On a Tan Product Conjecture

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1 Introduction

According to Joseph P. Buhler, the following is an open conjecture:

Conjecture 1 *Prove or disprove:*

$$\prod_{k=1}^{n} \tan k \quad \to \quad 0$$

Buhler notes that this has important consequences in the Atiyah theory of finitely generated groups. I argue here, via probabilistic analysis and via numerical computation, that this conjecture is almost certainly false, although this analysis certainly does not firmly rule on the conjecture one way or the other.

To see this, consider the function $F(x,n)$ defined for real x in $(0,\pi)$ and integers $n \geq 1$ as

$$F(x,n) \quad = \quad \log \left| \prod_{k=1}^{n} \tan(kx) \right|^{1/n} \quad = \quad \frac{1}{n} \sum_{k=1}^{n} \log|\tan(kx)|$$

The conjecture above is equivalent to the assertion that $\exp(nF(1,n)) \to 0$.

The Birkoff ergodic theorem can be stated as follows:

Lemma 1 *Let $f(t)$ be an integrable function on $[0,1)$, and let T be an "ergodic" transformation for μ (i.e. $T^{-1}A = A$ implies $\mu(A) = 0$ or 1). Then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k x) \quad = \quad \int f \, d\mu \quad \textit{for a.e. } x[\mu], \tag{1}$$

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Consider now the specific transformation $T_x(y) = x + y \bmod \pi$, for y in $(0, 1)$. Note that $T_x^n(y) = nx + y \bmod \pi$. The transformation T_x is easily seen to be both measure-preserving and ergodic. Thus the ergodic theorem holds for almost all y, including (almost certainly) $y = 0$, so that

$$F(x,n) \;=\; \frac{1}{n}\sum_{k=1}^{n} \log\tan(kx) \;\rightarrow\; \frac{1}{\pi}\int_0^{\pi} \log|\tan z|\,dz \;=\; 0$$

Indeed, one can further argue from the central limit theorem that $F(x, n)$, for a fixed n, not only have mean zero, but their variance is

$$V(nF(x,n)) \;=\; \frac{2n}{\pi}\int_0^{\pi/2} (\log|\tan z|))^2\;dz \;=\; \frac{n\pi^2}{4},$$

and thus the standard deviation of $F(x, n)$ is $\pi/(2\sqrt{n})$. In particular, we can expect $F(x, n)$ to range between $-\pi/(2\sqrt{n})$ and $\pi/(2\sqrt{n})$ infinitely often. This means among other things that for almost all x, we have $\prod_{n\geq1} \tan(kx) = (F(x, n))^n$ to be greater than one (by a substantial amount) infinitely often (and thus not converge to zero).

So far, these considerations are for a general x in $(0, \pi)$, whereas the original conjecture deals with the specific case $x = 1$. But it is hard to see why $x = 1$ should be a special case, given that 1 is algebraically independent of any rational multiple of π, which are the "natural" units here.

What's more, explicit numerical computations confirm that for n up to as high as one billion, the behavior of $F(1, n)$ is characteristic of "almost all" $F(x, n)$, as described above. In particular, the table below shows both $F(1, n)$ and $\exp(nF(1, n))$ for n ranging from 100,000,000 to 1,000,000,000. These computations were performed with exceedingly high accuracy (64-digit arithmetic) to ensure numerical reliability.

n	$F(1,n)$	$\exp(nF(1,n))$
$100,000,000$	$2.873246149 \times 10^{-7}$	$3.008496269 \times 10^{12}$
$200,000,000$	$-1.768128500 \times 10^{-7}$	$4.387639933 \times 10^{-16}$
$300,000,000$	$2.240989386 \times 10^{-7}$	$1.575722518 \times 10^{29}$
$400,000,000$	$-1.168045245 \times 10^{-7}$	$5.116533413 \times 10^{-21}$
$500,000,000$	$1.593448560 \times 10^{-7}$	$3.992968190 \times 10^{34}$
$600,000,000$	$-1.074963587 \times 10^{-7}$	$9.748880469 \times 10^{-29}$
$700,000,000$	$1.246094426 \times 10^{-7}$	$7.621410090 \times 10^{37}$
$800,000,000$	$-8.703131299 \times 10^{-8}$	$5.783953756 \times 10^{-31}$
$900,000,000$	$1.076407385 \times 10^{-7}$	$1.183044043 \times 10^{42}$
$1,000,000,000$	$-8.100794032 \times 10^{-8}$	$6.587164784 \times 10^{-36}$

Needless to say, there is no indication that $\exp(nF(1, n))$ are converging to zero — far from it. But $F(1, n)$ are converging to zero, as would be expected as the behavior of $F(x, n)$ for "almost all" x.

References

[1] Patrick Billingsley, *Ergodic Theory and Information*, John Wiley, New York, 1965.

[2] Halsey L. Royden, *Measure Theory*, Addison-Wesley, 1968.