# On the normality of Stoneham numbers, pi and log(2)

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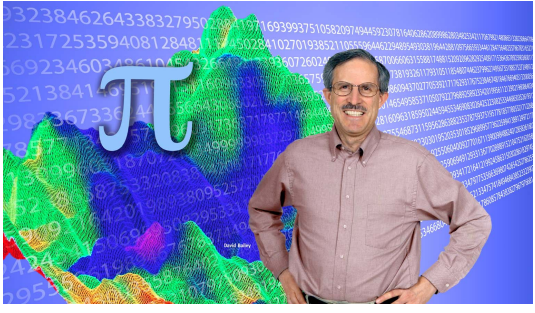
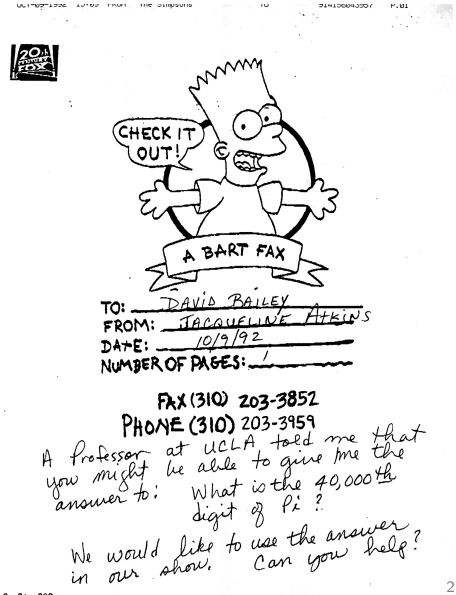


# DHB receives a fax from the Simpsons

On 9 October 1992, I received this fax from the staff of the Simpsons TV show, requesting the 40,000th digit of $\pi$. I computed the first 40,000 digits, and faxed the result back (noting that the 40,000th digit is a 1).

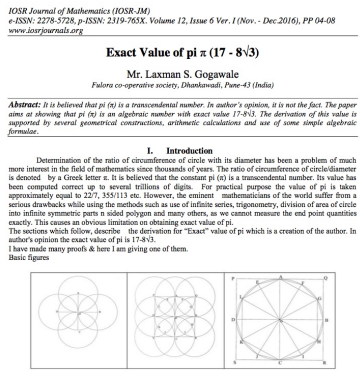
In the episode airing May 6, 1993, Apu, the manager of a convenience store, was challenged by Marge's attorney in a courtroom. He replied that he has a perfect memory. For example, he said, "I can recite 40,000 digits of $\pi$, and the last digit is a 1.



# "Exact" value of $\pi = 17 - 8\sqrt{3} = 3.1435935394\ldots$

In November 2016, the *IOSR Journal of Mathematics* published the following paper:

> **Abstract**: *It is believed that pi ($\pi$) is a transcendental number. In author's opinion, it is not the fact. The paper aims at showing that pi ($\pi$) is an algebraic number with exact value $17 - 8\sqrt{3}$. The derivation of this value is supported by several geometrical constructions, arithmetic calculations and use of some simple algebraic formulae.*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**Exact Value of pi π (17 - 8√3)**

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**Abstract:** *It is believed that pi (π) is a transcendental number. In author's opinion, it is not the fact. The paper aims at showing that pi (π) is an algebraic number with exact value 17-8√3. The derivation of this value is supported by several geometrical constructions, arithmetic calculations and use of some simple algebraic formulae.*

**I. Introduction**

Determination of the ratio of circumference of circle with its diameter has been a problem of much more interest in the field of mathematics since thousands of years. The ratio of circumference of circle/diameter is denoted by a Greek letter π. It is believed that the constant pi (π) is a transcendental number. Its value has been computed correct up to several trillions of digits. For practical purpose the value of pi is taken approximately equal to 22/7, 355/113 etc. However, the eminent mathematicians of the world suffer from a serious drawbacks while using the methods such as use of infinite series, trigonometry, division of area of circle into infinite symmetric parts n sided polygon and many others, as we cannot measure the end point quantities exactly. This causes an obvious limitation on obtaining exact value of pi.
The sections which follow, describe the derivation for "Exact" value of pi which is a creation of the author. In author's opinion the exact value of pi is 17-8√3.
I have made many proofs & here I am giving one of them.
Basic figures

Another author has been distributing manuscripts insisting that $\pi = (14 - \sqrt{2})/4 = 3.1464466094\ldots$


- Laxman S. Gogawale, "Exact value of pi $\pi(17 - 8\sqrt{3})$," *IOSR Journal of Mathematics*, vol. 12 (Nov.-Dec. 2016), `http://www.iosrjournals.org/iosr-jm/papers/Vol12-issue6/Version-1/B1206010408.pdf`.

