

# Computer discovery and analysis of large Poisson polynomials

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## Standing on the shoulders of giants

The following study is a good example of **multidisciplinary experimental mathematics**, as it crucially relies on many highly talented contributions:

- ▶ Crandall's application of the Poisson equation to image enhancement.
- ▶ Numerous studies involving elliptic curves, theta functions, ideals and fields.
- ▶ Ferguson's PSLQ algorithm (the largest computations ever done using PSLQ).
- ▶ Key observations by audience members in earlier presentations.
- ▶ A concluding proof by Watson Ladd, a graduate student at U.C. Berkeley.
- ▶ An enormous software infrastructure behind our computer code:
  - ▶ GNU compilers and Apple (Berkeley Unix) software (to handle 190,000-line code).
  - ▶ Custom datatypes and operator overloading (to handle multiprecision code).
  - ▶ The MPFR package (to handle 64,000-digit arithmetic).
  - ▶ OpenMP (to handle 20,000 core-hours of parallel processing).

## The PSLQ integer relation algorithm

Let  $X = (x_k)$  be an  $(m + 1)$ -long real or complex vector. An integer relation algorithm such as PSLQ finds a nontrivial integer vector  $A = (a_k)$  such that

$$a_0x_0 + a_1x_1 + \cdots + a_mx_m = 0.$$

- ▶ The multipair PSLQ algorithm is a more efficient and moderately parallelizable variant of PSLQ, the most widely used integer relation algorithm (although some use a variant of LLL).
- ▶ Integer relation detection (by any algorithm) requires very high precision: at least  $(m + 1) \cdot \max_k \log_{10} |a_k|$  digits, both in the input data and the algorithm.
- ▶ H. R. P. Ferguson, D. H. Bailey and S. Arno, "Analysis of PSLQ, an integer relation finding algorithm," *Mathematics of Computation*, vol. 68, no. 225 (Jan 1999), pg. 351–369.
- ▶ D. H. Bailey and D. J. Broadhurst, "Parallel integer relation detection: Techniques and applications," *Mathematics of Computation*, vol. 70, no. 236 (Oct 2000), 1719–1736.

