Experimental mathematics and integration

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Identifying definite integrals

Using computational methods to identify integrals is one of the most productive application of the experimental mathematics paradigm. Some particularly useful instances include:

- Ising integrals.
- Box integrals.
- Ramble integrals.
- Integrals of elliptic integral functions.

The key challenge is evaluating the integral to several hundred or several thousand digit precision.

Most quadrature (i.e., numerical integration) schemes studied in numerical analysis fail miserably for very high precision applications.

Gaussian quadrature

Gaussian quadrature is often the most efficient scheme for completely regular functions (including endpoints) and modest precision (less than 1000 digits):

$$\int_{-1}^{1} f(x)\,\mathrm{d}x \approx \sum_{j=1}^{n} w_j f(x_j)$$

The abscissas (x_j) are the roots of the n-th degree Legendre polynomial $P_n(x)$ on $[-1, 1]$. The weights (w_j) are given by

$$w_j = \frac{-2}{(n+1)P_n'(x_j)P_{n+1}(x_j)}$$

The abscissas (x_j) are computed by Newton iterations, with starting values $\cos[\pi(j - 1/4)/(n + 1/2)]$. Legendre polynomials and their derivatives can be computed using the formulas $P_0(x) = 0, P_1(x) = 1$, and

$$(k+1)P_{k+1}(x) = (2k+1)xP_k(x) - kP_{k-1}(x)$$

$$P_n'(x) = n(xP_n(x) - P_{n-1}(x))/(x^2 - 1)$$

Gaussian quadrature, continued

Advantages:

- For regular functions, it is typically the fastest quadrature scheme available.
- Decreasing h by a factor of two (i.e., increasing n by a factor of two) typically doubles the number of correct digits (provided all computations are performed to at least this precision level).

Disadvantages:

- Gaussian quadrature fails if the function or its higher derivatives has a singularity.
- Computation cost of abscissas and weights increases quadratically with n. This limits the scheme to less than 1000 digits or so.
- Abscissas and weights for a given n cannot be used for $2n$ or any other value.

The Euler-Maclaurin summation formula

$$\int_a^b f(x)\,\mathrm{d}x = h\sum_{j=0}^{n} f(x_j) - \frac{h}{2}(f(a)+f(b))$$
$$- \sum_{i=1}^{m} \frac{h^{2i}B_{2i}}{(2i)!}\left(D^{2i-1}f(b) - D^{2i-1}f(a)\right) - E(h)$$
$$|E(h)| \le 2(b-1)(h/(2\pi))^{2m+2} \max_{a\le x\le b} |D^{2m+2}f(x)|$$

Here $h = (b-a)/n$ and $x_j = a + jh$; B_{2i} are Bernoulli numbers; $D^m f(x)$ is the m-th derivative of $f(x)$.

The E-M formula can be thought of as providing high-order correction terms to the trapezoidal rule.

Note that in the case when $f(x)$ and all of its derivatives are zero at the endpoints a and b (as in a bell-shaped curve), the error $E(h)$ of a simple trapezoidal approximation to the integral goes to zero more rapidly than any power of h.

Tanh-sinh quadrature

Given $f(x)$ defined on $(-1, 1)$, define $g(t) = \tanh(\pi/2 \cdot \sinh t)$, so that $g'(t) = \pi/2 \cdot \cosh(t)/\cosh^2(\pi/2 \cdot \sinh(t))$. Then substituting $x = g(t)$ yields

$$\int_{-1}^{1} f(x)\,dx = \int_{-\infty}^{\infty} f(g(t))g'(t)\,dt \;\approx\; h\sum_{j=-N}^{N} w_j f(x_j),$$

where $x_j = g(hj)$ and $w_j = g'(hj)$. These abscissas and weights can be precomputed.

Since $g'(t)$ goes to zero very rapidly for large t, the integrand $f(g(t))g'(t)$ typically is a nice bell-shaped function for which the Euler-Maclaurin formula applies. As a consequence, for most integrand functions $f(t)$, the simple summation above (which is the tanh-sinh scheme) is remarkably accurate.

- D. H. Bailey, X. S. Li and K. Jeyabalan, "A comparison of three high-precision quadrature schemes," *Experimental Mathematics*, vol. 14 (2005), 317–329.