# Normal numbers

Given integer $b \geq 2$, a real number $x$ is $b$-normal (or "normal base $b$") if every $m$-long string of digits appears in the base-$b$ expansion of $x$ with limiting frequency $1/b^m$.

Using measure theory, it can be shown that almost all real numbers are $b$-normal for a given integer base $b$ (in fact, for all $b$ simultaneously).

These are widely believed to be $b$-normal, for all integer bases $b \geq 2$:

- $\pi = 3.14159265358979323846\ldots$
- $e = 2.7182818284590452354\ldots$
- $\sqrt{2} = 1.4142135623730950488\ldots$
- $\log(2) = 0.69314718055994530942\ldots$
- *Every* irrational algebraic number (this conjecture is due to Borel).

But there are *no proofs of normality* for any of the above constants, for any base $b$, or even for $m = 1$, much less for all $m$ or all $b$.

Until recently, normality proofs were known only for a few constants, such as Champernowne's constant $= 0.12345678910111213141516\ldots$ (which is 10-normal).

# One (very weak) result for algebraic numbers

If $x$ is algebraic of degree $d > 1$, then its binary expansion through position $n$ must have at least $Cn^{1/d}$ 1-bits, for all sufficiently large $n$ and for some $C$ that depends on $x$.

Example: The first $n$ binary digits of $\sqrt{2}$ must have at least $\sqrt{n}$ one bits.

In this case, the result follows by noting that the one-bit count of the product of two integers is less than or equal to the product of the one-bit counts of the two integers. The more general result above requires a more sophisticated approach.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | | | | 1 | 0 | 1 | 1 | 1 |
| | | | | × | 1 | 1 | 0 | 1 | 1 |
| 1 | 0 | 0 | 1 | 1 | 0 | 1 | 1 | 0 | 1 |

However, note that these results are still a far cry from even single-digit normality.

- D. H. Bailey, J. M. Borwein, R. E. Crandall and C. Pomerance, "On the binary expansions of algebraic numbers," *Journal of Number Theory Bordeaux*, vol. 16 (2004), 487–518.

# The PSLQ integer relation algorithm

Let $(x_n)$ be a given vector of real numbers. An integer relation algorithm either finds integers $(a_n)$ such that

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n \quad = \quad 0$$

(to within the "epsilon" of the arithmetic being used), or else finds bounds within which no relation can exist.

The "PSLQ" algorithm of mathematician-sculptor Helaman Ferguson is the most widely used integer relation algorithm.

Integer relation detection requires very high precision (at least $n \times d$ digits, where $d$ is the size in digits of the largest $a_k$), both in the input data and in the operation of the algorithm.


1. H. R. P. Ferguson, D. H. Bailey and S. Arno, "Analysis of PSLQ, an integer relation finding algorithm," *Mathematics of Computation*, vol. 68, no. 225 (Jan 1999), 351–369.
2. D. H. Bailey and D. J. Broadhurst, "Parallel integer relation detection: Techniques and applications," *Mathematics of Computation*, vol. 70, no. 236 (Oct 2000), 1719–1736.