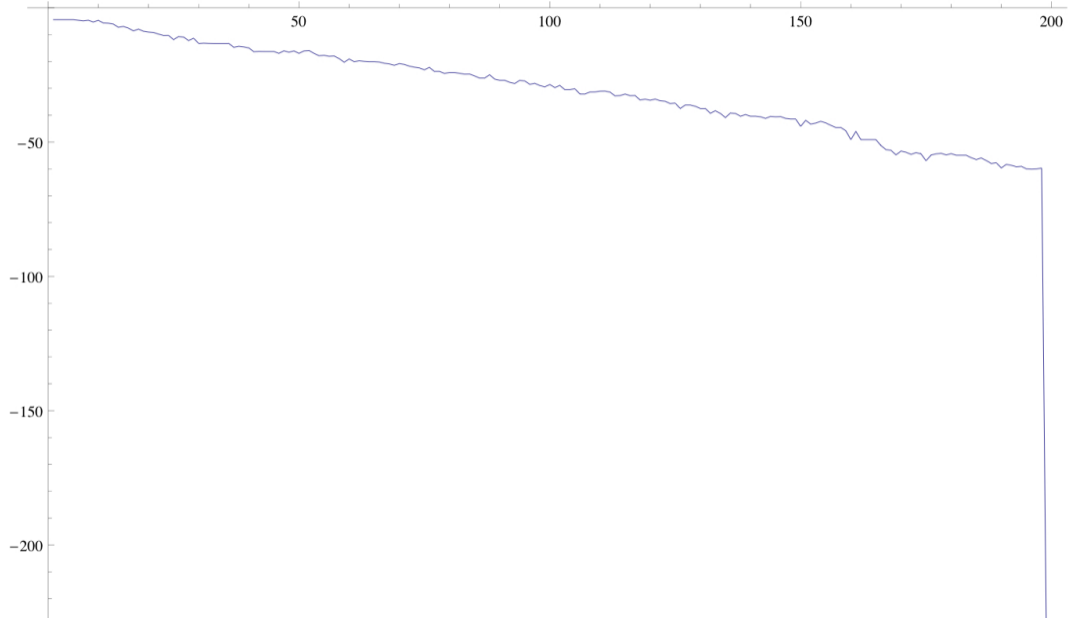
## PSLQ, continued

- ▶ PSLQ constructs a sequence of integer-valued matrices  $B_n$  that reduce the vector  $y = x \cdot B_n$ , until either the relation is found (as one of the columns of matrix  $B_n$ ), or else precision is exhausted.
- ▶ A relation is detected when the size of smallest entry of the  $y$  vector suddenly drops to roughly “epsilon” (i.e.  $10^{-p}$ , where  $p$  is the number of digits of precision).
- ▶ The size of this drop can be viewed as a “confidence level” that the relation is not a numerical artifact: a drop of 20+ orders of magnitude almost always indicates a real relation.

### Efficient variants of PSLQ:

- ▶ 2-level and 3-level PSLQ perform almost all iterations with only double precision, updating full-precision arrays as needed. They are hundreds of times faster than the original PSLQ.
- ▶ Multi-pair PSLQ dramatically reduces the number of iterations required. It was designed for parallel systems, but runs faster even on 1 CPU.

## Decrease of $\log_{10}(\min |y_i|)$ in multipair PSLQ run



## Application of multipair PSLQ

One simple but important application of multipair PSLQ is to recognize a computed numerical value as the root of an integer polynomial of degree  $m$ .

Example: The following constant is suspected to be an algebraic number:

$$\alpha = 1.232688913061443445331472869611255647068988824547930576057634684778\dots$$

What is its minimal polynomial?

Method: Compute the vector  $(1, \alpha, \alpha^2, \dots, \alpha^m)$  for  $m = 30$ , then input this vector to multipair PSLQ.

Answer (using 250-digit arithmetic):

$$\begin{aligned} 0 = & 697 - 1440\alpha - 20520\alpha^2 - 98280\alpha^3 - 102060\alpha^4 - 1458\alpha^5 + 80\alpha^6 - 43920\alpha^7 \\ & + 538380\alpha^8 - 336420\alpha^9 + 1215\alpha^{10} - 80\alpha^{12} - 56160\alpha^{13} - 135540\alpha^{14} - 540\alpha^{15} \\ & + 40\alpha^{18} - 7380\alpha^{19} + 135\alpha^{20} - 10\alpha^{24} - 18\alpha^{25} + \alpha^{30} \end{aligned}$$

## High-precision numerical integration

Given  $f(x)$  defined on  $(-1, 1)$ , define  $g(t) = \tanh(\pi/2 \sinh t)$ . Then setting  $x = g(t)$  yields

$$\int_{-1}^1 f(x) dx = \int_{-\infty}^{\infty} f(g(t))g'(t) dt \approx h \sum_{j=-N}^N w_j f(x_j),$$

where  $x_j = g(h_j)$  and  $w_j = g'(h_j)$ . Since  $g'(t)$  goes to zero very rapidly for large  $t$ , the product  $f(g(t))g'(t)$  typically is a nice bell-shaped function, so that the simple summation above converges very rapidly. Reducing  $h$  by half typically doubles the number of correct digits.

We have found that tanh-sinh is the best general-purpose integration scheme for functions with vertical derivatives or singularities at endpoints, or for any function at very high precision ( $> 1000$  digits). Otherwise we use Gaussian quadrature.

1. D. H. Bailey, X. S. Li and K. Jeyabalan, "A Comparison of Three High-Precision Quadrature Schemes," *Experimental Mathematics*, vol. 14 (2005), no. 3, pg. 317–329.
2. H. Takahasi and M. Mori, "Double Exponential Formulas for Numerical Integration," *Publications of RIMS*, Kyoto University, vol. 9 (1974), pg. 721–D741.

## Ising integrals from mathematical physics

We applied our methods to study three classes of integrals:  $C_n$  are connected to quantum field theory,  $D_n$  arise in the Ising theory of mathematical physics, while the  $E_n$  integrands are derived from  $D_n$ :

$$C_n := \frac{4}{n!} \int_0^\infty \cdots \int_0^\infty \frac{1}{\left(\sum_{j=1}^n (u_j + 1/u_j)\right)^2} \frac{du_1}{u_1} \cdots \frac{du_n}{u_n}$$

$$D_n := \frac{4}{n!} \int_0^\infty \cdots \int_0^\infty \frac{\prod_{i<j} \left(\frac{u_i - u_j}{u_i + u_j}\right)^2}{\left(\sum_{j=1}^n (u_j + 1/u_j)\right)^2} \frac{du_1}{u_1} \cdots \frac{du_n}{u_n}$$

$$E_n = 2 \int_0^1 \cdots \int_0^1 \left( \prod_{1 \leq j < k \leq n} \frac{u_k - u_j}{u_k + u_j} \right)^2 dt_2 dt_3 \cdots dt_n$$

where in the last line  $u_k = t_1 t_2 \cdots t_k$ .