Tanh-sinh quadrature, continued

Advantages:

- Tanh-sinh is not bothered by singularities at the endpoints (either for the function itself or any higher derivative). If there is a singularity within the interval, just perform two integrals.
- Computation cost of abscissas and weights only increases linearly with n, and so computations with many thousands of digits are entirely feasible.
- Abscissas and weights computed for a given n can be used for the $2n$ set. They are simply the even-indexed elements of the $2n$ set.
- Decreasing h by a factor of two (i.e., increasing n by a factor of two) typically doubles the number of correct digits (provided all computations are performed to at least this precision level).

Disadvantages:

- For regular functions that can also be evaluated using Gaussian quadrature, tanh-sinh is typically several times slower.

I confess: tanh-sinh is my favorite!

How the tanh-sinh transformation handles singularities at endpoints

Upper plot: Original integrand on $[-1, 1]$:

$$f(x) = -\log\cos\left(\frac{\pi x}{2}\right)$$

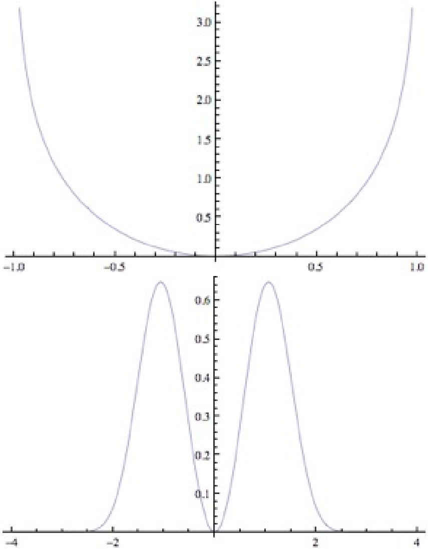
Note the singularities at the endpoints.

Lower plot: Transformed using $x = g(t)$:

$$f(g(t))g'(t) =$$

$$-\log\cos(\pi/2 \cdot \tanh(\sinh t))\left(\frac{\cosh(t)}{\cosh(\sinh t)^2}\right)$$

This is now a nice smooth bell-shaped function, so the E-M formula implies that a trapezoidal approximation is very accurate.

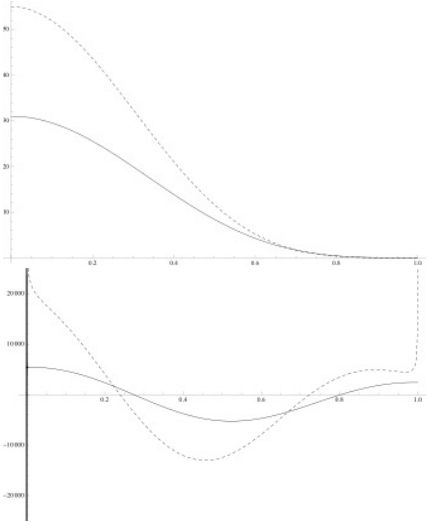


Regular function or not?

Plots of $f(x) = \sin^p(\pi x)\zeta(p, x)$ (upper) and its fourth derivative (lower), for $p = 3$ (solid) and $p = 7/2$ (dashed). Here ζ is Hurwitz zeta function.

When $p = 7/2$, the function itself appears completely regular, but the fourth derivative blows up at both endpoints. As a result, Gaussian quadrature works very poorly for this function.

But tanh-sinh works fine.

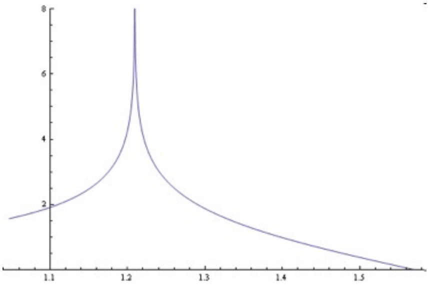


A log-tan integral identity verified with tanh-sinh quadrature

$$\frac{24}{7\sqrt{7}} \int_{\pi/3}^{\pi/2} \log \left| \frac{\tan t + \sqrt{7}}{\tan t - \sqrt{7}} \right| \mathrm{d}t = L_{-7}(2) =$$

$$\sum_{n=0}^{\infty} \left[\frac{1}{(7n+1)^2} + \frac{1}{(7n+2)^2} - \frac{1}{(7n+3)^2} + \frac{1}{(7n+4)^2} - \frac{1}{(7n+5)^2} - \frac{1}{(7n+6)^2} \right]$$

This identity arises from analysis of volumes of knot complements in hyperbolic space. This is simplest of 998 related identities.



