Lattice Sums Arising from the Poisson Equation

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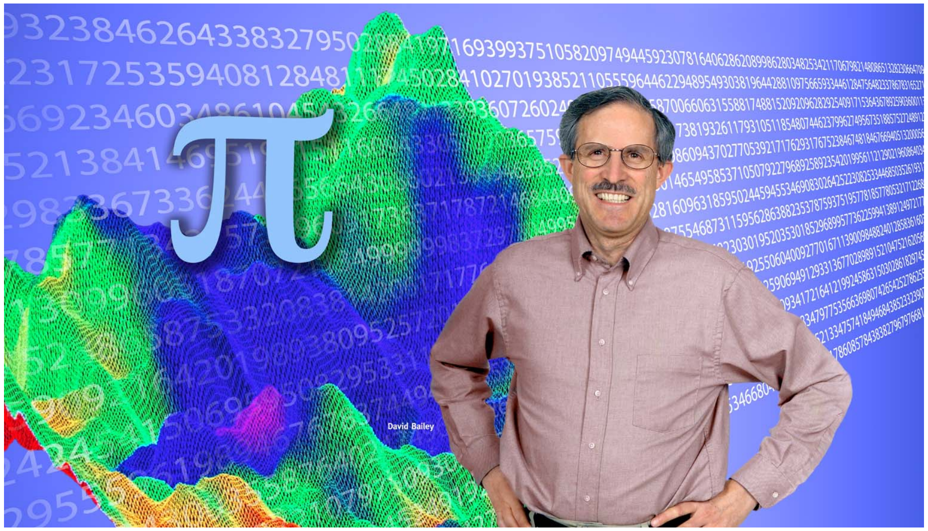
This talk is available at:

http://www.davidhbailey.com/dhbtalks/dhb-lattice.pdf



Lattice sum solutions to the Poisson equation

The Poisson equation for an n-dimensional point-charge source gives rise to an electric static potential of the form

$$\phi_n(\mathbf{r}) = \frac{2^n}{\pi^2} \sum_{m_1,\ldots,m_n > 0,\ \mathrm{odd}} \frac{\cos(\pi m_1 r_1) \cdots \cos(\pi m_n r_n)}{m_1^2 + \cdots m_n^2}$$

Sums of this type have been studied widely in mathematical physics and also have applications in image processing (to sharpen images).

In two recent papers (see below), we found closed forms for these sums in many cases, by taking advantage of connections with Jacobi theta functions.

1. DHB, J. M. Borwein, R. E. Crandall and J. Zucker, "Lattice sums arising from the Poisson equation," manuscript, http://www.davidhbailey.com/dhbpapers/PoissonLattice.pdf
2. DHB and J. M. Borwein, "Compressed lattice sums arising from the Poisson equation: Dedicated to Professor Hari Sirvastava," manuscript, http://www.davidhbailey.com/dhbpapers/Poissond.pdf.

Algebraic numbers in Poisson potential functions associated with lattice sums

In particular, we discovered numerically, and then **proved**, that for rational (*x*, *y*), the two-dimensional Poisson potential function satisfies

$$\phi_2(x,y) = \frac{1}{\pi^2} \sum_{m,n \text{ odd}} \frac{\cos(m\pi x)\cos(n\pi y)}{m^2+n^2} = \frac{1}{\pi}\log\alpha$$

where α is an *algebraic number*, i.e., the root of an integer polynomial:

$$0 = a_0 + a_1\alpha + a_2\alpha^2 + \cdots + a_n\alpha^n$$

Numerous other results, both particular and general, are presented in the two papers.

Fast series for computing $\phi_n(x,y)$

Our approach was to compute $\phi_n(x,y)$ to very high precision (hundreds or thousands of digits), then apply the PSLQ algorithm to identify the computed values. We discovered that these are always of the form $1/\pi \log(\alpha)$.