D. H. Bailey, J. M. Borwein and R. E. Crandall, "Integrals of the Ising class," *Journal of Physics A: Mathematical and General*, vol. 39 (2006), pg. 12271–12302.



## Limiting value of $C_n$ : What is this number?

Key observation: The  $C_n$  integrals can be converted to one-dimensional integrals involving the modified Bessel function  $K_0(t)$ :

$$C_n = \frac{2^n}{n!} \int_0^\infty t K_0^n(t) dt$$

High-precision numerical values, computed using this formula and tanh-sinh quadrature, approach a limit. For example:

$$C_{1024} = 0.6304735033743867961220401927108789043545870787 \dots$$

What is this number? We copied the first 50 digits into the online Inverse Symbolic Calculator (ISC) at <https://isc.carma.newcastle.edu.au>. The result was:

$$\lim_{n \rightarrow \infty} C_n = 2e^{-2\gamma}.$$

where  $\gamma$  denotes Euler's constant. This is now proven.

## Other Ising integral evaluations found using PSLQ

$$D_3 = 8 + 4\pi^2/3 - 27L_{-3}(2)$$

$$D_4 = 4\pi^2/9 - 1/6 - 7\zeta(3)/2$$

$$E_2 = 6 - 8\log 2$$

$$E_3 = 10 - 2\pi^2 - 8\log 2 + 32\log^2 2$$

$$E_4 = 22 - 82\zeta(3) - 24\log 2 + 176\log^2 2 - 256(\log^3 2)/3 \\ + 16\pi^2\log 2 - 22\pi^2/3$$

$$E_5 = 42 - 1984\text{Li}_4(1/2) + 189\pi^4/10 - 74\zeta(3) - 1272\zeta(3)\log 2 + 40\pi^2\log^2 2 \\ - 62\pi^2/3 + 40(\pi^2\log 2)/3 + 88\log^4 2 + 464\log^2 2 - 40\log 2$$

where  $\zeta$  is the Riemann zeta function and  $Li_n(x)$  is the polylog function.  $E_5$  remained a “numerical conjecture” for several years, but was proven in March 2014 by Erik Panzer.

$E_5$  was reduced to a 3-D integral of a 60-line integrand, which was evaluated using tanh-sinh quadrature to 250-digit arithmetic using over 1000 CPU-hours on a highly parallel system. The PSLQ calculation required only seconds.

## The Poisson potential function

In 2012, Richard Crandall, while investigating techniques to sharpen images, noted that each pixel was given by a form of the 2-D Poisson potential function:

$$\phi_2(x, y) = \frac{1}{\pi^2} \sum_{m, n \text{ odd}} \frac{\cos(m\pi x) \cos(n\pi y)}{m^2 + n^2}$$

In a 2013 study, we numerically discovered, and then proved the intriguing fact that for rational  $(x, y)$ ,

$$\phi_2(x, y) = \frac{1}{\pi} \log \alpha$$

where  $\alpha$  is *algebraic*, i.e., the root of a some integer polynomial of degree  $m$ .

By computing high-precision numerical values of  $\phi_2(x, y)$  for various specific rational  $x$  and  $y$ , and applying a multipair PSLQ program, we were able to produce the explicit minimal polynomials for  $\alpha$  in numerous specific cases.

- ▶ D. H. Bailey, J. M. Borwein, R. E. Crandall and J. Zucker, "Lattice sums arising from the Poisson equation," *Journal of Physics A: Mathematical and Theoretical*, vol. 46 (2013), 115201.