# Helaman Ferguson's "Syzygy" sculpture at Hamilton College, New York

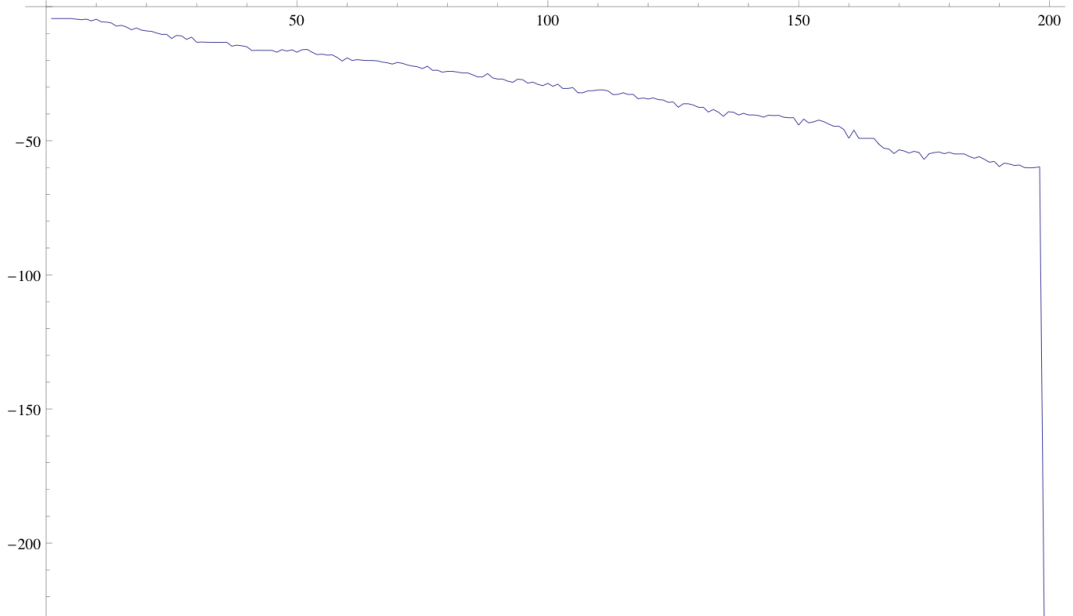
# PSLQ, continued

- PSLQ constructs a sequence of integer-valued matrices $B_n$ that reduce the vector $y = x \cdot B_n$, until either the relation is found (as one of the columns of matrix $B_n$), or else precision is exhausted.
- A relation is detected when the size of smallest entry of the $y$ vector suddenly drops to roughly "epsilon" (i.e. $10^{-p}$, where $p$ is the precision level in digits).
- The size of this drop can be viewed as a "confidence level" that the relation is not a numerical artifact: a drop of 20+ orders of magnitude almost always indicates a real relation.

Efficient variants of PSLQ:

- 2-level and 3-level PSLQ perform almost all iterations with only double precision, updating full-precision arrays as needed. They are hundreds of times faster than the original PSLQ.
- Multi-pair PSLQ dramatically reduces the number of iterations required. It was designed for parallel systems, but runs faster even on 1 CPU.

# Decrease of $\log 10(\min |y_i|)$ in multipair PSLQ run



# The binary algorithm for exponentiation

Problem: What is $3^{17}$ mod 10?

Algorithm A:
$3^{17} = 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times \times 3 = 129140163$,
so answer $= 3$.

Algorithm B (faster): $3^{17} = ((((3^2)^2)^2)^2) \times 3 = 129140163$, so answer $= 3$.

Algorithm C (fastest):
$3^{17} = ((((3^2 \bmod 10)^2 \bmod 10)^2 \bmod 10)^2 \bmod 10) \times 3 \bmod 10 = 3$.

Note that in Algorithm C, we never have to deal with integers larger than $9 \times 9 = 81$.
In general, if reducing mod $n$, we never have to deal with integers larger than $(n-1)^2$.

# How to compute binary digits of $\log 2$ beginning at an arbitrary position

1996 result: Consider this well-known formula for $\log 2$:

$$\log 2 = \sum_{n=1}^{\infty} \frac{1}{n2^n} = 0.10110001011100100001011111110111110100011100111101 1\ldots_2$$

Note that the binary digits of $\log 2$ beginning after position $d$ can be written as $\{2^d \log 2\}$, where $\{\cdot\}$ denotes fractional part. Thus we can write:

$$\begin{aligned}\{2^d \log 2\} &= \left\{ \sum_{n=1}^{d} \frac{2^{d-n}}{n} \right\} + \left\{ \sum_{n=d+1}^{\infty} \frac{2^{d-n}}{n} \right\} \\ &= \left\{ \sum_{n=1}^{d} \frac{2^{d-n} \bmod n}{n} \right\} + \left\{ \sum_{n=d+1}^{\infty} \frac{2^{d-n}}{n} \right\},\end{aligned}$$

where we have inserted $\bmod n$ since were are only interested in the fractional part when divided by $n$. Now note that the numerator $2^{d-n} \bmod n$ can be calculated very rapidly using the binary algorithm for exponentiation, using only standard arithmetic.

