^2 E_0[\pi_t]}{\partial t^2} < 0$, $\frac{\partial^2 \sqrt{V_0[\pi_t]}}{\partial t^2}$ or both. As t increases, $\varphi^t \to 0$, and $\frac{\partial^2 \sqrt{V_0[\pi_t]}}{\partial t^2} \approx -\frac{\sigma}{4|\varphi-1|\left(\frac{-1}{\varphi^2-1}+2\frac{1}{\varphi-1}+t+1\right)^{3/4}} < 0$ while $\frac{\partial^2 E_0[\pi_t]}{\partial t^2} \approx 0$, so in any case $Q_{\alpha,t}$ is convex after a sufficient number of bets. $Q_{\alpha,t}$ remains convex but increasingly linear, with $\lim_{t\to\infty} \frac{\partial^2 \sqrt{V_0[\pi_t]}}{\partial t^2} = 0^-$ (a convergence to zero from the left).

Putting all the pieces together, we know that $Q_{\alpha,t}$ could be initially a concave function of t, but eventually it becomes convex. It is guaranteed that $Q_{\alpha,t}$ has either zero or one inflexion point. As long as $\mu > 0$, $Q_{\alpha,t}$ is unimodal and a global minimum exists. The following section shows how to compute $MaxQL_\alpha = \max\{0, -MinQ_\alpha\}$.■

The Python code that numerically verifies the accuracy of our analytical solution can be found at http://ssrn.com/abstract=2511599 and www.QuantResearch.info/Software.htm

TABLES

PARAMETERS	PM1	PM2
Expected annual PnL	10,000,000	15,000,000
Expected Std annual PnL	10,000,000	10,000,000
# Independent trades per year	12	12
Confidence	95%	95%

Table 1 – Trading parameters for evaluating the stop-out

HFR Index	Bloomberg Code	Currency
HFRI Fund of Funds Composite Index	HFRIFOF Index	USD
HFRI Fund Weighted Composite Index	HFRIFWI Index	USD
HFRI Equity Hedge (Total) Index	HFRIEHI Index	USD
HFRI Macro (Total) Index	HFRIMI Index	USD
HFRI FOF: Diversified Index	HFRIFOFD Index	USD
HFRI ED: Distressed/Restructuring Index	HFRIDSI Index	USD
HFRI EH: Equity Market Neutral Index	HFRIEMNI Index	USD
HFRI FOF: Conservative Index	HFRIFOFC Index	USD
HFRI Event-Driven (Total) Index	HFRIEDI Index	USD
HFRI Macro: Systematic Diversified Index	HFRIMTI Index	USD
HFRI RV: Fixed Income-Corporate Index	HFRIFIHY Index	USD
HFRI RV: Multi-Strategy Index	HFRIFI Index	USD
HFRI Relative Value (Total) Index	HFRIRVA Index	USD
HFRI ED: Merger Arbitrage Index	HFRIMAI Index	USD
HFRI RV: Fixed Income-Convertible Arbitrage Index	HFRICAI Index	USD
HFRI Emerging Markets (Total) Index	HFRIEM Index	USD
HFRI Emerging Markets: Asia ex-Japan Index	HFRIEMA Index	USD
HFRI EH: Short Bias Index	HFRISHSE Index	USD
HFRI Emerging Markets: Latin America Index	HFRIEMLA Index	USD
HFRI FOF: Strategic Index	HFRIFOFS Index	USD
HFRI EH: Quantitative Directional	HFRIENHI Index	USD
HFRI Fund Weighted Composite Index - GBP	HFRIFWIG Index	GBP
HFRI FOF: Market Defensive Index	HFRIFOFM Index	USD
HFRI Fund Weighted Composite Index - CHF	HFRIFWIC Index	CHF
HFRI Fund Weighted Composite Index - JPY	HFRIFWIJ Index	JPY
HFRI EH: Sector - Technology/Healthcare Index	HFRISTI Index	USD

Table 2 – Selected Hedge Fund Research Indices

Table 2 lists the hedge fund indices in the HFR database with a history from 01/01/1990 to 01/01/2013. These are the indices that we have used in the empirical study presented in Section 10.