## Samples of minimal polynomials found by multipair PSLQ

- $s$  Minimal polynomial corresponding to  $x = y = 1/s$ :
- 5  $1 + 52\alpha - 26\alpha^2 - 12\alpha^3 + \alpha^4$
  - 6  $1 - 28\alpha + 6\alpha^2 - 28\alpha^3 + \alpha^4$
  - 7  $-1 - 196\alpha + 1302\alpha^2 - 14756\alpha^3 + 15673\alpha^4 + 42168\alpha^5 - 111916\alpha^6 + 82264\alpha^7 - 35231\alpha^8 + 19852\alpha^9 - 2954\alpha^{10} - 308\alpha^{11} + 7\alpha^{12}$
  - 8  $1 - 88\alpha + 92\alpha^2 - 872\alpha^3 + 1990\alpha^4 - 872\alpha^5 + 92\alpha^6 - 88\alpha^7 + \alpha^8$
  - 9  $-1 - 534\alpha + 10923\alpha^2 - 342864\alpha^3 + 2304684\alpha^4 - 7820712\alpha^5 + 13729068\alpha^6 - 22321584\alpha^7 + 39775986\alpha^8 - 44431044\alpha^9 + 19899882\alpha^{10} + 3546576\alpha^{11} - 8458020\alpha^{12} + 4009176\alpha^{13} - 273348\alpha^{14} + 121392\alpha^{15} - 11385\alpha^{16} - 342\alpha^{17} + 3\alpha^{18}$
  - 10  $1 - 216\alpha + 860\alpha^2 - 744\alpha^3 + 454\alpha^4 - 744\alpha^5 + 860\alpha^6 - 216\alpha^7 + \alpha^8$

These computations are very expensive. The case  $x = y = 1/32$ , for instance, required 10,000-digit arithmetic and ran for 45 hours. Other runs, using even higher precision, ultimately failed, evidently due to subtle program bugs. [Help!](#)

## Kimberley's formula for the degree of the polynomial

Based on our preliminary results, Jason Kimberley of the University of Newcastle, Australia observed that the degree  $m(s)$  of the minimal polynomial associated with the case  $x = y = 1/s$  appears to be given by the following:

Set  $m(2) = 1/2$ . Otherwise for primes  $p$  congruent to 1 mod 4, set  $m(p) = \text{int}^2(p/2)$ , where  $\text{int}$  denotes greatest integer, and for primes  $p$  congruent to 3 mod 4, set  $m(p) = \text{int}(p/2)(\text{int}(p/2) + 1)$ . Then for any other positive integer  $s$  whose prime factorization is  $s = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}$ ,

$$m(s) \stackrel{?}{=} 4^{r-1} \prod_{i=1}^r p_i^{2(e_i-1)} m(p_i).$$

Does Kimberley's formula hold for larger  $s$ ? Why?

What is the true mathematical connection between the pair of rationals  $(x, y)$  and the algebraic number  $\alpha$ ?

## Three improvements to the Poisson polynomial computation program

1. MPFUN2015: A new thread-safe multiprecision package.
  - ▶ Speedup: 3X
2. A new 3-level multipair PSLQ program.
  - ▶ Speedup: 4.2X
3. Parallel implementation on a 16-core system.
  - ▶ Speedup: 12.2X

Overall speedup: 156X

## Thread safety: A major challenge for high-precision software

- ▶ Because of greatly magnified run times (often 1000X or more) of high-precision computations, highly parallel implementations are in order. However, most high-precision packages are *not* thread-safe.
- ▶ Many packages employ global read/write variables, e.g., for transcendental function evaluation, which ruin thread safety.
- ▶ The working precision level, a global variable that is changed frequently within the package itself and often by users also, is particularly troublesome.

One thread-safe package: The MPFR C++ package, which calls the lower-level MPFR package, is thread-safe (provided MPFR is compiled with the thread-safe build option).

MPFR is a very well-designed, features correct rounding and is the fastest low-level package currently available, but is only a low-level package.

There is no thread-safe high-level Fortran package.

## MPFUN2015: A new thread-safe arbitrary precision package

DHB has written a new package (approx. 50,000 lines of code) for arbitrary precision floating-point computation. It is available in two versions:

- ▶ MPFUN-Fort: Completely self-contained, all-Fortran version. Compilation is a simple one-line command, which completes in a few seconds.
- ▶ MPFUN-MPFR: Calls the MPFR package for lower-level operations. Installation is significantly more complicated (since GMP and MPFR must first be installed), but performance is roughly 3X faster.