These computations were enabled by finding the following very rapidly convergent series for $\phi_n(x,y)$:

$$\phi_2(x,y) = \frac{1}{4\pi}\log\frac{\cosh(\pi x)+\cos(\pi y)}{\cosh(\pi x)-\cos(\pi y)} - \frac{2}{\pi}\sum_{m>0 \text{ odd}} \frac{\cosh(\pi m x)\cos(\pi m y)}{m(1+e^{\pi m})}$$

The minimal polynomials for $\alpha = \exp(\pi\, \phi_2(x,y))$ were found by PSLQ calculations, with the $(n+1)$-long vector $(1, \alpha, \alpha^2, \ldots, \alpha^n)$ as input.

PSLQ returns the vector of integer coefficients $(a_0, a_1, a_2, \ldots, a_n)$ as output.

The PSLQ integer relation algorithm

Let (x_n) be a given vector of real numbers. An integer relation algorithm finds integers (a_n) such that

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = 0$$

(or within "epsilon" of zero, where epsilon = 10^{-p} and p is the precision).

At the present time the "PSLQ" algorithm of mathematician-sculptor Helaman Ferguson is the most widely used integer relation algorithm. It was named one of ten "algorithms of the century" by *Computing in Science and Engineering*.

Integer relation detection requires very high precision (at least $n*d$ digits, where d is the size in digits of the largest a_k), both in the input data and in the operation of the algorithm.

1. H. R. P. Ferguson, DHB and S. Arno, "Analysis of PSLQ, An Integer Relation Finding Algorithm," *Mathematics of Computation*, vol. 68, no. 225 (Jan 1999), pg. 351-369.
2. DHB and D. J. Broadhurst, "Parallel Integer Relation Detection: Techniques and Applications," *Mathematics of Computation*, vol. 70, no. 236 (Oct 2000), pg. 1719-1736.