We verified this numerically to 20,000 digits, using tanh-sinh quadrature on a highly parallel computer. A proof was known, but we were not aware of this at the time.

Ising integrals from mathematical physics

We recently applied our methods to study three classes of integrals (one of which was referred to us by Craig Tracy of U.C. Davis) that arise in the Ising theory of mathematical physics:

$$C_n := \frac{4}{n!}\int_0^\infty \cdots \int_0^\infty \frac{1}{\left(\sum_{j=1}^n (u_j + 1/u_j)\right)^2}\frac{\mathrm{d}u_1}{u_1}\cdots\frac{\mathrm{d}u_n}{u_n}$$

$$D_n := \frac{4}{n!}\int_0^\infty \cdots \int_0^\infty \frac{\prod_{i<j}\left(\frac{u_i-u_j}{u_i+u_j}\right)^2}{\left(\sum_{j=1}^n (u_j + 1/u_j)\right)^2}\frac{\mathrm{d}u_1}{u_1}\cdots\frac{\mathrm{d}u_n}{u_n}$$

$$E_n = 2\int_0^1 \cdots \int_0^1 \left(\prod_{1\le j<k\le n}\frac{u_k - u_j}{u_k + u_j}\right)^2 \mathrm{d}t_2\, dt_3 \cdots \mathrm{d}t_n$$

where in the last line $u_k = t_1 t_2 \cdots t_k$.

- D. H. Bailey, J. M. Borwein and R. E. Crandall, “Integrals of the Ising class,” *Journal of Physics A: Mathematical and General*, vol. 39 (2006), pg. 12271–12302.

Limiting value of C_n: What is this number?

Key observation: The C_n integrals can be converted to one- dimensional integrals involving the modified Bessel function $K_0(t)$:

$$C_n = \frac{2^n}{n!} \int_0^\infty t K_0^n(t) \, \mathrm{d}t$$

High-precision numerical values, computed using this formula and tanh-sinh quadrature, approach a limit. For example:

$$C_{1024} = 0.6304735033743867961220401927108789043545870787\ldots$$

What is this number? We copied the first 50 digits into the online Inverse Symbolic Calculator (ISC) at https://isc.carma.newcastle.edu.au. The result was:

$$\lim_{n \to \infty} C_n = 2e^{-2\gamma}.$$

where γ denotes Euler's constant. This is now proven.

Other Ising integral evaluations found using PSLQ

$$
\begin{aligned}
D_3 &= 8 + 4\pi^2/3 - 27\,\mathrm{Li}_{-3}(2) \\
D_4 &= 4\pi^2/9 - 1/6 - 7\zeta(3)/2 \\
E_2 &= 6 - 8\log 2 \\
E_3 &= 10 - 2\pi^2 - 8\log 2 + 32\log^2 2 \\
E_4 &= 22 - 82\zeta(3) - 24\log 2 + 176\log^2 2 - 256(\log^3 2)/3 \\
&\quad + 16\pi^2\log 2 - 22\pi^2/3 \\
E_5 &= 42 - 1984\,\mathrm{Li}_4(1/2) + 189\pi^4/10 - 74\zeta(3) - 1272\zeta(3)\log 2 \\
&\quad + 40\pi^2\log^2 2 - 62\pi^2/3 + 40(\pi^2\log 2)/3 + 88\log^4 2 \\
&\quad + 464\log^2 2 - 40\log 2
\end{aligned}
$$

where $\zeta(x)$ is the Riemann zeta function and $\mathrm{Li}_n(x)$ is the polylogarithm function.

The Ising integral E_5

We were able to reduce E_5 to an extremely complicated 3-D integral (see right).

We computed this integral to 250-digit precision, using a highly parallel, high-precision 3-D quadrature program. Then we used a PSLQ program to discover the evaluation given on the previous page.

We also computed D_5 to 500 digits, but were unable to identify it. In March 2014, Erik Panzer proved our formula for E_5 and found a similar evaluation for D_5.