Both versions include a high-level language interface, using custom datatypes and operator overloading. For most codes, only minor changes are required.

A C++ version is in the works.

We used MPFUN-MPFR in this study.

Full details of design, algorithms, installation and usage are given in

- ▶ D. H. Bailey, "MPFUN2015: A thread-safe arbitrary precision computation package," manuscript, 1 Oct 2015. <http://www.davidhbailey.com/dhbpapers/mpfun2015.pdf>. The software is available at <http://www.davidhbailey.com/dhbsoftware>.



## New three-level multipair PSLQ program

Employs three levels of numeric precision:

- ▶ Ordinary double precision.
- ▶ Medium precision, typically 100–2000 digits.
- ▶ Full precision, typically many thousands of digits.

When an entry of the double precision reduced vector is smaller than  $10^{-14}$ , or when an entry of one of the integer-valued double precision arrays exceeds  $2^{53} \approx 9.007 \cdot 10^{15}$ , the medium precision arrays are updated by matrix multiplication.

Similarly, when an entry of the medium precision reduced vector is smaller than the medium precision “epsilon,” the full-precision arrays are updated.

Substantial care must be taken to manage this three-level hierarchy, and to correctly handle numerous atypical scenarios.

## A fast algorithm to compute the Poisson potential function $\phi_2(x, y)$

$$\phi_2(x, y) = \frac{1}{2\pi} \log \left| \frac{\theta_2(z, q)\theta_4(z, q)}{\theta_1(z, q)\theta_3(z, q)} \right|,$$

where  $q = e^{-\pi}$  and  $z = \frac{\pi}{2}(y + ix)$ . Compute the four theta functions using the following rapidly convergent formulas involving complex variables:

$$\theta_1(z, q) = 2 \sum_{k=1}^{\infty} (-1)^{k-1} q^{(2k-1)^2/4} \sin((2k-1)z),$$

$$\theta_2(z, q) = 2 \sum_{k=1}^{\infty} q^{(2k-1)^2/4} \cos((2k-1)z),$$

$$\theta_3(z, q) = 1 + 2 \sum_{k=1}^{\infty} q^{k^2} \cos(2kz),$$

$$\theta_4(z, q) = 1 + 2 \sum_{k=1}^{\infty} (-1)^k q^{k^2} \cos(2kz).$$

## High-level computational algorithm

1. Given rationals  $x = p/s$  and  $y = q/s$ , select a conjectured minimal polynomial degree  $m(s)$  (using Kimberley's formula) and other parameters for the run.
2. Calculate  $\phi_2(x, y)$  to  $P_2$ -digit precision using the formulas from two viewgraphs above. When done, calculate  $\alpha = \exp(8\pi\phi_2(x, y))$  and generate the  $(m + 1)$ -long vector  $X = (1, \alpha, \alpha^2, \dots, \alpha^m)$ , to  $P_2$ -digit precision.
3. Apply the three-level multipair PSLQ algorithm to  $X$ . For larger problems, employ a parallel version of the three-level multipair PSLQ code, using the OpenMP construct `DO PARALLEL` to perform certain time-intensive loops in parallel.
4. If no numerically significant relation is found, try again with a larger degree  $m$  or higher precision  $P_2$ . If a relation is found, employ the polynomial factorization facilities in *Mathematica* and *Maple* to ensure that the polynomial is irreducible.

## Application program and libraries for the Poisson calculations

Description	Language	Lines of code
Poisson polynomial program*	Fortran	2,000
MPFUN-MPFR package	Fortran	12,000
MPFR package	C	93,000
GMP package	C	83,000
Total		190,000

\*This includes the computation of  $\phi_2(x, y)$  and the 3-level multipair PSLQ program.



## Timings for the case $x = y = 1/35$

Multiprecision software	PSLQ code	Cores	Run time	Speedup
ARPREC	2-level	1	$1.599 \cdot 10^6$	1.00
MPFUN-MPFR	2-level	1	$5.249 \cdot 10^5$	3.05
MPFUN-MPFR	3-level	1	$1.240 \cdot 10^5$	12.90
		2	$7.585 \cdot 10^4$	21.08
		4	$4.121 \cdot 10^4$	38.80
		8	$2.476 \cdot 10^4$	64.58
		16	$1.021 \cdot 10^4$	156.61

The run times are wall-clock run times (in seconds), measured on a 16-core 2.4 GHz MacPro, in a typically busy environment with similar jobs running on other cores.