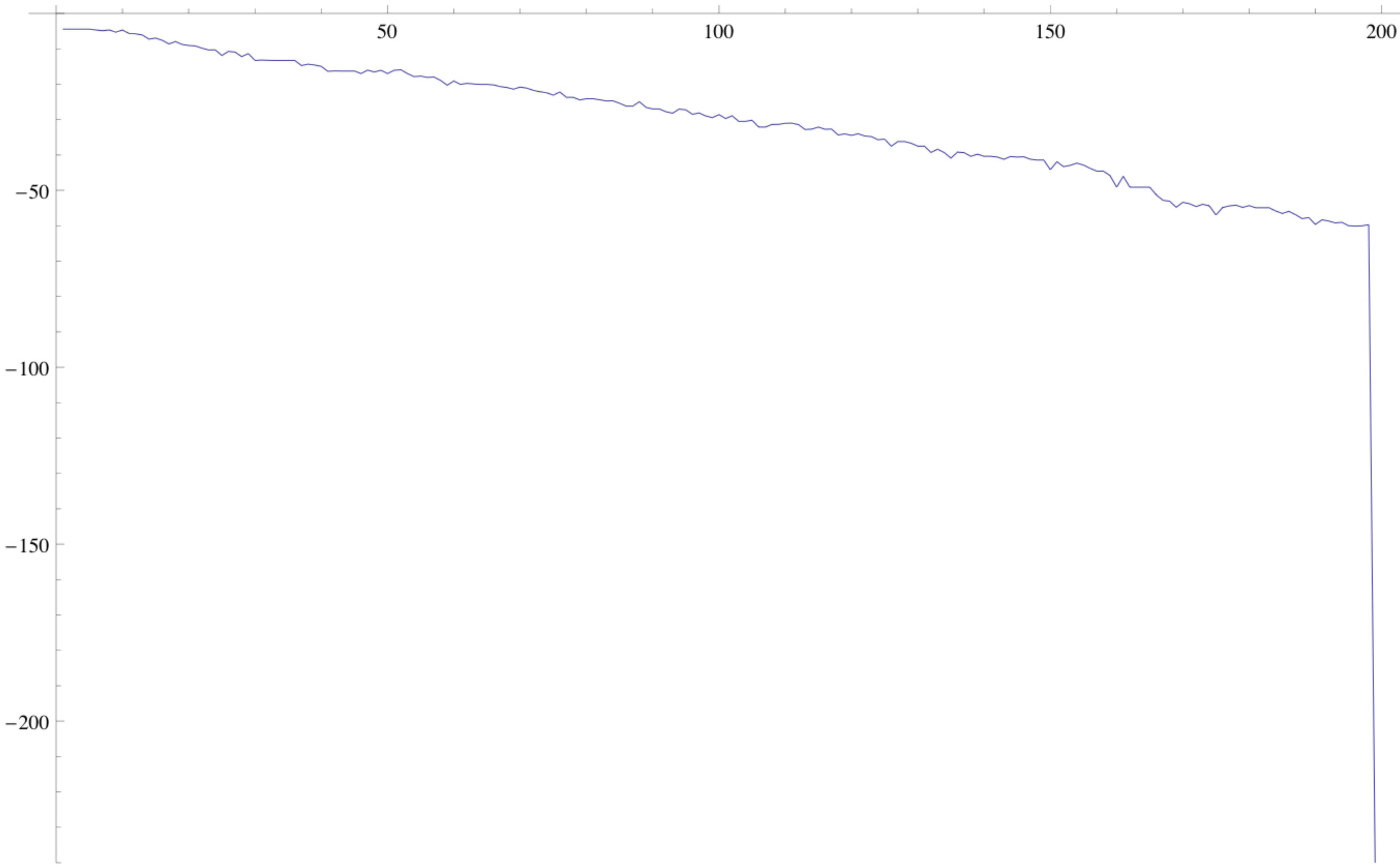
PSLQ, continued

- PSLQ constructs a sequence of integer-valued matrices B_n that reduces the vector $y = x * B_n$, until either the relation is found (as one of the columns of B_n), or else precision is exhausted.
- At the same time, PSLQ generates a steadily growing bound on the size of any possible relation.
- When a relation is found, the size of smallest entry of the *y* vector suddenly drops to roughly "epsilon" (i.e. 10^{-p}, where *p* is the number of digits of precision).
- The size of this drop can be viewed as a "confidence level" that the relation is real and not merely a numerical artifact -- a drop of 20+ orders of magnitude almost always indicates a real relation.

Several efficient variants of PSLQ are available:

- 2-level and 3-level PSLQ: performs almost all PSLQ iterations with only double precision, updating full-precision arrays as needed. Hundreds of times faster than the original full-precision PSLQ algorithm.
- Multi-pair PSLQ: dramatically reduces the number of iterations required. Designed for parallel system, but runs faster even on 1 CPU.

Decrease of $\log_{10}(\min |y_i|)$ in a multipair PSLQ run



Samples of minimal polynomials discovered by PSLQ computations

k	Minimal polynomial for exp $(8\pi\,\phi_2(1/k,1/k))$
5	$1 + 52\alpha - 26\alpha^2 - 12\alpha^3 + \alpha^4$
6	$1 - 28\alpha + 6\alpha^2 - 28\alpha^3 + \alpha^4$
7	$-1 - 196\alpha + 1302\alpha^2 - 14756\alpha^3 + 15673\alpha^4 + 42168\alpha^5 - 111916\alpha^6 + 82264\alpha^7 - 35231\alpha^8 + 19852\alpha^9 - 2954\alpha^{10} - 308\alpha^{11} + 7\alpha^{12}$
8	$1 - 88\alpha + 92\alpha^2 - 872\alpha^3 + 1990\alpha^4 - 872\alpha^5 + 92\alpha^6 - 88\alpha^7 + \alpha^8$
9	$-1 - 534\alpha + 10923\alpha^2 - 342864\alpha^3 + 2304684\alpha^4 - 7820712\alpha^5 + 13729068\alpha^6 - 22321584\alpha^7 + 39775986\alpha^8 - 44431044\alpha^9 + 19899882\alpha^{10} + 3546576\alpha^{11} - 8458020\alpha^{12} + 4009176\alpha^{13} - 273348\alpha^{14} + 121392\alpha^{15} - 11385\alpha^{16} - 342\alpha^{17} + 3\alpha^{18}$
10	$1 - 216\alpha + 860\alpha^2 - 744\alpha^3 + 454\alpha^4 - 744\alpha^5 + 860\alpha^6 - 216\alpha^7 + \alpha^8$

The minimal polynomial for exp $(8\pi\,\phi_2(1/32,1/32))$ has degree 128, with individual coefficients ranging from 1 to over 10^{56}. This PSLQ computation required 10,000-digit precision. See next slide.