Code	Mean	StDev	Phi	Sigma	t-Stat(Phi)	MaxQL	t*	TuW	Penance
HFRIFOF Index	0.0055	0.0170	0.3594	0.0158	6.2461	6.65%	14.5551	52.1831	2.5852
HFRIFWI Index	0.0089	0.0202	0.3048	0.0192	5.1907	4.74%	7.3222	24.4918	2.3449
HFRIEHI Index	0.0099	0.0264	0.2651	0.0255	4.4601	7.27%	9.0236	32.1120	2.5587
HFRIMI Index	0.0095	0.0215	0.1844	0.0211	3.0419	4.15%	5.4157	19.1093	2.5285
HFRIFOFD Index	0.0052	0.0174	0.3535	0.0163	6.1295	7.52%	16.9638	61.9700	2.6531
HFRIDSI Index	0.0096	0.0188	0.5458	0.0158	10.5612	5.40%	10.7065	30.4208	1.8413
HFRIEMNI Index	0.0052	0.0094	0.1644	0.0093	2.7035	1.33%	3.4722	11.6921	2.3674
HFRIFOFC Index	0.0048	0.0116	0.4557	0.0103	8.3023	4.00%	11.9696	39.0229	2.2602
HFRIEDI Index	0.0095	0.0192	0.3916	0.0177	6.9021	4.34%	7.3855	22.6758	2.0703
HFRIMTI Index	0.0085	0.0216	-0.0188	0.0216	-0.3051	--	--	--	--
HFRIFIHY Index	0.0072	0.0177	0.4838	0.0155	8.9720	6.69%	13.3986	43.7383	2.2644
HFRIFI Index	0.0069	0.0129	0.5059	0.0111	9.5874	3.12%	8.9080	25.0456	1.8116
HFRIRVA Index	0.0080	0.0130	0.4528	0.0116	8.2430	2.00%	5.9134	15.3920	1.6029
HFRIMAI Index	0.0071	0.0104	0.2982	0.0100	5.0670	1.08%	3.2508	8.9163	1.7428
HFRICAI Index	0.0071	0.0200	0.5780	0.0163	11.4865	11.60%	22.1308	74.4170	2.3626
HFRIEM Index	0.0104	0.0410	0.3593	0.0383	6.2431	21.71%	23.4821	87.9134	2.7439
HFRIEMA Index	0.0080	0.0382	0.3112	0.0363	5.3109	22.57%	30.2969	116.2881	2.8383
HFRISHSE Index	-0.0017	0.0535	0.0907	0.0533	1.4776	--	--	--	--
HFRIEMLA Index	0.0111	0.0508	0.1969	0.0499	3.2575	22.77%	21.7061	84.0775	2.8735
HFRIFOFS Index	0.0068	0.0248	0.3231	0.0235	5.5360	11.00%	18.2415	67.7961	2.7166
HFRIENHI Index	0.0101	0.0367	0.2011	0.0359	3.3299	12.84%	13.8963	52.7651	2.7971
HFRIFWIG Index	0.0094	0.0360	0.2314	0.0350	3.8573	14.15%	16.4723	62.5481	2.7972
HFRIFOFM Index	0.0056	0.0159	0.0422	0.0159	0.6842	3.25%	6.0074	23.5097	2.9135
HFRIFWIC Index	0.0089	0.0390	0.0505	0.0390	0.8200	12.59%	14.3295	56.6921	2.9563
HFRIFWIJ Index	0.0084	0.0363	0.0954	0.0361	1.5542	12.55%	15.4084	60.4123	2.9207
HFRISTI Index	0.0111	0.0464	0.1608	0.0458	2.6428	17.61%	16.8089	65.0637	2.8708

Table 3 – Maximum Quantile-Loss and Time under Water considering first-order serial correlation

Table 3 reports the descriptive statistics computed on the hedge fund indices listed in Table 2, when we take into account first-order auto-correlation. *MaxQL* is the maximum quantile-loss at a $\alpha = 0.05$ significance level, which occurs after $t*$ observations. At that same significance level, these hedge fund indices remain under water for the period reported in column *TuW*. *Penance* measures how long it takes to recover from the maximum quantile-loss as a multiple of the time it took to reach the bottom. As we can appreciate, *Penance* ranges between 1.6 and 3, and it is smaller the higher φ (*Phi*) and the higher the ratio $\frac{\mu}{\sigma}$ (*Mean* divided by *Sigma*). Although positive serial correlation leads to greater downside potential, longer t_α^* and longer periods under water, the *Penance* is smaller.

Code	Mean	Phi	Sigma	MaxQL	t*	TuW	Penance
HFRIFOF Index	0.0055	0.0000	0.0170	3.53%	6.3996	25.5985	3.0000
HFRIFWI Index	0.0089	0.0000	0.0202	3.10%	3.4905	13.9621	3.0000
HFRIEHI Index	0.0099	0.0000	0.0264	4.80%	4.8667	19.4669	3.0000
HFRIMI Index	0.0095	0.0000	0.0215	3.28%	3.4435	13.7740	3.0000
HFRIFOFD Index	0.0052	0.0000	0.0174	3.96%	7.6477	30.5909	3.0000
HFRIDSI Index	0.0096	0.0000	0.0188	2.48%	2.5827	10.3309	3.0000
HFRIEMNI Index	0.0052	0.0000	0.0094	1.16%	2.2389	8.9554	3.0000
HFRIFOFC Index	0.0048	0.0000	0.0116	1.90%	3.9492	15.7968	3.0000
HFRIEDI Index	0.0095	0.0000	0.0192	2.63%	2.7554	11.0216	3.0000
HFRIMTI Index	0.0085	0.0000	0.0216	3.69%	4.3218	17.2870	3.0000
HFRIFIHY Index	0.0072	0.0000	0.0177	2.95%	4.1164	16.4656	3.0000
HFRIFI Index	0.0069	0.0000	0.0129	1.64%	2.3883	9.5530	3.0000
HFRIRVA Index	0.0080	0.0000	0.0130	1.42%	1.7701	7.0803	3.0000
HFRIMAI Index	0.0071	0.0000	0.0104	1.03%	1.4444	5.7777	3.0000
HFRICAI Index	0.0071	0.0000	0.0200	3.79%	5.3200	21.2800	3.0000
HFRIEM Index	0.0104	0.0000	0.0410	10.98%	10.6100	42.4399	3.0000
HFRIEMA Index	0.0080	0.0000	0.0382	12.38%	15.4963	61.9851	3.0000
HFRISHSE Index	-0.0017	0.0000	0.0535	--	--	--	--
HFRIEMLA Index	0.0111	0.0000	0.0508	15.79%	14.2615	57.0458	3.0000
HFRIFOFS Index	0.0068	0.0000	0.0248	6.09%	8.9046	35.6185	3.0000
HFRIENHI Index	0.0101	0.0000	0.0367	9.02%	8.9357	35.7430	3.0000
HFRIFWIG Index	0.0094	0.0000	0.0360	9.33%	9.9416	39.7662	3.0000
HFRIFOFM Index	0.0056	0.0000	0.0159	3.05%	5.4422	21.7686	3.0000
HFRIFWIC Index	0.0089	0.0000	0.0390	11.50%	12.8580	51.4319	3.0000
HFRIFWIJ Index	0.0084	0.0000	0.0363	10.58%	12.5579	50.2317	3.0000
HFRISTI Index	0.0111	0.0000	0.0464	13.17%	11.8933	47.5731	3.0000

