Computation and analysis of arbitrary digits of Pi and other mathematical constants

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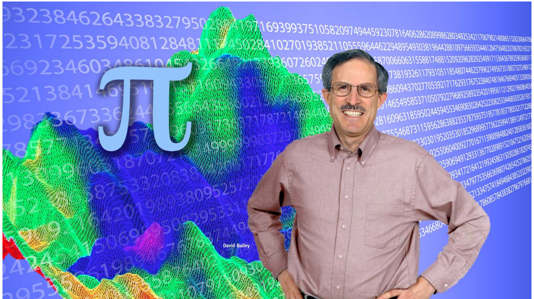


Jonathan M. Borwein, 1951–2016

- 388 published journal articles; another 103 in refereed conference proceedings.
- ISI Web of Knowledge lists 6,593 citations from 351 items; one paper has been cited 666 times.
- His work spanned pure mathematics, applied mathematics, optimization theory, computer science, mathematical finance, and experimental mathematics.
- Borwein sought to do research that is accessible, and to highlight aspects of his work that a broad audience (including both researchers and the lay public) could appreciate.
- More information, including memorials and links to nearly 1700 publications, preprints and talks: http://www.jonborwein.org.

New York Times PiDay 2007 (March 14, 2007) crossword puzzle

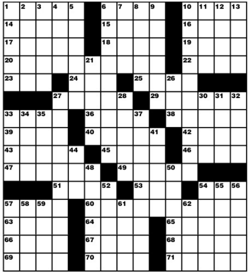
The New York Times Crossword | Edited by Will Shortz | No. 0314

Across

1 Enlighten
6 A couple CBS spinoffs
10 1972 Broadway musical
14 Metal giant
15 Evict
16 Area
17 Surface again, as a road
18 Pirate or Padre, briefly
19 Camera feature
20 Barracks artwork, perhaps
22 River to the Ligurian Sea
23 Keg necessity
24 "… ___ he drove out of sight"
25 ___ St. Louis, Ill.
27 Preen
29 Greek peak
33 Vice president after Hubert
36 Patient wife of Sir Geraint
38 Action to an ante
39 Gain ___
40 French artist Odilon ___
42 Grape for winemaking
43 Single-dish meal
45 Broad valley
46 See 21-Down
47 Artery inserts
49 Offspring
51 Mexican mouse catcher
53 Medical procedure, in brief
54 "Wheel of Fortune" option
57 Animal with striped legs
60 Editorial
63 It gets bigger at night
64 "Hold your horses!"
65 Idiots
66 Europe/Asia border river
67 Suffix with launder
68 Leaning
69 Brownback and Obama, e.g.: Abbr.
70 Rick with the 1976 #1 hit "Disco Duck"
71 Yegg's targets

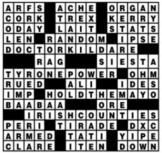
Down

1 Mastodon trap
2 "Mefistofele" soprano
3 Misbehave
4 Pen
5 More pleased
6 Treated with disdain
7 Enterprise crewman
8 Rhone feeder
9 Many a webcast
10 Mushroom, for one
11 Unfortunate
12 Nevada's state tree
13 Disney fish
21 Colonial figure with 46-Across
26 Poker champion Ungar
27 Self-medicating excessively
28 March 14, to mathematicians
30 Book part
31 Powder, e.g.
32 007 and others: Abbr.
33 Drains
34 Stove feature
35 Feet per second, e.g.
37 Italian range
41 Prefix with surgery
44 Captain's announcement, for short
48 Tucked away
50 Stealthy fighters
52 Sedative
54 Letter feature
55 Jam
56 Settles in
57 Symphony or sonata
58 Japanese city bombed in W.W. II
59 Beelike
61 Evening, in ads
62 Religious artwork



Puzzle by Peter A. Collins

ANSWER TO PREVIOUS PUZZLE



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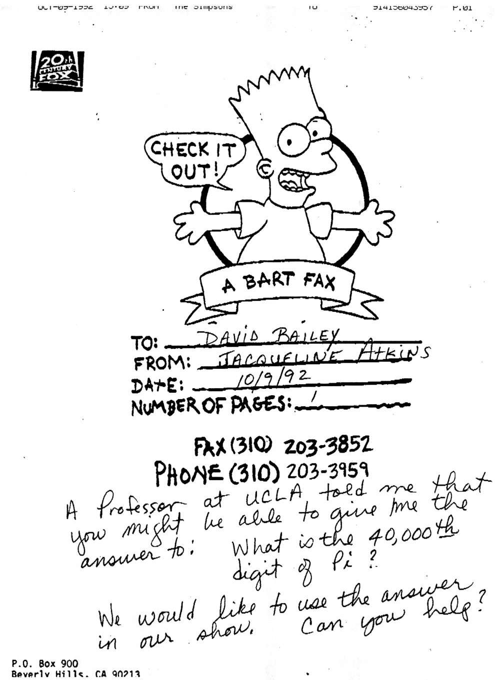
Answers to crossword