## Selected runs (degrees, precision, timings, etc.) for $x = y = 1/s$

$s$	$m$	$\log_{10}(D)$	$P_1$	$P_2$	$N$	$M$	$C$	$T$ (sec.)	$C \cdot T$ (sec.)
20	32	-463.84	160	700	1967	0.81	1	$2.19 \cdot 10^0$	$2.19 \cdot 10^0$
24	64	-1883.78	320	2200	9297	11.33	1	$7.73 \cdot 10^1$	$7.73 \cdot 10^1$
30	64	-1868.01	350	2300	9064	11.33	1	$1.02 \cdot 10^2$	$1.02 \cdot 10^2$
32	128	-7577.07	650	8200	45893	168.20	1	$5.13 \cdot 10^3$	$5.13 \cdot 10^3$
34	128	-7574.93	650	8200	45914	168.20	1	$5.16 \cdot 10^3$	$5.16 \cdot 10^3$
36	144	-9570.86	750	10300	62282	267.10	1	$9.54 \cdot 10^3$	$9.54 \cdot 10^3$
37	324	-48431.32	1650	51000	931254	6579.66	16	$4.84 \cdot 10^5$	$7.74 \cdot 10^6$
38	180	-14951.64	900	16000	120984	642.98	1	$3.88 \cdot 10^4$	$3.88 \cdot 10^4$
39	288	-38330.14	1450	40000	667153	4124.24	16	$2.68 \cdot 10^5$	$4.29 \cdot 10^6$
40	128	-7580.00	650	8200	45655	168.20	1	$5.02 \cdot 10^3$	$5.02 \cdot 10^3$
42	192	-16993.99	1000	18000	150364	829.41	8	$1.57 \cdot 10^4$	$1.26 \cdot 10^5$
44	240	-26604.14	1200	28000	323762	2003.33	8	$7.43 \cdot 10^4$	$5.94 \cdot 10^5$
45	288	-38315.08	1450	40000	660001	4124.24	16	$2.09 \cdot 10^5$	$3.35 \cdot 10^6$
46	264	-32036.34	1350	34000	476902	2921.57	16	$1.06 \cdot 10^5$	$1.70 \cdot 10^6$
48	256	-30248.55	1350	32000	415316	2586.39	16	$8.98 \cdot 10^4$	$1.44 \cdot 10^6$
50	200	-18421.18	1000	20000	168947	974.44	8	$2.12 \cdot 10^4$	$1.69 \cdot 10^5$
52	288	-38414.49	1550	41000	655291	4124.24	16	$2.12 \cdot 10^5$	$3.40 \cdot 10^6$
*60	256	-14477.99	800	16000	90371	336.41	1	$5.28 \cdot 10^3$	$5.28 \cdot 10^3$
*64	512	-57816.90	1600	64000	802361	5172.79	16	$3.78 \cdot 10^5$	$2.42 \cdot 10^6$