Table 4 – Maximum Quantile-Loss and Time under Water ignoring serial correlation

It is common to assume returns independence (thus disregarding evidence of serial correlation) to simplify calculations. Table 4 reports the maximum quantile-loss ($MaxQL$), the observation at which the maximum quantile-loss occurs (t^*_α) and the quantile-time under water (TuW) for a significance level of $\alpha = 0.05$. Wrongly assuming $\varphi = 0$ (see column *Phi*) may lead to a gross underestimation of the downside potential, in some cases by as much as 70%. As predicted by the "triple penance rule", the number of observations it takes to resurface after the maximum quantile-loss is exactly 3 times the number of observations it took to reach that maximum quantile-loss, with the same confidence level.

Code	MaxQL	t*	Alpha1	Mean2	Phi2	Sigma2	Alpha2
HFRIFOF Index	0.0353	6.3996	0.0500	0.0055	0.3594	0.0158	0.1205
HFRIFWI Index	0.0310	3.4905	0.0500	0.0089	0.3048	0.0192	0.1014
HFRIEHI Index	0.0480	4.8667	0.0500	0.0099	0.2651	0.0255	0.0975
HFRIMI Index	0.0328	3.4435	0.0500	0.0095	0.1844	0.0211	0.0796
HFRIFOFD Index	0.0396	7.6477	0.0500	0.0052	0.3535	0.0163	0.1207
HFRIDSI Index	0.0248	2.5827	0.0500	0.0096	0.5458	0.0158	0.1312
HFRIEMNI Index	0.0116	2.2389	0.0500	0.0052	0.1644	0.0093	0.0728
HFRIFOFC Index	0.0190	3.9492	0.0500	0.0048	0.4557	0.0103	0.1331
HFRIEDI Index	0.0263	2.7554	0.0500	0.0095	0.3916	0.0177	0.1114
HFRIMTI Index	0.0369	4.3218	0.0500	0.0085	-0.0188	0.0216	--
HFRIFIHY Index	0.0295	4.1164	0.0500	0.0072	0.4838	0.0155	0.1400
HFRIFI Index	0.0164	2.3883	0.0500	0.0069	0.5059	0.0111	0.1224
HFRIRVA Index	0.0142	1.7701	0.0500	0.0080	0.4528	0.0116	0.1029
HFRIMAI Index	0.0103	1.4444	0.0500	0.0071	0.2982	0.0100	0.0814
HFRICAI Index	0.0379	5.3200	0.0500	0.0071	0.5780	0.0163	0.1688
HFRIEM Index	0.1098	10.6100	0.0500	0.0104	0.3593	0.0383	0.1243
HFRIEMA Index	0.1238	15.4963	0.0500	0.0080	0.3112	0.0363	0.1139
HFRISHSE Index	--	--	0.0500	-0.0017	0.0907	0.0533	--
HFRIEMLA Index	0.1579	14.2615	0.0500	0.0111	0.1969	0.0499	0.0873
HFRIFOFS Index	0.0609	8.9046	0.0500	0.0068	0.3231	0.0235	0.1145
HFRIENHI Index	0.0902	8.9357	0.0500	0.0101	0.2011	0.0359	0.0872
HFRIFWIG Index	0.0933	9.9416	0.0500	0.0094	0.2314	0.0350	0.0940
HFRIFOFM Index	0.0305	5.4422	0.0500	0.0056	0.0422	0.0159	0.0567
HFRIFWIC Index	0.1150	12.8580	0.0500	0.0089	0.0505	0.0390	0.0586
HFRIFWIJ Index	0.1058	12.5579	0.0500	0.0084	0.0954	0.0361	0.0667
HFRISTI Index	0.1317	11.8933	0.0500	0.0111	0.1608	0.0458	0.0795

Table 5 – Intended vs. actual probability of false positives

Table 5 reports the effective probability of false positives when portfolio managers or strategies are stopped-out based on quantile-loss limits that ignore first-order auto-correlation. For all hedge fund styles, the actual probability of false positives is considerably greater than the one intended. Because many firms evaluate their managers' performance assuming independent returns (e.g., Sharpe ratio), they are improperly stopping them out. In some cases, they may be firing more than three times the number of skillful portfolio managers, compared to the number they were willing to accept under the (wrong) assumption of returns independence.