ANSWER TO PREVIOUS PUZZLE

T	E	A	C	H		C	S	I	S		π	P	π	N
A	L	C	O	A		O	U	S	T		Z	O	N	E
R	E	T	O	P		N	L	E	R		Z	O	O	M
π	N	U	P	π	C	T	U	R	E		A	R	N	O
T	A	P		E	R	E		E	A	S	T			
			P	R	I	M	P		M	T	O	S	S	A
S	π	R	O		E	N	I	D		U	P	π	N	G
A	L	A	P		R	E	D	O	N		π	N	O	T
P	O	T	π	E		D	A	L	E		N	E	W	S
S	T	E	N	T	S		Y	O	U	N	G			
			G	A	T	O		M	R	I		S	π	N
O	K	A	π		O	π	N	I	O	N	π	E	C	E
P	U	π	L		W	A	I	T		J	E	R	K	S
U	R	A	L		E	T	T	E		A	T	I	L	T
S	E	N	S		D	E	E	S		S	A	F	E	S

I once received a strange fax

- In October 1992, I received this fax from the Simpsons TV show.
- They wanted the 40,000th digit of π.
- I faxed back the result: it is a "1."
- This was used in the Simpsons show, dated 6 May 1993, "Marge in Chains."



OCT-09-1992 FROM The Simpsons TO 914156043957 P.01

CHECK IT OUT!

A BART FAX

TO: DAVID BAILEY

FROM: JACQUELINE ATKINS

DATE: 10/9/92

NUMBER OF PAGES: 1

FAX (310) 203-3852

PHONE (310) 203-3959

A Professor at UCLA told me that you might be able to give me the answer to: What is the 40,000th digit of Pi?

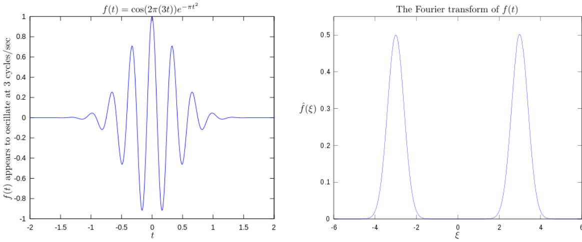
We would like to use the answer in our show. Can you help?

P.O. Box 900

Beverly Hills, CA 90213

Smartphones and π

- Every smartphone or mobile phone crucially relies on computations (e.g., the fast Fourier transform) that involve π to resolve microwave signals.



- π appears in the fundamental equations of quantum mechanics, which are used to design smartphone electronics. For example, Heisenberg's uncertainty principle:

$$\left(\int_{-\infty}^{\infty} s^2|f(s)|^2\,\mathrm{d}s\right)\left(\int_{-\infty}^{\infty} t^2|f(t)|^2\,\mathrm{d}t\right) \geq \frac{||f||_2^4}{16\pi^2}$$

- π appears in the equations of general relativity, used in GPS:

$$R_{\mu\nu} - \frac{Rg_{\mu}\nu}{2} = \frac{8\pi G}{c^4}T_{\mu\nu}$$

Why compute π?

- Question: Do we need to know π to thousands or millions of digits in everyday science and engineering?

  Answer: No. 10–15 digits suffice for most scientific calculations.

- However, some research problems in mathematics and physics require hundreds or thousands of digits.
- I have personally done computations that required π to 64,000-digit precision.
- Billions and even trillions of digits have been computed by mathematicians, in part to explore the unanswered question "Are the digits of π 'random'?"

The first 1000 decimal digits of π

3.14159265358979323846264338327950288419716939937510582097494459230781
6406286208998628034825342117067982148086513282306647093844609550582231
7253594081284811174502841027019385211055596446229489549303819644288109
7566593344612847564823378678316527120190914564856692346034861045432664
8213393607260249141273724587006606315588174881520920962829254091715364
3678925903600113305305488204665213841469519415116094330572703657595919
5309218611738193261179310511854807446237996274956735188575272489122793
8183011949129833673362440656643086021394946395224737190702179860943702
7705392171762931767523846748184676694051320005681271452635608277857713
4275778960917363717872146844090122495343014654958537105079227968925892
3542019956112129021960864034418159813629774771309960518707211349999998
3729780499510597317328160963185950244594553469083026425223082533446850
3526193118817101000313783875288658753320838142061717766914730359825349
0428755468731159562863882353787593751957781857780532171226806613001927
876611195909216420198...