Degree-128 minimal polynomial for $\exp(8\pi\,\phi_2(1/32,1/32))$

$$
\begin{aligned}
&-1 + 21888\alpha + 5893184\alpha^{2} + 15077928064\alpha^{3} - 3696628330464\alpha^{4} - 287791501240448\alpha^{5} - 30287462976198976\alpha^{6} \\
&+4426867843186404992\alpha^{7} - 554156920878198587888\alpha^{8} + 10731545733669133574528\alpha^{9} \\
&+120048731928709050250048\alpha^{10} + 4376999211577765512726656\alpha^{11} - 279045693458194222125366432\alpha^{12} \\
&+18747586287780118903854334848\alpha^{13} - 643310226865188446831485766208\alpha^{14} \\
&+12047117225922787728443496655488\alpha^{15} - 117230595100328033884939566091384\alpha^{16} \\
&+667772184328316952814362214365568\alpha^{17} - 4130661734713288144037409932696512\alpha^{18} \\
&+72313626239383964765274946226530432\alpha^{19} - 1891420571205861612091802761809141088\alpha^{20} \\
&+38770881730553471470590641060872686464\alpha^{21} - 577943965397394779947709633563006963008\alpha^{22} \\
&+6279796382074485140847650604801614559872\alpha^{23} - 50438907678331243798448849245156136801232\alpha^{24} \\
&+305806320133365055812520453224169520739712\alpha^{25} - 1441007171934715336769224848138270812591296\alpha^{26} \\
&+5554617356232728647085822946642640269497472\alpha^{27} - 20280024430170705107000630261773759070647328\alpha^{28} \\
&+99541720739995105011861264308551867164583808\alpha^{29} - 754081464712315412970559119390477134883548736\alpha^{30} \\
&+6271958646895434365874802435136411922022336128\alpha^{31} - 45931349314815625339442690290912948480194150172\alpha^{32} \\
&+280907040806572157908285324812126135484630889344\alpha^{33} - 1427273782916972532576299009596755423149111059136\alpha^{34} \\
&+6055180299673737231932804443230077408291723908736\alpha^{35} - 216099109391645533161019943019529887930132911355584\alpha^{36} \\
&+65433275736596914909292838375737685959952141180288\alpha^{37} - 169928170513492897108417040254326115991438719391296\alpha^{38} \\
&+385709310577705218843549196766620216295554031550592\alpha^{39} - 801233230832691550861608914233661767474963249815792\alpha^{40} \\
&+1706210557291030772074402183123327251333271061516160\alpha^{41} - 4421210594351357102505784181831242174063263551938496\alpha^{42} \\
&+14444199585866329915643888187597383540233619718619776\alpha^{43} - 50968478530199956388487913417905125665738409426112032\alpha^{44} \\
&+169891313454945514927724813351516976839425267825908096\alpha^{45} - 506612996672385619931633440499093959534203673546181440\alpha^{46} \\
&+1330573388204326565144545192834096788469932897185696896\alpha^{47} - 3069501638444045841407951432645059776135089489403138888\alpha^{48} \\
&+6226636397646752257692349351542872634032398917736673152\alpha^{49} - 11133383491631126059761752734485434504397040890449485504\alpha^{50} \\
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&+468944248086031450001465269696090117959962662732817675648\alpha^{61} - 622467103741378906100611838210632752408312516281305008960\alpha^{62} \\
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&+738516443137003178837650661261546833168555909499151978624\alpha^{65} - 622467103741378906100611838210632752408312516281305008960\alpha^{66} \\
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&+40292583920117898286863491450657424717015372825433076864\alpha^{73} - 35982430389670551550204799905599476866868765647852189248\alpha^{74} \\
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&+17601823309919260471943648355479182983209248554083752576\alpha^{77} - 11133383491631126059761752734485434504397040890449485504\alpha^{78} \\
&+6226636397646752257692349351542872634032398917736673152\alpha^{79} - 3069501638444045841407951432645059776135089489403138888\alpha^{80} \\
&+1330573388204326565144545192834096788469932897185696896\alpha^{81} - 506612996672385619931633440499093959534203673546181440\alpha^{82} \\
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&+6055180299673737231932804443230077408291723908736\alpha^{93} - 1427273782916972532576299009596755423149111059136\alpha^{94} \\
&+280907040806572157908285324812126135484630889344\alpha^{95} - 45931349314815625339442690290912948480194150172\alpha^{96} \\
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&+18747586287780118903854334848\alpha^{115} - 279045693458194222125366432\alpha^{116} \\
&+4376999211577765512726656\alpha^{117} + 120048731928709050250048\alpha^{118} + 10731545733669133574528\alpha^{119} \\
&-554156920878198587888\alpha^{120} + 4426867843186404992\alpha^{121} - 30287462976198976\alpha^{122} \\
&-287791501240448\alpha^{123} - 3696628330464\alpha^{124} + 15077928064\alpha^{125} + 5893184\alpha^{126} + 21888\alpha^{127} - \alpha^{128}
\end{aligned}
$$