FIGURES

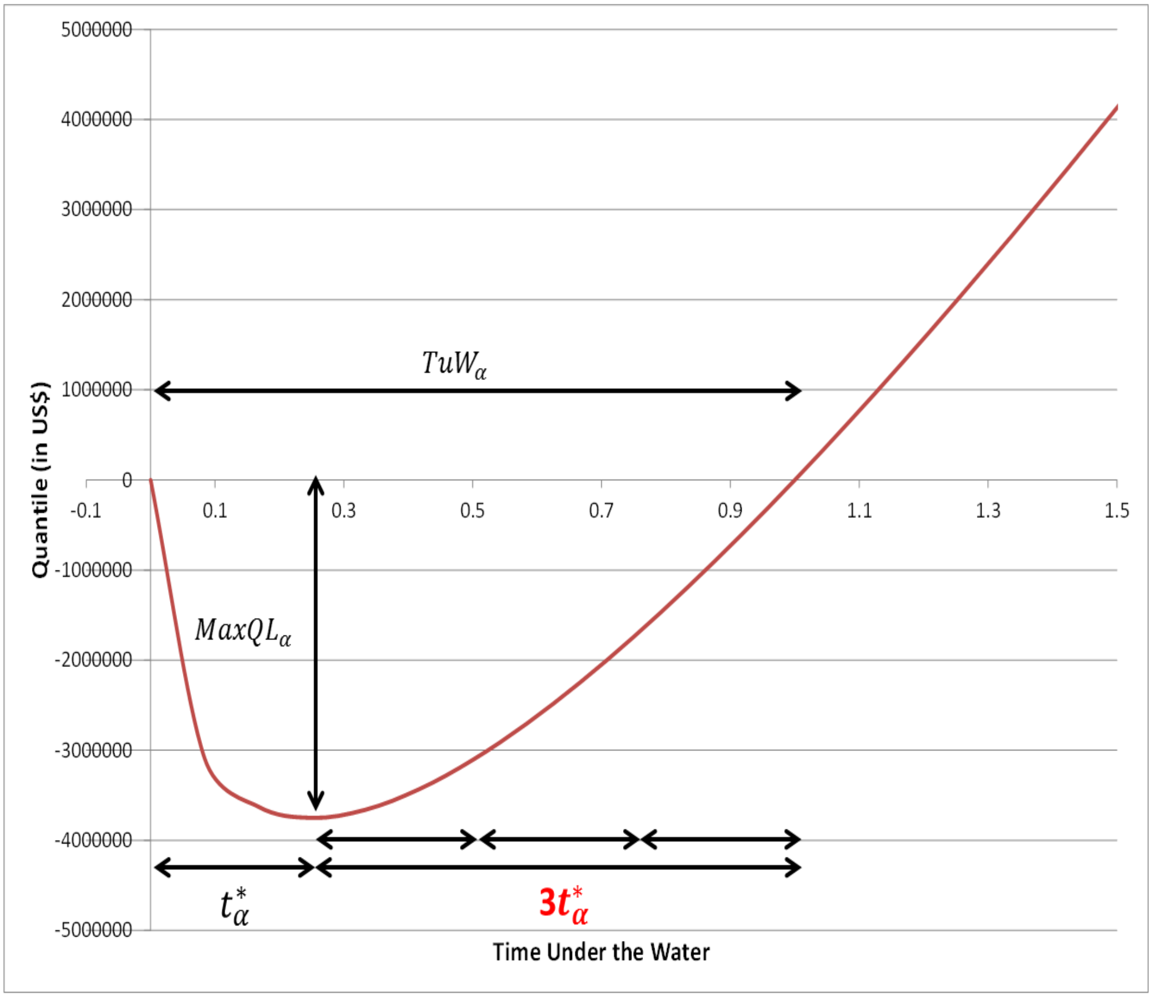

Figure 1 – The Triple Penance rule

Figure 1 provides a graphical representation of the Triple Penance rule. It takes three time longer to recover from the maximum quantile-loss ($MaxTuW_\alpha$) than the time it took to produce it (t^*_α), for a given significance level $\alpha < \frac{1}{2}$, regardless of the PM's Sharpe ratio.