Pre-computer history of π calculations

Name	Year	Digits
Archimedes	-250?	3
Ptolemy	150?	3
Liu Hui	265?	5
Aryabhata	480?	5
Tsu Ch'ung Chi	480?	7
Madhava	1400?	13
Al-Kashi	1429	14
Romanus	1593	15
Van Ceulen	1615	35
Sharp and Halley	1699	71
Machin	1706	100
Strassnitzky and Dase	1844	200
Rutherford	1853	440
W. Shanks	1874	*707
Ferguson (mechanical calculator)	1947	808

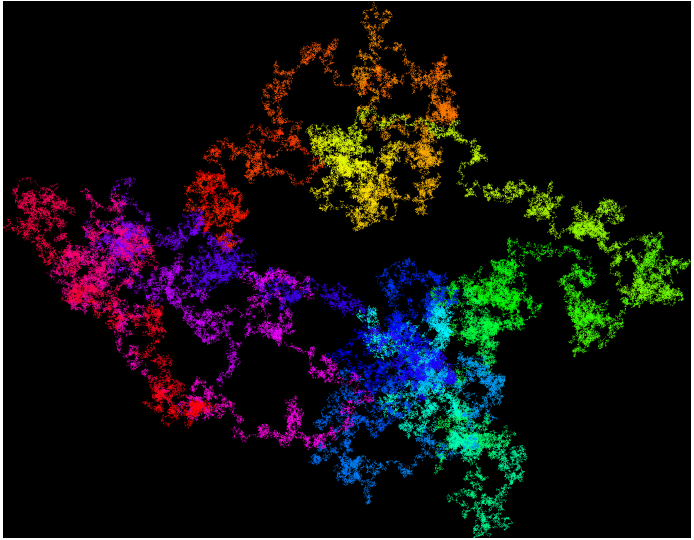
*Only the first 527 were correct.

Computer-era π calculations

Name	Year	Digits
Miyoshi and Kanada	1981	2,000,036
Kanada-Yoshino-Tamura	1982	16,777,206
Gosper	1985	17,526,200
Bailey	1986	29,360,111
Kanada et. al	1987	134,217,700
Kanada and Tamura	1989	1,073,741,799
Chudnovskys	1994	4,044,000,000
Kanada and Takahashi	1997	51,539,600,000
Kanada and Takahashi	1999	206,158,430,000
Kanada-Ushiro-Kuroda	2002	1,241,100,000,000
Takahashi	2009	2,576,980,377,524
Bellard	2009	2,699,999,990,000
Kondo and Yee	2010	5,000,000,000,000
Trueb	2016	22,459,157,718,361

If 22 trillion digits were printed in 12-point type, they would stretch nearly to Mars.

A random walk on the first 100 billion base-4 digits of π



This dataset can be explored online: `http://gigapan.com/gigapans/106803`

Some formulas for computing π

$$\pi = \frac{3\sqrt{3}}{4} - 24\sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{(2n+3)(2n-1)4^{2n+1}} \quad \text{(Newton, 1660)}$$

$$\frac{\pi}{4} = 4\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)5^{2n+1}} - \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)239^{2n+1}} \quad \text{(Machin, 1730)}$$

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801}\sum_{n=0}^{\infty} \frac{(4n)!(1103+26390n)}{(n!)^4 396^{4n}} \quad \text{(Ramanujan, 1930)}$$

Set $a_0 = 6 - 4\sqrt{2}$ and $y_0 = \sqrt{2} - 1$. Iterate

$$y_{k+1} = \frac{1-(1-y_k^4)^{1/4}}{1+(1-y_k^4)^{1/4}}$$

$$a_{k+1} = a_k(1+y_{k+1})^4 - 2^{2k+3}(1+y_{k+1}+y_{k+1}^2).$$

Then a_k converge quartically to $1/\pi$: each iteration quadruples the number of correct digits.
20 iterations are sufficient to compute π to 2.9 trillion digits, provided all iterations are done to this precision. (Jonathan and Peter Borwein, 1984)

How does one do arithmetic to extremely high precision?

Computing π or anything else to extremely high precision requires special software:

- High-precision numbers are stored as a sequence of computer words.
- Addition and subtraction are performed using relatively simple methods.
- Multiplication is performed using a fast Fourier transform, which is thousands or even millions of times faster than conventional methods.
- Division and square roots are performed using Newton iterations, based on multiplication and addition.
- Exponential and trigonometric functions are evaluated using special algorithms.

Software packages to perform these operations are readily available on the Internet, or by using systems such as *Mathematica*, *Maple* or *Sage*.

The PSLQ integer relation algorithm

Given a vector (x_n) of real numbers, an integer relation algorithm finds integers (a_n) such that

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n \quad = \quad 0$$

(to within the precision of the arithmetic being used), or else finds bounds within which no relation can exist.

Helaman Ferguson's PSLQ algorithm is the most widely used integer relation algorithm.

Integer relation detection (using PSLQ or any other algorithm) requires very high numeric precision, both in the input data and in the operation of the algorithm.