- D. H. Bailey, J. M. Borwein and R. E. Crandall, "Integrals of the Ising class," *J. Physics A: Math. and Gen.*, vol. 39 (2006), 12271–12302.

$$
\begin{aligned}
E_5 = {} & \int_0^1 \int_0^1 \int_0^1 [2(1-x)^2(1-y)^2(1-xy)^2(1-z)^2(1-yz)^2(1-xyz)^2 \\
& (-[4(x+1)(xy+1)\log(2)(y^5z^3x^7 - y^4z^2(4(y+1)z+3)x^6 - y^3z((y^2+1)z^2+4(y+ \\
& 1)z+5)x^5 + y^2(4y(y+1)z^3+3(y^2+1)z^2+4(y+1)z-1)x^4 + y(z(z^2+4z \\
& +5)y^2+4(z^2+1)y+5z+4)x^3 + ((-3z^2-4z+1)y^2-4zy+1)x^2 - (y(5z+4) \\
& +4)x-1)] / [(x-1)^3(xy-1)^3(xyz-1)^3] + [3(y-1)^2y^4(z-1)^2z^2(yz \\
& -1)^2x^6 + 2y^3z(3(z-1)^2z^3y^5 + z^2(5z^3+3z^2+3z+5)y^4 + (z-1)^2z \\
& (5z^2+16z+5)y^3 + (3z^5+3z^4-22z^3-22z^2+3z+3)y^2 + 3(-2z^4+z^3+2 \\
& z^2+z-2)y + 3z^3+5z^2+5z+3)x^5 + y^2(7(z-1)^2z^4y^6 - 2z^3(z^3+15z^2 \\
& +15z+1)y^5 + 2z^2(-21z^4+6z^3+14z^2+6z-21)y^4 - 2z(z^5-6z^4-27z^3 \\
& -27z^2-6z+1)y^3 + (7z^6-30z^5+28z^4+54z^3+28z^2-30z+7)y^2 - 2(7z^5 \\
& +15z^4-6z^3-6z^2+15z+7)y + 7z^4-2z^3-42z^2-2z+7)x^4 - 2y(z^3(z^3 \\
& -9z^2-9z+1)y^6 + z^2(7z^4-14z^3-18z^2-14z+7)y^5 + z(7z^5+14z^4+3 \\
& z^3+3z^2+14z+7)y^4 + (z^6-14z^5+3z^4+84z^3+3z^2-14z+1)y^3 - 3(3z^5 \\
& +6z^4-z^3-z^2+6z+3)y^2 - (9z^4+14z^3-14z^2+14z+9)y + z^3+7z^2+7z \\
& +1)x^3 + (z^2(11z^4+6z^3-66z^2+6z+11)y^6 + 2z(5z^5+13z^4-2z^3-2z^2 \\
& +13z+5)y^5 + (11z^6+26z^5+44z^4-66z^3+44z^2+26z+11)y^4 + (6z^5-4 \\
& z^4-66z^3-66z^2-4z+6)y^3 - 2(33z^4+2z^3-22z^2+2z+33)y^2 + (6z^3+26 \\
& z^2+26z+6)y + 11z^2+10z+11)x^2 - 2(z^2(5z^3+3z^2+3z+5)y^5 + z(22z^4 \\
& +5z^3-22z^2+5z+22)y^4 + (5z^5+5z^4-26z^3-26z^2+5z+5)y^3 + (3z^4- \\
& 22z^3-26z^2-22z+3)y^2 + (3z^3+5z^2+5z+3)y + 5z^2+22z+5)x + 15z^2+2z \\
& +2y(z-1)^2(z+1) + 2y^3(z-1)^2z(z+1) + y^4z^2(15z^2+2z+15) + y^2(15z^4 \\
& -2z^3-90z^2-2z+15) + 15] / [(x-1)^2(y-1)^2(xy-1)^2(z-1)^2(yz-1)^2 \\
& (xyz-1)^2] - [4(x+1)(y+1)(yz+1)(-z^2y^4+4z(z+1)y^3+(z^2+1)y^2 \\
& -4(z+1)y + 4x(y^2-1)(y^2z^2-1) + x^2(z^2y^4-4z(z+1)y^3-(z^2+1)y^2 \\
& +4(z+1)y+1)-1)\log(x+1)] / [(x-1)^3x(y-1)^3(yz-1)^3] - [4(y+1)(xy \\
& +1)(z+1)(x^2(z^2-4z-1)y^4 + 4x(x+1)(z^2-1)y^3 - (x^2+1)(z^2-4z-1) \\
& y^2 - 4(x+1)(z^2-1)y + z^2-4z-1)\log(xy+1)] / [x(y-1)^3y(xy-1)^3(z- \\
& 1)^3] - [4(z+1)(yz+1)(x^3y^5z^7 + x^2y^4(4x(y+1)+5)z^6 - xy^3((y^2+ \\
& 1)x^2-4(y+1)x-3)z^5 - y^2(4y(y+1)x^3+5(y^2+1)x^2+4(y+1)x+1)z^4 + \\
& y(y^2x^3-4y(y+1)x^2-3(y^2+1)x-4(y+1))z^3 + (5x^2y^2+y^2+4x(y+1) \\
& y+1)z^2 + ((3x+4)y+4)z-1)\log(xyz+1)] / [xy(z-1)^3z(yz-1)^3(xyz-1)^3])] \\
& / [(x+1)^2(y+1)^2(xy+1)^2(z+1)^2(yz+1)^2(xyz+1)^2] \, dx\, dy\, dz
\end{aligned}
$$