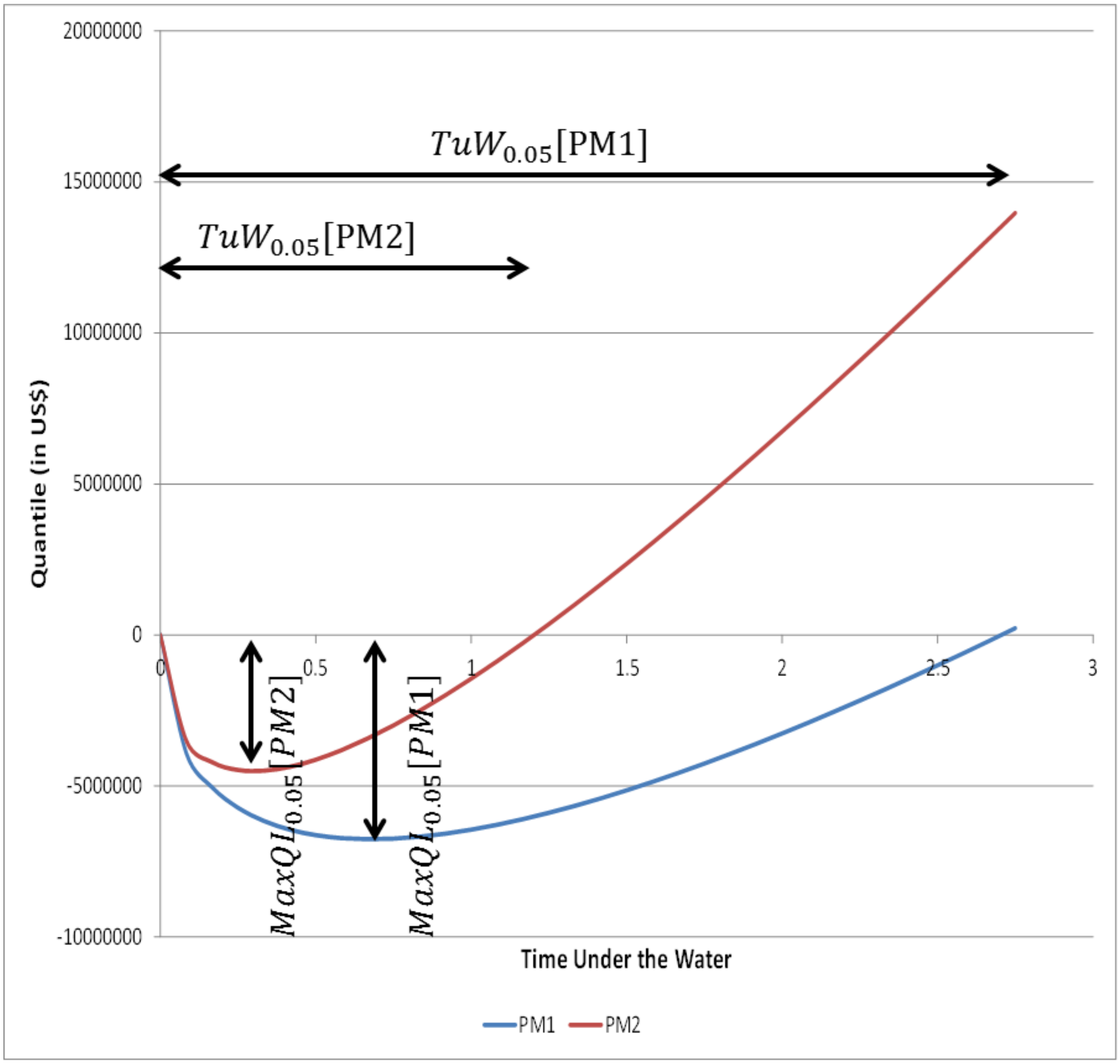

Figure 2 – Quantile and Time under water for PM1 and PM2, with the same confidence level (95%)

Figure 2 plots the quantile-loss function $Q_{\alpha,t}$ as time passes, for $\alpha = 0.05$, where PM1 has an annualized Sharpe ratio of 1 (annual mean and standard deviation of US$10m), and PM2 has an annualized Sharpe ratio of 1.5 (annual mean of US$15m, and annual standard deviation of US$10m). For that 95% confidence level, PM1 reaches a maximum quantile-loss at US$6,763,858.64 after 0.676 years, and remains up to 2.706 years under water, whereas PM2 reaches a maximum quantile-loss at US$4,509,239.09 after 0.3 years, and remains 1.202 years under water. These results are consistent with the "triple penance" rule.