## Selected runs (degrees, precision, timings, etc.) for $x = 1/s, y = q/s$

$s$	$q$	$m$	$\log_{10}(D)$	$P_1$	$P_2$	$N$	$M$	$C$	$T$ (sec.)	$C \cdot T$ (sec.)
20	3	32	-359.23	160	700	1637	0.81	1	$3.06 \cdot 10^0$	$3.06 \cdot 10^0$
24	5	32	-336.46	320	2200	1613	11.33	1	$2.83 \cdot 10^0$	$2.83 \cdot 10^0$
30	7	64	-1291.21	350	2300	6867	11.33	1	$1.01 \cdot 10^2$	$1.01 \cdot 10^2$
32	3	128	-5578.06	650	8200	33829	168.20	1	$4.16 \cdot 10^3$	$4.16 \cdot 10^3$
34	3	128	-5411.70	650	8200	32842	168.20	1	$2.53 \cdot 10^3$	$2.53 \cdot 10^3$
36	5	144	-6831.83	750	10300	45559	267.10	1	$6.93 \cdot 10^3$	$6.93 \cdot 10^3$
37	2	324	-35412.09	1650	51000	691277	6579.66	16	$3.42 \cdot 10^5$	$5.47 \cdot 10^6$
38	3	180	-11011.83	900	16000	89722	642.98	1	$7.42 \cdot 10^3$	$7.42 \cdot 10^3$
39	2	288	-27943.70	1450	40000	458238	4124.24	16	$1.84 \cdot 10^5$	$2.94 \cdot 10^6$
40	3	128	-5674.61	650	8200	34273	168.20	1	$4.13 \cdot 10^3$	$4.13 \cdot 10^3$
42	5	192	-12183.50	1000	18000	106770	829.41	8	$1.16 \cdot 10^4$	$9.27 \cdot 10^4$
44	3	240	-19581.93	1200	28000	232713	2003.33	8	$6.18 \cdot 10^4$	$4.95 \cdot 10^5$
45	2	288	-27857.00	1450	40000	482959	4124.24	16	$1.47 \cdot 10^5$	$2.35 \cdot 10^6$
46	3	264	-23318.37	1350	34000	346987	2921.57	16	$7.28 \cdot 10^4$	$1.17 \cdot 10^6$
48	5	256	-21480.15	1350	32000	292974	2586.39	16	$5.93 \cdot 10^4$	$9.50 \cdot 10^5$
50	3	200	-13409.44	1000	20000	122468	974.44	8	$1.63 \cdot 10^4$	$1.30 \cdot 10^5$

$s$  = denominator;  $m$  = degree;  $D$  = detection level;  $P_1$  = medium precision;  $P_2$  = full precision;  $N$  = number of iterations;  $M$  = Mbytes;  $C$  = cores;  $T$  = wall clock time;  $C \cdot T$  = total core-seconds.



## Palindromic polynomials

From our results, in the case  $(1/s, 1/s)$  where  $s$  is even, the resulting polynomial is always palindromic ( $a_k = a_{m-k}$ ). For instance, when  $s = 16$ ,

$$\begin{aligned} p_{16}(\alpha) = & 1 - 1376\alpha^1 - 12560\alpha^2 - 3550496\alpha^3 + 81241720\alpha^4 - 169589984\alpha^5 \\ & + 1334964944\alpha^6 - 24307725984\alpha^7 + 238934926108\alpha^8 - 1043027124704\alpha^9 \\ & + 2328675366384\alpha^{10} - 3219896325280\alpha^{11} + 4238551472456\alpha^{12} \\ & - 10247414430048\alpha^{13} + 28552105805904\alpha^{14} - 55832851687968\alpha^{15} \\ & + 70020268309062\alpha^{16} \\ & - 55832851687968\alpha^{17} + 28552105805904\alpha^{18} - 10247414430048\alpha^{19} \\ & + 4238551472456\alpha^{20} - 3219896325280\alpha^{21} + 2328675366384\alpha^{22} \\ & - 1043027124704\alpha^{23} + 238934926108\alpha^{24} - 24307725984\alpha^{25} + 1334964944\alpha^{26} \\ & - 169589984\alpha^{27} + 81241720\alpha^{28} - 3550496\alpha^{29} - 12560\alpha^{30} - 1376\alpha^{31} + \alpha^{32} \end{aligned}$$

Nitya Mani, an undergraduate student at Stanford University, observed that if  $\alpha$  is a root of a palindromic polynomial such as this, then  $\alpha + 1/\alpha$  is a root of a transformed polynomial of half the degree. This fact can be used to significantly accelerate the computation of Poisson polynomials in the even case (runs denoted by \* in the table).

## New observations for the case $(1/s, 1/s)$

After doing some Google searches on the coefficients of the polynomials  $p_{11}$  and  $p_{13}$ , we found the coefficient 387221579866 in  $p_{11}$  appears in a 2010 preprint by Savin and Quarfoot, and the coefficient 221753896032 in  $p_{13}$  appears in a manuscript, also dated 2010, by Bostan, Boukraa, Hassani, Maillard, Weil, Zenine and Abarenkova.