Box integrals

The following integrals appear in numerous applications:

$$B_n(s) := \int_0^1 \cdots \int_0^1 \left(r_1^2 + \cdots + r_n^2\right)^{s/2} \, \mathrm{d}R$$

$$\Delta_n(s) := \int_0^1 \cdots \int_0^1 \left((r_1 - q_1)^2 + \cdots + (r_n - q_n)^2\right)^{s/2} \, \mathrm{d}R\mathrm{d}Q$$

- $B_n(1)$ is average distance of a random point from the origin.
- $\Delta_n(1)$ is average distance between two random points.
- $B_n(-n+2)$ is average electrostatic potential in an n-cube whose origin has a unit charge.
- $\Delta_n(-n+2)$ is average electrostatic energy between two points in a uniform n-cube of charged "jellium."
- Recently integrals of this type have arisen in neuroscience, e.g. the average distance between synapses in a mouse brain.

- D. H. Bailey, J. M. Borwein and R. E. Crandall, "Box integrals," *Journal of Computational and Applied Mathematics*, vol. 206 (2007), pg. 196–208.

Sample evaluations of box integrals

n	s	$B_n(s)$
any	even $s \geq 0$	rational, e.g., : $B_2(2) = 2/3$
1	$s \neq -1$	$\frac{1}{s+1}$
2	-4	$-\frac{1}{4} - \frac{\pi}{8}$
2	-3	$-\sqrt{2}$
2	-1	$2\log(1+\sqrt{2})$
2	1	$\frac{1}{3}\sqrt{2} + \frac{1}{3}\log(1+\sqrt{2})$
2	3	$\frac{7}{5}\sqrt{2} + \frac{3}{20}\log(1+\sqrt{2})$
2	$s \neq -2$	$\frac{2}{2+s}\, {}_2F_1\left(\frac{1}{2}, -\frac{s}{2}; \frac{3}{2}; -1\right)$
3	-5	$-\frac{1}{6}\sqrt{3} - \frac{1}{12}\pi$
3	-4	$-\frac{3}{2}\sqrt{2}\arctan\frac{1}{\sqrt{2}}$
3	-2	$-3G + \frac{3}{2}\pi\log(1+\sqrt{2}) + 3\ \mathrm{Ti}_2(3-2\sqrt{2})$
3	-1	$-\frac{1}{4}\pi + \frac{3}{2}\log\left(2+\sqrt{3}\right)$
3	1	$\frac{1}{4}\sqrt{3} - \frac{1}{24}\pi + \frac{1}{2}\log\left(2+\sqrt{3}\right)$
3	3	$\frac{2}{5}\sqrt{3} - \frac{1}{60}\pi - \frac{7}{20}\log\left(2+\sqrt{3}\right)$

Here F is hypergeometric function; G is Catalan; Ti is Lewin's inverse-tan function.

Sample evaluations of delta integrals

n	s	$\Delta_n(s)$
3	-7	$\frac{4}{5} - \frac{16\sqrt{2}}{15} + \frac{2\sqrt{3}}{5} + \frac{\pi}{15}$
3	–3,-4,-5,-6	∞
3	–2	$2\pi - 12\ G + 12\,\mathrm{Ti}_2\left(3 - 2\sqrt{2}\right) + 6\pi\log\left(1+\sqrt{2}\right) + 2\log 2 - \frac{5}{2}\log 3 - 8\sqrt{2}\arctan\left(\frac{1}{\sqrt{2}}\right)$
3	-1	$\frac{2}{5} - \frac{2}{3}\pi + \frac{2}{5}\sqrt{2} - \frac{4}{5}\sqrt{3} + 2\log\left(1+\sqrt{2}\right) + 12\log\left(\frac{1+\sqrt{3}}{\sqrt{2}}\right) - 4\log\left(2+\sqrt{3}\right)$
3	1	$-\frac{118}{21} - \frac{2}{3}\,\pi + \frac{34}{21}\,\sqrt{2} - \frac{4}{7}\,\sqrt{3} + 2\,\log\left(1+\sqrt{2}\right) + 8\,\log\left(\frac{1+\sqrt{3}}{\sqrt{2}}\right)$
3	3	$-\frac{1}{105} - \frac{2}{105}\,\pi + \frac{73}{840}\,\sqrt{2} + \frac{1}{35}\,\sqrt{3} + \frac{3}{56}\,\log\left(1+\sqrt{2}\right) + \frac{13}{35}\,\log\left(\frac{1+\sqrt{3}}{\sqrt{2}}\right)$