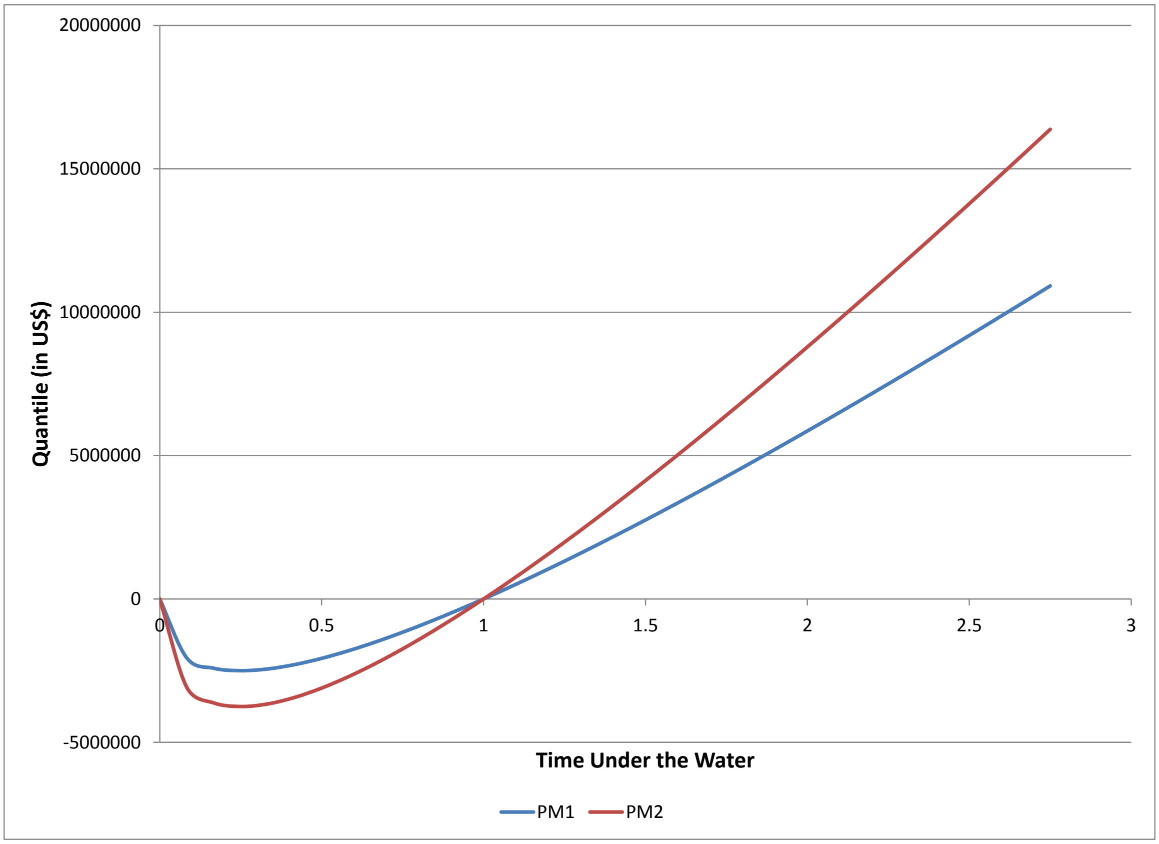

Figure 3 – Quantile and Time under water for PM1 and PM2, with confidence levels that aim at a maximum of 1 year under water

Figure 3 plots the quantile-loss function $Q_{\alpha,t}$ as time passes, where $\bar{\alpha}$ has been computed individually to meet the goal of being under water a maximum of 1 year. As a result, whereas before we had a tighter stop-out for the portfolio manager with higher Sharpe ratio, now the stricter stop-out is imposed on the portfolio manager with lower Sharpe ratio. This is consistent with the business reality that a hedge fund faces the risk of seeing good portfolio managers defecting if a performance bonus cannot be paid within a certain period.