1. H. R. P. Ferguson, D. H. Bailey and S. Arno, "Analysis of PSLQ, an integer relation finding algorithm," *Mathematics of Computation*, vol. 68, no. 225 (Jan 1999), 351–369.
2. D. H. Bailey and D. J. Broadhurst, "Parallel integer relation detection: Techniques and applications," *Mathematics of Computation*, vol. 70, no. 236 (Oct 2000), 1719–1736.

Helaman Ferguson's "Umbilic Torus SC" sculpture at Stony Brook Univ.

Computing binary digits of log(2) beginning at an arbitrary position

1996 result: Consider this well-known formula for log(2):

$$\log(2) = \sum_{n=1}^{\infty} \frac{1}{n2^n} \; = \; 0.10110001011100100001011111110111110100011100111101 1\ldots_2$$

Note that the binary digits of $\log 2$ beginning after position d can be written as $\{2^d \log 2\}$, where $\{\cdot\}$ denotes fractional part. Thus we can write:

$$\begin{aligned}\{2^d \log(2)\} &= \left\{\sum_{n=1}^{d} \frac{2^{d-n}}{n}\right\} + \left\{\sum_{n=d+1}^{\infty} \frac{2^{d-n}}{n}\right\} \\ &= \left\{\sum_{n=1}^{d} \frac{2^{d-n} \bmod n}{n}\right\} + \left\{\sum_{n=d+1}^{\infty} \frac{2^{d-n}}{n}\right\}\end{aligned}$$

We have inserted "mod n" since were are only interested in the fractional part when divided by n. Now note that the numerator $2^{d-n} \bmod n$ can be calculated very rapidly using the binary algorithm for exponentiation.

The binary algorithm for exponentiation

Problem: What is $3^{17} \bmod 10$? (i.e., what is the last decimal digit of 3^{17}?)

Algorithm A:
$3^{17} = 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 = 129140163$,
so answer $= 3$.

Algorithm B (faster): $3^{17} = ((((3^2)^2)^2)^2) \times 3 = 129140163$, so answer $= 3$.

Algorithm C (fastest):
$3^{17} = ((((3^2 \bmod 10)^2 \bmod 10)^2 \bmod 10)^2 \bmod 10) \times 3 \bmod 10 = 3$.

Note that in Algorithm C, we never have to deal with integers larger than $9 \times 9 = 81$, so the entire operation can be performed very rapidly on a computer.

General BBP-type formulas

The same “trick” that was used for $\log(2)$ can be applied for any real constant α that can be written in the form

$$\alpha = \sum_{n=0}^{\infty} \frac{p(n)}{b^n q(n)}$$

where p and q are integer polynomials, $\deg p < \deg q$, and q has no zeroes for $n \geq 0$, or as a linear sum of such formulas.

What other well-known mathematical constants can be written by such a formula?

Can π be written in this form? None was known at the time (1996).

The BBP formula for π

In 1996, a PSLQ program discovered this new formula for π:

$$\pi = \sum_{k=0}^{\infty} \frac{1}{16^k} \left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6} \right)$$

Indeed, this formula permits one to compute base-16 (or binary) digits of π beginning at an arbitrary starting position. The proof is simple.

This is the first known instance of a computer program discovering a fundamentally new formula for π.

BBP-type formulas (also discovered using PSLQ) are now known for numerous other mathematical constants.

Sadly, there is no such similar formula for base-10 digits of π.

- D. H. Bailey, P. B. Borwein and S. Plouffe, "On the rapid computation of various polylogarithmic constants," *Mathematics of Computation*, vol. 66 (Apr 1997), 903–913.

Some other BBP-type formulas found using PSLQ

$$\pi^2 = \frac{1}{8}\sum_{k=0}^{\infty}\frac{1}{64^k}\left(\frac{144}{(6k+1)^2} - \frac{216}{(6k+2)^2} - \frac{72}{(6k+3)^2} - \frac{54}{(6k+4)^2} + \frac{9}{(6k+5)^2}\right)$$

$$\pi^2 = \frac{2}{27}\sum_{k=0}^{\infty}\frac{1}{729^k}\left(\frac{243}{(12k+1)^2} - \frac{405}{(12k+2)^2} - \frac{81}{(12k+4)^2} - \frac{27}{(27k+5)^2}\right.$$
$$\left. - \frac{72}{(12k+6)^2} - \frac{9}{(12k+7)^2} - \frac{9}{(12k+8)^2} - \frac{5}{(12k+10)^2} + \frac{1}{(12k+11)^2}\right)$$

$$\pi^2 \log(2) = \frac{1}{32}\sum_{k=0}^{\infty}\frac{1}{4096^k}\left(\frac{18432}{(24k+2)^3} - \frac{69120}{(24k+3)^3} + \frac{18432}{(24k+4)^3} + \frac{25344}{(24k+6)^3} + \frac{27648}{(24k+8)^3}\right.$$
$$+ \frac{8640}{(24k+9)^3} + \frac{1152}{(24k+10)^3} + \frac{2880}{(24k+12)^3} + \frac{288}{(24k+14)^3} + \frac{1080}{(24k+15)^3} + \frac{1728}{(24k+16)^3}$$
$$\left. + \frac{396}{(24k+18)^3} + \frac{72}{(24k+20)^3} - \frac{135}{(24k+21)^3} + \frac{18}{(24k+22)^3}\right)$$

- David H. Bailey, “A compendium of BBP-type formulas for mathematical constants,” updated 15 Aug 2017, `http://www.davidhbailey.com/dhbpapers/bbp-formulas.pdf`.