# General BBP-type formulas

For any real constant $\alpha$ that can be written in the form

$$\alpha = \sum_{n=0}^{\infty} \frac{p(n)}{b^n q(n)},$$

where $p$ and $q$ are integer polynomials, $\deg p < \deg q$, and $q$ has no zeroes for $n \geq 0$, the same technique can be employed to compute base-$b$ digits beginning at an arbitrary position.

What other well-known mathematical constants can be written by such a formula?

Can $\pi$ be written in this form?

# The BBP formula for $\pi$

In 1996, a PSLQ program discovered this new formula for $\pi$:

$$\pi = \sum_{n=0}^{\infty} \frac{1}{16^k} \left( \frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6} \right).$$

This formula permits one to compute binary (or hexadecimal) digits of $\pi$ beginning at an arbitrary starting position.

This is the first known instance of a computer program discovering a fundamentally new formula for $\pi$.

BBP-type formulas (also discovered using PSLQ) are now known for numerous other mathematical constants.


- D. H. Bailey, P. B. Borwein and S. Plouffe, "On the rapid computation of various polylogarithmic constants," *Mathematics of Computation*, vol. 66 (Apr 1997), 903–913.

# Some other BBP-type formulas found using PSLQ

$$\pi^2 = \frac{1}{8}\sum_{k=0}^{\infty}\frac{1}{64^k}\left(\frac{144}{(6k+1)^2} - \frac{216}{(6k+2)^2} - \frac{72}{(6k+3)^2} - \frac{54}{(6k+4)^2} + \frac{9}{(6k+5)^2}\right)$$

$$\pi^2 = \frac{2}{27}\sum_{k=0}^{\infty}\frac{1}{729^k}\left(\frac{243}{(12k+1)^2} - \frac{405}{(12k+2)^2} - \frac{81}{(12k+4)^2} - \frac{27}{(27k+5)^2}\right.$$
$$\left. - \frac{72}{(12k+6)^2} - \frac{9}{(12k+7)^2} - \frac{9}{(12k+8)^2} - \frac{5}{(12k+10)^2} + \frac{1}{(12k+11)^2}\right)$$

$$\zeta(3) = \frac{1}{1792}\sum_{k=0}^{\infty}\frac{1}{2^{12k}}\left(\frac{6144}{(24k+1)^3} - \frac{43008}{(24k+2)^3} + \frac{24576}{(24k+3)^3} + \frac{30720}{(24k+4)^3} - \frac{1536}{(24k+5)^3}\right.$$
$$+ \frac{3072}{(24k+6)^3} + \frac{768}{(24k+7)^3} - \frac{3072}{(24k+9)^3} - \frac{2688}{(24k+10)^3} - \frac{192}{(24k+11)^3} - \frac{1536}{(24k+12)^3}$$
$$- \frac{96}{(24k+13)^3} - \frac{672}{(24k+14)^3} - \frac{384}{(24k+15)^3} + \frac{24}{(24k+17)^3} + \frac{48}{(24k+18)^3} - \frac{12}{(24k+19)^3}$$
$$\left. + \frac{120}{(24k+20)^3} + \frac{48}{(24k+21)^3} - \frac{42}{(24k+22)^3} + \frac{3}{(24k+23)^3}\right)$$


- D. H. Bailey, J. M. Borwein, A. Mattingly and G. Wightwick, "The computation of previously inaccessible digits of Pi$^2$ and Catalan's constant," *Notices of the AMS*, vol. 60 (2013), 844–854.

# BBP formulas and normality

Consider a general BBP-type constant (i.e., a formula that permits the BBP "trick"):

$$\alpha \;=\; \sum_{n=0}^{\infty} \frac{p(n)}{b^n q(n)},$$

where $p$ and $q$ are integer polynomials, $\deg p < \deg q$, and $q$ has no zeroes for $n \geq 0$.

In 2001, Richard Crandall and I proved that $\alpha$ is $b$-normal iff the sequence $x_0 = 0$, and

$$x_n \;=\; \left\{ b x_{n-1} + \frac{p(n)}{q(n)} \right\},$$

where $\{\cdot\}$ again denote fractional part, is equidistributed in the unit interval. Here "equidistributed" means that the sequence visits each subinterval $[c, d)$ with limiting frequency $d - c$.