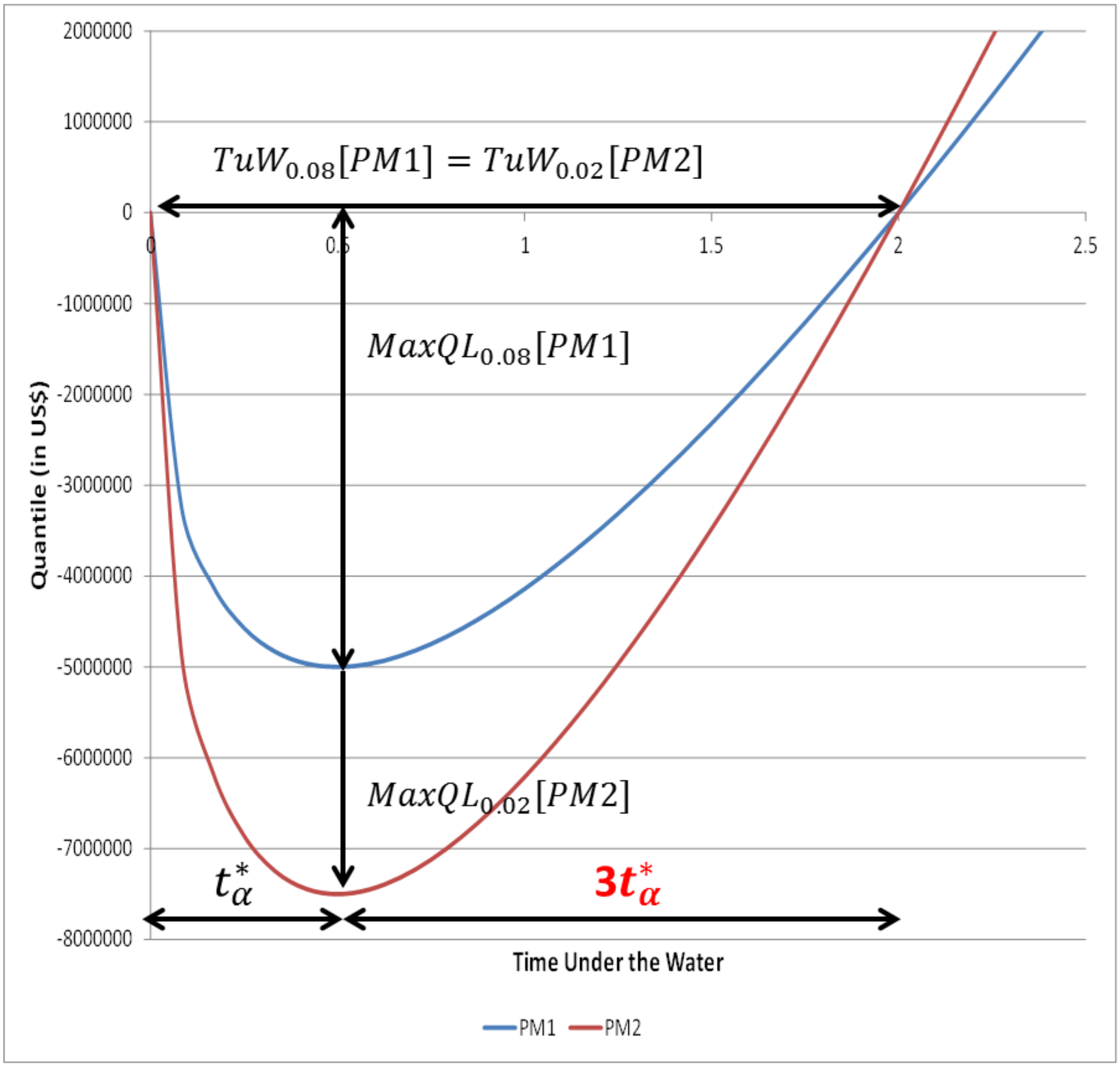

Figure 4 – Quantile-Loss and Time under water for PM1 and PM2, with confidence levels that aim at a maximum of 2 years under water

Figure 4 plots the quantile-loss function $Q_{\alpha,t}$ as time passes, where $\bar{\alpha}$ has been computed to meet the goal of being under water a maximum of 2 years. As a result of the longer maximum period (2 years instead of 1), we are even more permissive of in setting stop-out levels for PM2 than we were in Figure 2.

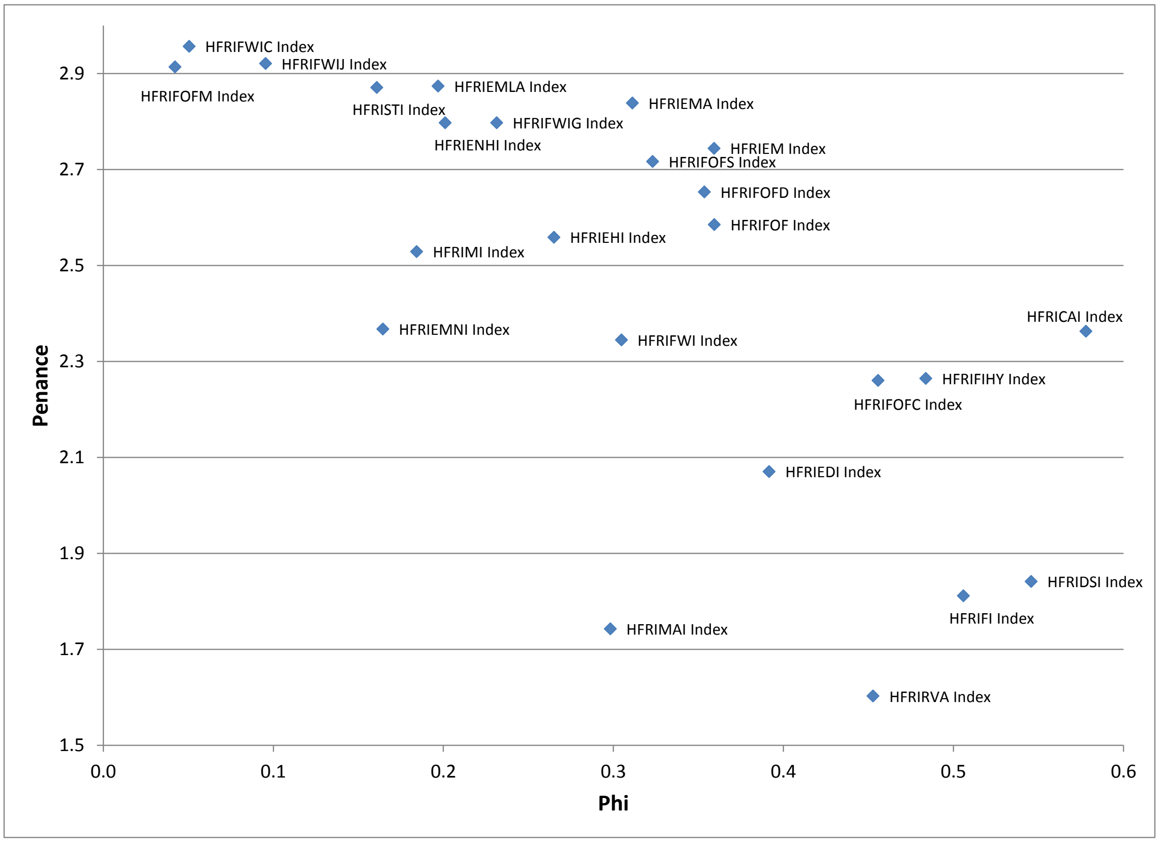

Figure 5 – The effect of higher serial correlation on Penance

Figure 5 plots *Penance* for hedge fund indices with various $\hat{\varphi}$. Although positive serial correlation leads to greater quantile-losses, longer t_{α}^{*} and longer periods under water, the *Penance* may be substantially smaller. In particular, *Penance* is smaller the higher $\hat{\varphi}$ (*Phi*) and the higher the ratio $\frac{\mu}{\sigma}$ (*Mean* divided by *Sigma*).