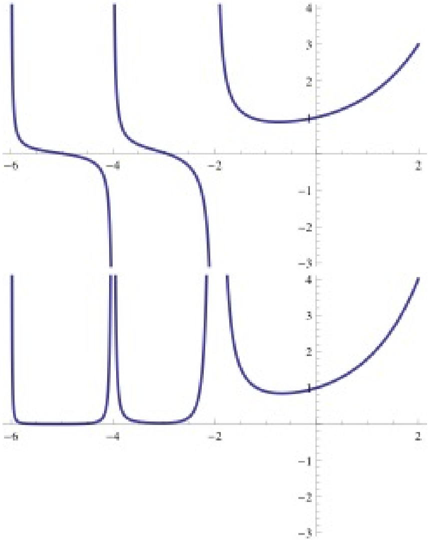
Ramble integrals

Continuing some earlier research [see refs below], we considered

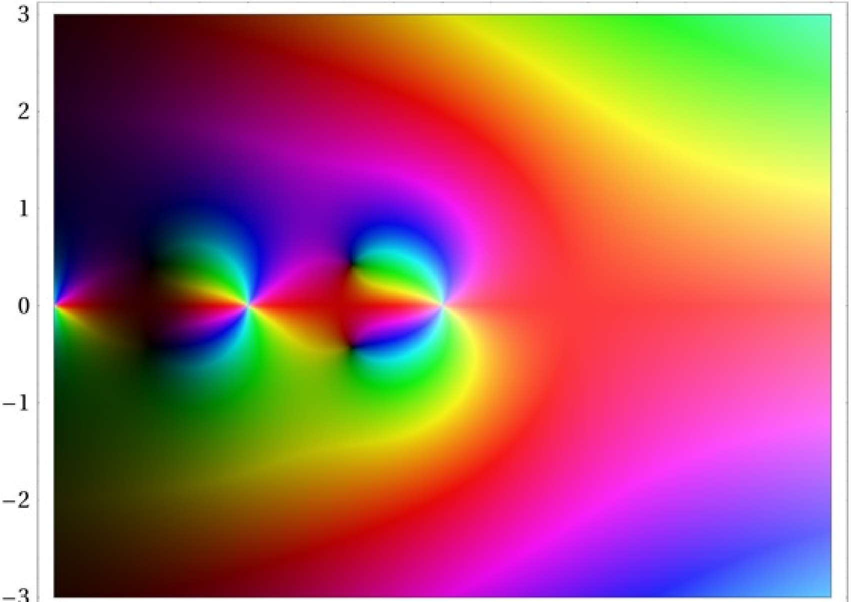
$$W_n(s) = \int_{[0,1]^n} \left| \sum_{k=1}^{n} e^{2\pi x_k i} \right|^s \mathrm{d}x$$

which is the s-th moment of the distance to the origin after n steps of a uniform random walk in the plane, with unit steps in a random direction.

1. J. M. Borwein, D. Nuyens, A. Straub and J. Wan, "Some arithmetic properties of short random walk integrals," *Ramanujan Journal*.
2. J .M. Borwein, A. Straub and J. Wan, "Three-step and four-step random walk integrals," *Experimental Mathematics*.