- D. H. Bailey and R. E. Crandall, "On the random character of fundamental constant expansions," *Experimental Mathematics*, vol. 10 (Jun 2001), 175–190.

# Two specific examples

Consider the sequence $x_0 = 0$ and

$$x_n = \left\{ 2x_{n-1} + \frac{1}{n} \right\}.$$

Then $\log 2$ is 2-normal iff this sequence (the "BBP sequence for $\log 2$") is equidistributed in the unit interval.

Similarly, consider the sequence $x_0 = 0$ and

$$x_n = \left\{ 16x_{n-1} + \frac{120n^2 - 89n + 16}{512n^4 - 1024n^3 + 712n^2 - 206n + 21} \right\}.$$

Then $\pi$ is 16-normal and hence 2-normal iff this sequence (the "BBP sequence for $\pi$") is equidistributed in the unit interval.

Sadly, we have not yet been able to prove equidistribution for either sequence.

# A class of provably normal constants

Crandall and I proved that an infinite class of constants are 2-normal, e.g.

$$\begin{aligned}\alpha_{2,3} &= \sum_{n=1}^{\infty} \frac{1}{3^n 2^{3^n}} \\ &= 0.041883680831502985071252898624571682426096\ldots_{10} \\ &= 0.0ab8e38f684bda12f684bf35ba781948b0fcd6e9e0\ldots_{16}\end{aligned}$$

This constant was proven 2-normal by Stoneham in 1971, but we have extended this to the case where $(2, 3)$ are any pair $(p, q)$ of relatively prime integers $\geq 2$. We also extended this result to an uncountable class: For any real $r$ in $[0, 1)$, the constant

$$\alpha_{2,3}(r) = \sum_{n=1}^{\infty} \frac{1}{3^n 2^{3^n + r_n}}$$

is 2-normal, where $r_n$ is the $n$-th bit in the binary expansion of $r$ in $[0, 1)$. These constants are all distinct, so the class is uncountable.


- D. H. Bailey and R. E. Crandall, "Random generators and normal numbers," *Experimental Mathematics*, vol. 11, no. 4 (2002), 527–546.

# A (weak) hot spot theorem

The previous result on the normality of Stoneham numbers can now be proven much more easily using this result from ergodic theory:

*The (weak) hot spot theorem*: Given the real constant $\alpha$, if there exists some $B$ such that for every subinterval $[c, d)$ of $[0, 1)$,

$$\limsup_{m \geq 1} \frac{\#_{0 \leq j \leq m} \left( \{b^j \alpha\} \in [c, d) \right)}{m(d - c)} \leq B$$

then $\alpha$ is $b$-normal.

In other words, if $\alpha$ is not $b$-normal, then:

- There is some interval $[c_1, d_1)$ that is visited 10 times too often by shifts of the base-$b$ expansion of alpha;
- The is some other interval $[c_2, d_2)$ that is visited 100 times too often;
- There is some other interval $[c_3, d_3)$ that is visited 1000 times too often; etc.

However, one cannot conclude that these intervals are necessarily nested.

- L. Kuipers and H. Niederreiter, *Uniform Distribution of Sequences*, Dover, 1974, 77.

# A strong hot spot theorem

In 2006 Michal Misiurewicz and I proved a stronger version of this result, using methods of ergodic theory:

*The strong hot spot theorem*: Let $0.x_1x_2\ldots x_n$ be the base-$b$ expansion of $x$ out to position $n$, so that $[c_n(x), d_n(x)) = [0.x_1x_2\ldots x_n,\ 0.x_1x_2\ldots(x_n+1))$ is the $n$-long digit interval containing $x$. If for every $x$ in $[0,1)$,

$$\liminf_{n\geq 1} \limsup_{m\geq 1} \frac{\#_{0\leq j<m}\left[\{b^j\alpha\} \in [c_n(x), d_n(x))\right]}{mb^{-n}} < \infty$$

then $\alpha$ is $b$-normal.

In other words: If $\alpha$ is not $b$-normal, then there is at least one "hot spot," namely some $x \in [0,1)$ such that shifts of the base-$b$ expansion of $\alpha$ visit all sufficiently small digit neighborhoods of $x$ too often, by an arbitrarily large factor.