The BBP formula for π in action

In our 1996 paper on the BBP algorithm, we computed base-16 digits of π starting at position 10 billion.

In July 2010, Tsz-Wo Sze used a variant of the BBP formula to compute the base-16 digits of π starting at position 500 trillion (corresponding to binary position 2 quadrillion). The run required 16 billion CPU-seconds of computing.

The result was checked by repeating the calculation to find digits starting at position 500 trillion + 1. The two results were (in base-16 digits):

```
0 E6C1294A ED40403F 56D2D764 026265BC A98511D0 FCFFAA10 F4D28B1B
  E6C1294A ED40403F 56D2D764 026265BC A98511D0 FCFFAA10 F4D28B1B
```

Note that the results precisely overlap. The probability that two randomly generated 56-long strings of base-16 digits perfectly agree is approximately 3.7×10^{-68}.

BBP-type formulas for π^2

Whereas only base-2 (binary) BBP-type formulas exist for π, there are both binary (base-2) and ternary (base-3) formulas for π^2, both discovered by PSLQ:

$$\pi^2 = \frac{9}{8}\sum_{k=0}^{\infty}\frac{1}{64^k}\left(\frac{16}{(6k+1)^2} - \frac{24}{(6k+2)^2} - \frac{8}{(6k+3)^2} - \frac{6}{(6k+4)^2} + \frac{1}{(6k+5)^2}\right)$$

$$\pi^2 = \frac{2}{27}\sum_{k=0}^{\infty}\frac{1}{729^k}\left(\frac{243}{(12k+1)^2} - \frac{405}{(12k+2)^2} - \frac{81}{(12k+4)^2} - \frac{27}{(12k+5)^2}\right.$$
$$\left. - \frac{72}{(12k+6)^2} - \frac{9}{(12k+7)^2} - \frac{9}{(12k+8)^2} - \frac{5}{(12k+10)^2} + \frac{1}{(12k+11)^2}\right)$$

We decided to use these formulas to compute base-64 and base-729 digits of π^2, starting at position ten trillion.

BBP-type formula for Catalan's constant

We also decided to calculate digits of Catalan's constant:

$$G = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} = 0.91596559417722\ldots$$

which is closely related to π^2:

$$\frac{\pi^2}{8} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = 1.2337005501362\ldots$$

We employed this formula, which we discovered using the PSLQ algorithm:

$$G = \frac{1}{4096} \sum_{k=0}^{\infty} \frac{1}{4096^k} \left(\frac{36864}{(24k+2)^2} - \frac{30720}{(24k+3)^2} - \frac{30720}{(24k+4)^2} - \frac{6144}{(24k+6)^2} - \frac{1536}{(24k+7)^2} \right.$$
$$+ \frac{2304}{(24k+9)^2} + \frac{2304}{(24k+10)^2} + \frac{768}{(24k+14)^2} + \frac{480}{(24k+15)^2} + \frac{384}{(24k+11)^2} + \frac{1536}{(24k+12)^2}$$
$$\left. + \frac{24}{(24k+19)^2} - \frac{120}{(24k+20)^2} - \frac{36}{(24k+21)^2} + \frac{48}{(24k+22)^2} - \frac{6}{(24k+23)^2} \right).$$

Andrew Mattingly, Glenn Wightwick, and the IBM BlueGene

For the actual computations, Jonathan Borwein and I turned to our colleagues Andrew Mattingly and Glenn Wightwick at IBM Australia, who were willing to help with programming and tuning. They received permission from IBM to use an IBM BlueGene supercomputer for this purpose.

Our results — two double-checking runs each

1. *Base-64 digits of π^2 beginning at position 10 trillion (a base-64 digit is a pair of base-8 digits)*:

   ```
   75|60114505303236475724500005743262754530363052416350634
     |60114505303236475724500005743262754530363052416350634
   ```

2. *Base-729 digits of π^2 beginning at position 10 trillion (a base-729 digits is a triplet of base-9 digits)*:

   ```
   001|12264485064548583177111135210162856048323453468
      |12264485064548583177111135210162856048323453468
   ```

3. *Base-4096 digits of Catalan's constant beginning at position 10 trillion (a base-4096 digit is a quadruplet of base-8 digits)*:

   ```
   0176|34705053774777051122613371620125257327217324522
       |34705053774777051122613371620125257327217324522
   ```

These runs required 22.1 billion CPU-seconds.