Complex plane plot of W_4



Some ramble integral results

$$W_3'(0) = \int_{1/6}^{5/6} \log(2\sin(\pi y))\mathrm{d}y \;=\; \frac{1}{\pi}\,\mathrm{Cl}\left(\frac{\pi}{3}\right)$$

$$W_3'(2) = 2 + \frac{3}{\pi}\mathrm{Cl}\left(\frac{\pi}{3}\right) - \frac{3\sqrt{3}}{2\pi}$$

$$W_4'(0) = \frac{7}{2}\frac{\zeta(3)}{\pi^2}$$

$$W_n'(0) = \log(2) - \gamma - \int_0^1 \left(J_0^n(x) - 1\right)\frac{\mathrm{d}x}{x} - \int_1^\infty J_0^n(x)\frac{\mathrm{d}x}{x}$$

$$= \log(2) - \gamma - n\int_0^\infty \log(x)J_0^{n-1}(x)J_1(x)\mathrm{d}x$$

$$W_n''(0) = n\int_0^\infty \left(\log\left(\frac{2}{x}\right) - \gamma\right)^2 J_0^{n-1}(x)J_1(x)\mathrm{d}x$$

$$W_n'(-1) = (\log 2 - \gamma)W_n(-1) - \int_0^\infty \log(x)J_0^n(x)\mathrm{d}x$$

$$W_n'(1) = \int^\infty \frac{n}{x}J_0^{n-1}(x)J_1(x)\left(1 - \gamma - \log(2x)\right)\mathrm{d}x$$

1000-digit computations of $W_n'(0)$ using Sidi's extrapolation scheme

n	Precision	Iterations	Time	30-digit numerical values
3	200	159	123	0.323065947219450514093636510723 8...
	400	320	2046	
	1000	802	106860	
5	200	159	249	0.544412561752185585195878062745 0...
	400	319	2052	
	1000	801	106860	
7	200	157	249	0.702926292476967266787823944395 2...
	400	318	2050	
	1000	800	106860	
9	200	156	248	0.824156239532388694820522824849 6...
	400	317	2120	
	1000	799	106800	
11	200	155	247	0.921850886732653697565891527970 3...
	400	316	4123	
	1000	796	213480	
13	200	154	246	1.003583530489320110604453874320 8...
	400	314	4113	
	1000	796	213540	
15	200	152	245	1.073826217256856036184252781500 3...
	400	313	4096	
	1000	795	213480	
17	200	151	244	1.135410703767411072953239250042 9...
	400	312	4104	
	1000	794	213360	

Elliptic integral functions

The research with ramble integrals led us to study integrals of the form:

$$I(n_0, n_1, n_2, n_3, n_4) := \int_0^1 x^{n_0} K^{n_1}(x) K'^{n_2}(x) E^{n_3}(x) E'^{n_4}(x) \mathrm{d}x,$$

where K, K', E, E' are elliptic integral functions:

$$K(x) := \int_0^1 \frac{\mathrm{d}t}{\sqrt{(1-t^2)(1-x^2t^2)}}$$

$$K'(x) := K(\sqrt{1-x^2})$$

$$E(x) := \int_0^1 \frac{\sqrt{1-x^2t^2}}{\sqrt{1-t^2}} \mathrm{d}t$$

$$E'(x) := E(\sqrt{1-x^2})$$

Relations among the I integrals

Thousands of relations have been found among the I integrals, using PSLQ. For example, among the class with $n_0 \le D_1 = 4$ and $n_1 + n_2 + n_3 + n_4 = D_2 = 3$ (a set of 100 integrals), we found that all can be expressed in terms of an integer linear combination of 8 simple integrals. For example:

$$81 \int_0^1 x^3 K^2(x) E(x) \mathrm{d}x = -6 \int_0^1 K^3(x) \mathrm{d}x - 24 \int_0^1 x^2 K^3(x) \mathrm{d}x + 51 \int_0^1 x^3 K^3(x) \mathrm{d}x + 32 \int_0^1 x^4 K^3(x) \mathrm{d}x$$

$$-243 \int_0^1 x^3 K(x) E(x) K'(x) \mathrm{d}x = -59 \int_0^1 K^3(x) \mathrm{d}x + 468 \int_0^1 x^2 K^3(x) \mathrm{d}x + 156 \int_0^1 x^3 K^3(x) \mathrm{d}x - 624 \int_0^1 x^4 K^3(x) \mathrm{d}x$$

$$-20736 \int_0^1 x^4 E^2(x) K'(x) \mathrm{d}x = 3901 \int_0^1 K^3(x) \mathrm{d}x - 3852 \int_0^1 x^2 K^3(x) \mathrm{d}x - 1284 \int_0^1 x^3 K^3(x) \mathrm{d}x + 5136 \int x^4 K^3(x)$$

$$- 2592 \int_0^1 x^2 K^2(x) K'(x) \mathrm{d}x - 972 \int_0^1 K(x) E(x) K'(x) \mathrm{d}x - 8316 \int_0^1 x K(x) E(x) K'(x)$$