REFERENCES

- Bailey, D. and M. López de Prado (2012): "The Sharpe Ratio Efficient Frontier". *Journal of Risk*, 15(2), Winter, 3-44. Available at http://ssrn.com/abstract=1821643.
- Bailey, D. and M. López de Prado (2013): "The Strategy Approval Decision: A Sharpe Ratio Indifference Curve Approach". *Algorithmic Finance,2(1), 99-109*. Available at http://ssrn.com/abstract=2003638.
- Carr, P., H. Zhang and O. Hadjiliadis (2011): "Maximum Drawdown Insurance", *International Journal of Theoretical and Applied Finance*, 14(8), pp. 1195-1230.
- Chekhlov, A., S. Uryasev and M. Zabarankin (2003): "Portfolio optimization with drawdown constraints", in B. Scherer (Ed.): "Asset and liability management tools." Risk Books.
- Chekhlov, A., S. Uryasev and M. Zabarankin (2005): "Drawdown measure in portfolio optimization." *International Journal of Theoretical and Applied Finance*, Vol. 8(1), pp. 13-58.
- Chen, J. (2014): "Measuring Market Risk Under the Basel Accords: VaR, Stressed VaR, and Expected Shortfall". *The IEB International Journal of Finance*, Vol. 8, pp. 184-201. Available at SSRN: http://ssrn.com/abstract=2252463
- Getmansky, M., A. Lo and I. Makarov (2004): "An econometric model of serial correlation and illiquidity in hedge fund returns." *Journal of Financial Economics*, Vol. 74, pp. 529-609.
- Grinstead, C. and Snell (1997): "Introduction to Probability." American Mathematical Society, Chapter 7, 2nd Edition.
- Grossman, S. and Z. Zhou (1993): "Optimal Investment Strategies for controlling drawdowns." *Mathematical Finance*, Vol. 3, pp. 241-276.
- Hamilton, J. (1994): "Time Series Analysis." Princeton, Chapter 4.
- Hayes, B. (2006): "Maximum drawdowns of hedge funds with serial correlation." *Journal of Alternative Investments*, Vol. 8(4), pp. 26-38.
- Jorion, P. (2006): "Value at Risk: The new benchmark for managing financial risk." McGraw-Hill, 3rd Edition.
- Lin, C., W. Owens and J. Owers (2010): "The association between market risk disclosure reporting and firm risk: In impact of SEC FRR No. 48", *The Journal of Applied Business Research*, 26(4), July/August, pp. 35-46.
- Lo, A. (2002): "The Statistics of Sharpe Ratios." *Journal of Financial Analysts*, Vol. 58, No. 4, July/August.
- López de Prado, M. and A. Peijan (2004): "Measuring the Loss Potential of Hedge Fund Strategies." *Journal of Alternative Investments*, Vol. 7(1), pp. 7-31. Available at http://ssrn.com/abstract=641702.
- Magdon-Ismail, M. and A. Atiya (2004): "Maximum drawdown." *Risk Magazine*, October.

- Magdon-Ismail, M., A. Atiya, A. Pratap and Y. Abu-Mostafa (2004): "On the maximum drawdown of a Brownian motion." *Journal of Applied Probability*, Vol. 41(1).
- Markowitz, H.M. (1952): "Portfolio Selection." *Journal of Finance*, Vol. 7(1), pp. 77–91.
- Markowitz, H.M. (1956): "The Optimization of a Quadratic Function Subject to Linear Constraints." *Naval Research Logistics Quarterly*, Vol. 3, 111–133.
- Markowitz, H.M. (1959): "Portfolio Selection: Efficient Diversification of Investments." John Wiley and Sons.
- Mendes, M. and R. Leal (2005): "Maximum drawdown: Models and applications." *Journal of Alternative Investments*, Vol. 7, pp. 83-91.
- Meucci, A. (2005): "Risk and Asset Allocation". Springer.
- Pavlikov, K., S. Uryasev and M. Zabarankin (2012): "Capital Asset Pricing Model (CAPM) with drawdown measure." Research Report 2012-9, ISE Dept., University of Florida, September.
- Sharpe, W. (1975) "Adjusting for Risk in Portfolio Performance Measurement." *Journal of Portfolio Management*, Vol. 1(2), Winter, pp. 29-34.
- Sharpe, W. (1994) "The Sharpe ratio." *Journal of Portfolio Management*, Vol. 21(1), Fall, pp. 49-58.
- Yang, Z. and L. Zhong (2012): "Optimal portfolio strategy to control maximum drawdown: The case of risk-based dynamic asset allocation." SSRN Working Paper Series.
- Zhang, H., T. Leung and O. Hadjiliadis (2013): "Stochastic modeling and fair valuation of drawdown insurance." *Insurance, Mathematics and Economics*, 53(3), pp. 840-850.