New calculation of base-16 digits of π

In December 2016, Daisuke Takahashi finished the computation of hexadecimal (base-16) digits of π beginning at position 100 quadrillion, or 10^{17}.

The run used Bellard's formula (a variation of the BBP formula for π). Both the main run and the verification run each required 320 hours on 512 nodes of a Fujitsu cluster at the Joint Center for Advanced High Performance Computing (JCAHPC) in Japan.

The hexadecimal digits of π from position 10^{17} to $10^{17} + 15$ are: `A937EB59439E485E`

Are the digits of π random?

Given a positive integer b, a real number α is normal base b if every m-long string of digits appears in the base-b expansion of α with limiting frequency $1/b^m$. It can be shown that almost all real numbers are normal base b, for all bases b.

These constants are widely believed to be normal base b, for all bases b:

- $\pi = 3.14159265358979323846\ldots$
- $e = 2.71828182845904523540\ldots$
- $\sqrt{2} = 1.41421356237309504880\ldots$
- $\log(2) = 0.69314718055994530942\ldots$
- *Every* irrational algebraic number (this conjecture is due to Borel).

But there are no proofs of normality for any of the above — not even for $b = 2$ and $m = 1$ (i.e., equal numbers of zeroes and ones in the binary expansion).

Until recently, normality proofs were known only for a few constants, such as Champernowne's constant $= 0.12345678910111213141516\ldots$ (normal base 10).

BBP formulas and normality

Consider a general BBP-type constant (i.e., a formula that permits the BBP "trick"):

$$\alpha = \sum_{n=0}^{\infty} \frac{p(n)}{b^n q(n)},$$

where p and q are integer polynomials, $\deg p < \deg q$, and q has no zeroes for $n \geq 0$.

Richard Crandall (deceased 2012) and I proved that α is normal base b if and only if the sequence $x_0 = 0$, and

$$x_n = \left\{ b x_{n-1} + \frac{p(n)}{q(n)} \right\},$$

is equidistributed in the unit interval. Brackets $\{\cdot\}$ denote fractional part, as before.

Here equidistributed means that the sequence visits each subinterval (c, d) with limiting frequency $d - c$.

- D. H. Bailey and R. E. Crandall, "On the random character of fundamental constant expansions," *Experimental Mathematics*, vol. 10 (Jun 2001), 175–190.

Two specific examples: $\log(2)$ and π

Consider the sequence $x_0 = 0$ and

$$x_n = \left\{ 2x_{n-1} + \frac{1}{n} \right\}$$

Then $\log(2)$ is normal base 2 if and only if (x_n) is equidistributed in the unit interval.

Similarly, consider the sequence $y_0 = 0$ and

$$y_n = \left\{ 16y_{n-1} + \frac{120n^2 - 89n + 16}{512n^4 - 1024n^3 + 712n^2 - 206n + 21} \right\}$$

Then π is normal base 16 (and hence normal base 2) if and only if (y_n) is equidistributed in the unit interval.

Sadly, we have not yet been able to prove equidistribution for either sequence.

Curiously, the sequence (y_n), when mapped to the 16 divisions of the unit interval, appears to generate, digit by digit, the entire base-16 expansion of π, error-free.

A class of provably normal constants

Crandall and I also proved that the following constant is normal base 2:

$$\begin{aligned}\alpha_{2,3} &= \sum_{n=1}^{\infty} \frac{1}{3^n 2^{3^n}} \\ &= 0.0418836808315029850712528986245716824260\ldots_{10} \\ &= 0.0000101010111000111000111000111101101000001\ldots_{2}\end{aligned}$$

This constant was proven normal by Stoneham in 1971, but we have extended this to the case where $(2,3)$ are any pair (p,q) of relatively prime integers, and also to a larger, uncountably infinite class.

The original proof is difficult, but a subsequent proof using a "hot spot lemma" (via ergodic theory) is quite simple.

1. D. H. Bailey and R. E. Crandall, "Random generators and normal numbers," *Experimental Mathematics*, vol. 11 (2002), 527–546.
2. D. H. Bailey and M. Misiurewicz, "A strong hot spot theorem," *Proceedings of the American Mathematical Society*, vol. 134 (2006), 2495-2501.