Savin and Quarfoot define a sequence  $\psi_n$  of polynomials in  $x$  and  $y$ , based on the curve  $y^2 = x^3 + x$ , as follows:

$$\begin{aligned}\psi_1 &= 1 \\ \psi_2 &= 2y \\ \psi_3 &= 3x^4 + 6x^2 - 1 \\ \psi_4 &= 2y(2x^6 + 10x^4 - 10x^2 - 2),\end{aligned}\tag{1}$$

and, recursively,

$$\begin{aligned}\psi_{2n+1} &= \psi_{n+2} \cdot \psi_n^3 - \psi_{n-1} \cdot \psi_{n+1}^3 \quad \text{for } n \geq 2 \\ \psi_{2n} &= 1/(2y) \cdot \psi_n(\psi_{n+2} \cdot \psi_{n-1}^2 - \psi_{n-2} \cdot \psi_{n+1}^2) \quad \text{for } n \geq 3.\end{aligned}$$

## Our analysis

We constructed a related sequence  $J_s$  of integer coefficient polynomials in  $a$  by setting  $x = \sqrt{-a}$ , and so  $y^2 = x(x^2 + 1) = \sqrt{-a}(1 - a)$ ; we also remove the leading  $2y$ :

$$\begin{aligned}J_{2n+1}(a) &= \psi_{2n+1}(x, y) \\ J_{2n}(a) &= 1/(2y) \cdot \psi_{2n}(x, y)\end{aligned}$$

The initial values of  $J_s(a)$  are

$$\begin{aligned}J_1 &= 1 \\ J_2 &= 1 \\ J_3 &= 3a^2 - 6a - 1 \\ J_4 &= 2a^3 - 10a^2 - 10a + 2.\end{aligned}$$

After computation in *Magma*, we were able to prove that for each prime  $q \equiv 3 \pmod{4}$  the polynomial  $J_q$  has degree  $m(q)$ , where  $m(s)$  is Kimberley's formula.

## Additional conjectures

In fact, our computations support these conjectures:

- ▶ For each prime  $q \equiv 3 \pmod{4}$ , the polynomial  $J_q$  is precisely  $p_q$  as computed by PSLQ.
- ▶ For each integer  $s \geq 1$ ,  $p_s$  is the unique degree  $m(s)$  prime factor of  $J_s$ .
- ▶ The  $J$  function is a divisibility sequence:  $m \mid n$  implies  $J_m \mid J_n$ .
- ▶ For each positive integer  $s$ , both  $J_s$  and  $p_s$  have largest real root  $\alpha_s$ .

## Proofs of Kimberley's formula and the palindromic property

- ▶ On March 16, DHB presented our results at a seminar at the University of California, Berkeley.
- ▶ Following the presentation, Watson Ladd, a graduate student in mathematics, brought to our attention the fact that some of our conjectures should follow from known results in the theory of elliptic curves, Gaussian integers and ideals.
- ▶ After some effort, Ladd produced proofs of Kimberley's formula and the palindromic property, which proofs we then included in our paper and returned to the journal.
- ▶ A preprint is available here:  
<http://www.davidhbailey.com/dhbpapers/poisson-res.pdf>.

## Conclusions

- ▶ These computations, which employed up to 64,000-digit precision, producing polynomials with degrees up to 512 and integer coefficients up to  $10^{229}$ , constitute the largest successful integer relation computations to date.
- ▶ Kimberley's formula was affirmed in every case  $x = y = 1/s$ , for  $s$  up to 52 (except for  $s = 41, 43, 47, 49, 51$ ), and also for  $s = 60$  and  $s = 64$ .
- ▶ The resulting polynomial coefficients yielded clues as to why Kimberley's formula and other phenomena observed in the computed output holds.
- ▶ These efforts ultimately led to a proof of Kimberley's formula and the palindromic property, employing techniques of elliptic curves, Gaussian integers and ideals.
- ▶ Additional research is needed to understand many other combinations, e.g.,  $x = p/s$  and  $y = q/s$ , for different values of  $p$ ,  $q$  and  $s$ , which will require even more extreme computation.
- ▶ While a 12X parallel speedup is certainly welcome, a scheme to efficiently employ hundreds or thousands of processors for PSLQ computation is needed. A fundamentally new integer relation algorithm may be required.

# Thanks!

- ▶ This talk is available at <http://www.davidhbailey.com/dhbtalks/dhb-acmes-2016.pdf>.
- ▶ A technical report with full details, results and analysis is available at <http://www.davidhbailey.com/dhbpapers/poisson-res.pdf>.
- ▶ A technical report giving full details of MPFUN2015 package is available at <http://www.davidhbailey.com/dhbpapers/mpfun2015.pdf>.