$EE'KK'$ tanh-sinh / PSLQ results

D_1	D_2	Relations	Basis	Total	Precision	Basis norm bound	Max relation norm
0	1	1	3	4	1500	1.582082×10^{298}	2.236068×10^{0}
1	1	5	3	8	1500	2.155768×10^{297}	3.605551×10^{0}
2	1	9	3	12	1500	2.155768×10^{297}	5.916080×10^{0}
3	1	13	3	16	1500	2.155768×10^{297}	1.679286×10^{1}
4	1	17	3	20	1500	2.155768×10^{297}	6.592420×10^{1}
5	1	21	3	24	1500	2.155768×10^{297}	2.419628×10^{2}
0	2	4	6	10	1500	5.609665×10^{261}	2.109502×10^{1}
1	2	12	8	20	1500	4.877336×10^{196}	5.744563×10^{0}
2	2	22	8	30	1500	6.109876×10^{195}	2.293469×10^{1}
3	2	32	8	40	1500	6.109876×10^{195}	2.293469×10^{1}
4	2	42	8	50	1500	6.109876×10^{195}	1.639153×10^{3}
5	2	52	8	60	1500	6.109876×10^{195}	2.428260×10^{3}
0	3	14	6	20	1500	3.871282×10^{262}	2.664001×10^{2}
1	3	34	6	40	1500	2.164052×10^{261}	8.960469×10^{1}
2	3	52	8	60	1500	1.496420×10^{197}	9.666276×10^{2}
3	3	72	8	80	1500	2.829003×10^{196}	2.291372×10^{3}
4	3	92	8	100	1500	8.853827×10^{195}	5.860112×10^{3}
5	3	112	8	120	1500	8.853827×10^{195}	9.240898×10^{4}
0	4	20	15	35	1500	2.689124×10^{104}	1.963656×10^{4}
1	4	53	17	70	1500	6.195547×10^{91}	2.186030×10^{3}
2	4	88	17	105	1500	4.059577×10^{91}	2.970026×10^{4}
3	4	121	19	140	1500	8.856138×10^{81}	5.658994×10^{5}
4	4	156	19	175	1500	2.759846×10^{82}	5.571466×10^{6}

$EE'KK'$ tanh-sinh / PSLQ results, continued

D_1	D_2	Relations	Basis	Total	Precision	Basis norm bound	Max relation norm
0	5	45	11	56	1500	1.256977×10^{142}	1.061532×10^{5}
1	5	101	11	112	1500	2.602478×10^{142}	1.025453×10^{5}
2	5	155	13	168	1500	2.151577×10^{120}	3.953731×10^{5}
3	5	211	13	224	1500	1.314945×10^{120}	3.728547×10^{5}
4	5	265	15	280	1500	5.040597×10^{104}	8.658997×10^{6}
5	5	321	15	336	1500	4.186191×10^{104}	3.954175×10^{11}
0	6	56	28	84	3000	2.958413×10^{105}	1.748907×10^{6}
1	6	138	30	168	3000	2.018080×10^{98}	2.219430×10^{6}
2	6	222	30	252	3000	3.089318×10^{98}	6.301251×10^{8}
3	6	304	32	336	3000	1.324953×10^{92}	2.929549×10^{10}
4	6	388	32	420	3000	9.312061×10^{91}	6.168516×10^{12}
5	6	470	34	504	3000	6.616755×10^{86}	7.199329×10^{13}

For further details

David H. Bailey and Jonathan M. Borwein, "Hand-to-hand combat with thousand-digit integrals," *Journal of Computational Science*, vol. 3 (2012), 77–86, preprint at: http://www.davidhbailey.com/dhbpapers/combat.pdf

Summary

- Identifying definite integrals, by computing very high-precision numerical values and then applying PSLQ, is one of the most productive applications of the experimental mathematics paradigm.
- The key challenge for such applications is to compute the integral to extreme precision. Most schemes taught in numerical analysis courses are not suitable.
- Although Gaussian quadrature is faster for some problems, the tanh-sinh scheme has numerous advantages, including insensitivity to singularities at endpoints, and a cost for generating abscissas and weights that only increases linearly with n.
- We have used the tanh-sinh scheme to compute integrals to as high as 20,000 digit precision.
- In some applications, such as the $EE'KK'$ integrals, literally thousands of relations have been discovered by this approach.

Thanks!

This talk is available at
http://www.davidhbailey.com/dhbtalks/dhb-combat-2017.pdf