Binary digits of $\alpha_{2,3}$ versus base-6 digits of $\alpha_{2,3}$

0.	0.
0000101010111000111000111000111101101000	0130140430003334251130502130000001243555
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1110000001100101001000101100001111110011	0354501505352205204434045215150510241155
0101101110100111100000011001010010001011	2500425130051124454001044131150032420303
0000111111011110010010011011111101010111	2130000000000000000000000000000000000000
0111000010111001011010100110011100111110	0000014212034311121452013525445342113412
0010100000000100001101101100100000010100	2402205253010542044235524110554150155204
0100011110100011010010101100110001100000	3504145554003101453030335320025343404013
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0000111010111111111101001100001100111111	4551440430000000000000000000000000000000
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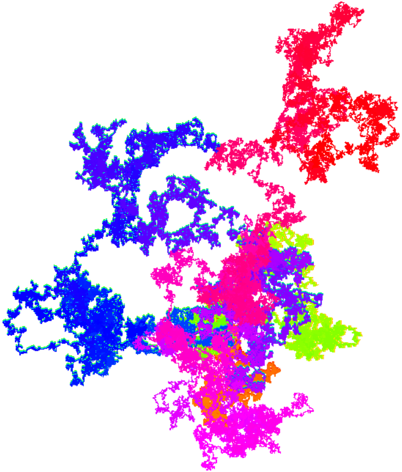
Binary digits of $\alpha_{2,3}$ versus base-6 digits of $\alpha_{2,3}$: Random walks



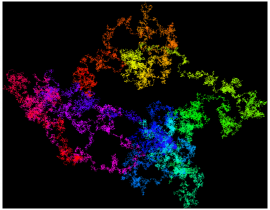
- F. J. Aragon Artacho, D. H. Bailey, J. M. Borwein and P. B. Borwein, "Walking on real numbers," *Mathematical Intelligencer*, vol. 35 (2013), 42-60.

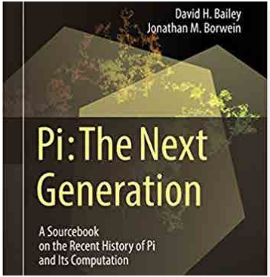
π and experimental mathematics

π continues to excite millions worldwide, leading many to pursue careers in math, science and engineering.

Experimental mathematics enables a much broader community to do real math research:

- High school and college students.
- Computer scientists.
- Computer graphics experts.
- Statisticians.
- Data scientists.





Winning the battle, but losing the war

Mathematicians and scientists may be winning battles to publish papers and obtain grants, but we are losing the war for the hearts and minds of the public:

- 51% in U.S. either do not believe in climate change, or do not believe there is any human connection.
- 42% in U.S. believe that humans were created within past 10,000 years.
- 38% in U.S. do not believe in evolution.
- 32% in U.S. do not believe vaccinations are safe.
- 48% in U.S. believe humans are being visited by extraterrestrial UFOs.
- 6% in U.S. believe NASA faked the Apollo moon landings.
- Some even dispute the value of π. (I frequently receive such email.)

Anti-science movements arise from both sides of the political spectrum:

- From the left: anti-vaccination and anti-fluoridation.
- From the right: anti-climate change and anti-evolution.

Carl Sagan's warning (*The Demon Haunted World*, 1995)

I have a foreboding of an America in my children's or my grandchildren's time — when the United States is a service and information economy; when nearly all the key manufacturing industries have slipped away to other countries; when awesome technological powers are in the hands of a very few, and no one representing the public interest can even grasp the issues; when the people have lost the ability to set their own agendas or knowledgeably question those in authority; when, clutching our crystals and nervously consulting our horoscopes, our critical faculties in decline, unable to distinguish between what feels good and what's true, we slide, almost without noticing, back into superstition and darkness. ...

We've arranged a global civilization in which most crucial elements ... profoundly depend on science and technology. We have also arranged things so that almost no one understands science and technology. This is a prescription for disaster. We might get away with it for a while, but sooner or later this combustible mixture of ignorance and power is going to blow up in our faces.

How can we turn the tide?

- Start a blog.
- Visit schools or give lectures.
- Write books for the general public.
- Write articles for science news forums.
- Write expository articles for scientific journals.
- Pursue research topics that have potentially wide appeal.
- Study the arts and humanities to sharpen communication skills.
- Recognize communication skills in hiring, promotion and research grant decisions.

Math Drudge

Two mathematicians contemplate the cosmos

NEW MATH SCHOLAR BLOG | EXPERIMENTAL MATH SITE | JON BORWEIN MEMORIAL SITE | FINANCIAL MATH SITE

ESSAYS | QUOTATIONS | BOOK REVIEWS | NEWS | DISCLAIMER AND COPYRIGHT

« Where is ET? Fermi's paradox turns 65 Is the nature of mathematical proof changing

I Prefer Pi: Background for Big Pi Day (3/14/15)

"I Prefer Pi" is appropriate title for **Pi Day** (3/14, i.e., March 14), as it is one of the few palindromes involv Pi = 3.141592653589793... (a palindrome is a phrase that reads the same forwards or backwards).

Pi Day is particularly memorable this year, since only once a century can one celebrate this event in a year where the longer version 3/14/15 continues two more correct digits of Pi. The Museum of Mathematics in New York City, among others, is taking Pi Day 2015 one step further, by celebrating at 9:26am, i.e., 3/14/15/926, adding three more digits. See **MoMath's website** for details.