Degrees of minimal polynomials found for exp(8 π ϕ_2(1/*d*,1/*d*))

Legend:

d	Argument
m(*d*)	Degree of minimal poly
z(*d*)	Zero coefficients
P	Precision in digits
T	Run time in seconds
$\log_{10} M$	Log10 of largest coeff.

Jason Kimberley noted that *m*(*d*) satisfies:

$$m\left(\prod_{i=1}^{k} p_i^{e_i}\right) \stackrel{?}{=} 4^{k-1} \prod_{i=1}^{k} p_i^{2(e_i-1)}\, m(p_i)$$

d	$m(d)$	$z(d)$	P	T	$\log_{10} M$	$\frac{\log_{10} M}{m(d)}$
5	4	0	400	0.40	1.4150	0.3537
6	4	0	400	0.39	0.7782	0.1945
7	12	0	400	0.71	5.0489	0.4207
8	8	0	400	0.43	3.2989	0.4124
9	18	0	400	1.81	7.6477	0.4249
10	8	0	400	0.54	2.6571	0.3321
11	30	0	1000	28.50	12.9873	0.4329
12	16	0	400	1.22	6.7880	0.4243
13	36	0	1000	44.04	15.6385	0.4344
14	24	0	1000	12.37	9.7245	0.4052
15	32	0	1000	34.58	12.8370	0.4012
16	32	0	1000	29.62	13.8452	0.4327
17	64	0	4000	3387.71	28.2396	0.4412
18	36	0	2000	274.60	13.8718	0.3853
19	90	0	6000	19559.37	39.8456	0.4427
20	32	0	2000	222.87	13.9705	0.4366
21	96	0	6000	25210.51	42.4696	0.4424
22	60	0	3000	1748.19	25.8002	0.4300
23	132	0	12000	212634.54	58.7280	0.4449
24	64	0	3000	2224.42	28.1624	0.4400
25	100	0	8000	58723.90	44.0690	0.4407
26	72	0	4000	4961.57	30.9611	0.4300
27	144		12000	**Failed**	**79.4540**	**0.5518**
28	96	0	8000	46795.52	42.5098	0.4428
29	144		12000	**Failed**	**79.5125**	**0.5522**
30	64	0	3000	2208.95	27.2294	0.4255
31	144		12000	**Failed**	**79.4119**	**0.5515**
32	128	0	10000	163662.83	56.8932	0.4445

For additional details

This talk is available at:

http://www.davidhbailey.com/dhbtalks/dhb-lattice.pdf

Full details are in given these two papers:

1. DHB, J. M. Borwein, R. E. Crandall and J. Zucker, “Lattice sums arising from the Poisson equation,” manuscript, available at **http://www.davidhbailey.com/dhbpapers/PoissonLattice.pdf**
2. DHB and J. M. Borwein, “Compressed lattice sums arising from the Poisson equation: Dedicated to Professor Hari Sirvastava,” manuscript, **http://www.davidhbailey.com/dhbpapers/Poissond.pdf**