Chicagoans plan to **celebrate** by running in a Pi-K race of 3.14 miles. Numerous city bakeries are offering special pies for the occasion at $3.14 per slice.

Not as well known perhaps is the fact that March 14 is also the 136th birthday of Albert Einstein, and that 2015 is the 100th anniversary of the publication of Einstein's paper on general relativity. To commemorate this year's doubly significant event, Princeton University is planning a **gala event**, including a pie eating contest, a performance by the Princeton Symphony, a contest to see who can recite the most correct digits of pi, and a guided Einstein tour.

Pi in the popular culture

Pi Day long ago extended its reach beyond a handful of mathematical zealots, to become a widely celebrated event. For example, the March 14, 2007 **New York Times crossword puzzle** featured clues, where, in numerous locations, a pi character (standing for PI) must be entered at the intersection of two words. For example, 33 across "Vice president after Hubert" (answer: SPIRO) intersects with 34 down "Stove feature" (answer: PILOT). Indeed 28 down, with clue "March 14, to mathematicians," was, appropriately enough, PIDAY, while PIPPIN is now a four-letter word.

In 2009, the U.S. House of Representatives passed **resolution** officially designating March 15 as as "National Pi Day," and encouraging "schools and educators to observe th day with appropriate activities that teach students about Pi and engage them about the study of mathematics." This may well be the first legislation on Pi Day to have been adopted by a national governmental body.

In general, pi is much more in the public eye than it was even five or ten years ago. On May 9, 2013, the North American quiz show Jeopardy! featured an entire category of questions on pi. The clues provided were:

1. ($200) Pi is the ratio of this measurement of a circle to its diameter.
2. ($400) Numerically, pi is considered this, like a type of "meditation".
3. ($600) For about $19,100 x pi, this "Black Swan" director made "Pi", his 1998 debut film about a math whiz.
4. ($800) In the 100s A.D. this Alexandrian astronomer calculated a more precise value of pi, the equivalent of 3.14166.
5. ($1000) You can find the area of this oval geometric shape with pi x A x B, if A & B are half of its longest & shortest diameter.

The clues and the answers (all were answered correctly by various contestants) are given **here** in the J-archive, an independent repository of clues and answers maintained by Jeopardy! fans.

Some other recent examples of the public's mania for pi include the following:

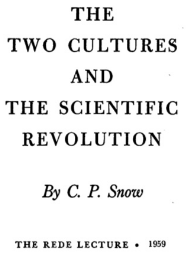
1. On September 12, 2012, five aircraft armed with dot-matrix-style skywriting technology **wrote 1000 digits of pi** in the sky above the San Francisco Bay Area as a spectacular if costly piece of *piformance* art.
2. On March 14, 2012 (appropriately enough), U.S. District Court Judge Michael H. Simon **dismissed** a copyright infringement suit relating to the lyrics of a song by rulii that "Pi is a non-copyrightable fact."
3. On August 18, 2005, Google **offered** 14,159,265 "new slices of rich technology" in their initial public stock offering. On January 29, 2013 they offered a pi-million doll prize for successful hacking of the Chrome Operating System on a specific Android phone.

Ending the war between science and the humanities

Given the growing tensions in society, and the impact of rapidly changing technology, we can no longer afford a war between the science-tech world and the humanities:

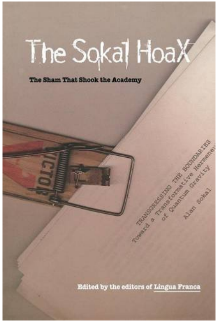
- Those in math, science and technology must learn more about the humanities, to better appreciate these fields, and to better communicate to the public.
- Those in the humanities must learn more about math, science and technology, to better appreciate these fields, and to better participate in dialogue on key issues.

It's in Apple's DNA that technology alone is not enough — that it's technology married with liberal arts, married with the humanities, that yields us the result that makes our hearts sing. [Steve Jobs]


THE
TWO CULTURES
AND
THE SCIENTIFIC
REVOLUTION

By C. P. Snow

THE REDE LECTURE • 1959



They should have sent a poet

In one memorable scene from the movie *Contact*, Jodi Foster views a galaxy from her spacecraft, and is so overcome with awe that she exclaims,

> They should have sent a poet. So beautiful. So beautiful... I had no idea.

In a similar way, those of us involved in scientific research are often stunned by the beauty and elegance of mathematics and science, along with the rather mysterious fact that we humans are able to comprehend these laws.

So why don't we do more to share this wonder? Why don't we write some poetry?

Thanks!

This talk is available here: `http://www.davidhbailey.com/dhbtalks/dhb-conant.pdf`

The Conant Prize paper is here: `http://www.ams.org/notices/201307/rnoti-p844.pdf`