Conversely, if one can establish that there is no such "hot spot," then $\alpha$ is $b$-normal.

- D. H. Bailey and M. Misiurewicz, "A strong hot spot theorem," *Proceedings of the American Mathematical Society*, vol. 134 (2006), 2495-2501.

# Simple examples of hot spots

Consider the rational fraction $1/28$:

$0.035714285714285714285714285714285714285714285714285714285714285\ldots$

Examine the sequence of shifts of its decimal expansion:

$0.357142857142857142857142857142857142857142857142857142857142857\ldots$

$0.571428571428571428571428571428571428571428571428571428571428571\ldots$

$0.714285714285714285714285714285714285714285714285714285714285714\ldots$

$0.142857142857142857142857142857142857142857142857142857142857142\ldots$

$0.428571428571428571428571428571428571428571428571428571428571428\ldots$

$0.285714285714285714285714285714285714285714285714285714285714285\ldots$

$0.857142857142857142857142857142857142857142857142857142857142857\ldots$

This sequence visits each of the six points $1/7, 2/7, 3/7, \cdots, 6/7$ one-sixth of the time, which, relative to the size of a sufficiently small neighborhood around each of these points, is too often by an arbitrarily large factor. Thus $1/28$ is not 10-normal.

In a similar way, the constant $1/10 + 1/10^4 + 1/10^9 + 1/10^{16} + 1/10^{25} + \cdots =$

$0.100100001000000100000000100000000001000000000000100000000000000010000000\ldots$

is irrational, but not 10-normal, since zero is clearly a hot spot for this constant.

# The BBP sequence associated with $\alpha_{2,3}$

It is fairly easy to show that the BBP sequence corresponding to

$$\alpha_{2,3} = \sum_{n=1}^{\infty} \frac{1}{3^n 2^{3^n}}$$

is the following. Note that each section is repeated three times, and the sequence evenly fills in the unit interval with all fractions of the form $k3^{-p}$:

$0, 0, 0,$

$\frac{1}{3}, \frac{2}{3}, \frac{1}{3}, \frac{2}{3}, \frac{1}{3}, \frac{2}{3},$

$\frac{4}{9}, \frac{8}{9}, \frac{7}{9}, \frac{5}{9}, \frac{1}{9}, \frac{2}{9},$ (repeated 3 times),

$\frac{13}{27}, \frac{26}{27}, \frac{25}{27}, \frac{23}{27}, \frac{19}{27}, \frac{11}{27}, \frac{22}{27}, \frac{17}{27}, \frac{7}{27}, \frac{14}{27}, \frac{1}{27}, \frac{2}{27}, \frac{4}{27}, \frac{8}{27}, \frac{16}{27}, \frac{5}{27}, \frac{10}{27}, \frac{20}{27},$

(repeated 3 times), etc.

# Proof that $\alpha_{2,3}$ is 2-normal using the hot spot theorem

Given any $x$ in $(0,1)$, let $c = 0.x_1x_2\ldots x_n$, and $d = 0.x_1x_2\ldots(x_n+1)$, so that $[c,d)$ is an $n$-digit interval containing $x$. Let $m$ be any integer $m > 2^{2n}$, and let $3^p$ be such that $3^p \le m < 3^{p+1}$.

Note that for $j > 2n$, we have $[c - 1/(2j), d + 1/(2j))$ is a subset of $[c2^{-n-1}, d + 2^{-n-1})$. Since the length of this interval is $2^{-n+1}$, it contains at most $3^p2^{-n+1} + 1$ instances of $k3^{-p}$ for $0 \le k < 3^p$.

Therefore

$$\frac{\#_{0\le j<m}\left(\{2^j\alpha\} \in [c,d)\right)}{m2^{-n}} \le \frac{2^n + \#_{2^n\le j<m}\left(z_j \in [c - 2^{-n-1}, d + 2^{-n-1})\right)}{m2^{-n}}$$
$$\le \frac{2^n + 3(3^p2^{-n+1} + 1)}{m2^{-n}} \; < \; 8$$

Thus by the hot spot theorem (either one), $\alpha_{2,3}$ is 2-normal.

# The constant $\alpha_{2,3}$ is *not* 6-normal

Consider the base-6 expansion of $\alpha_{2,3}$:


3130140430003334251130502130000001243555045432233011500243525320551352
3435410104300000000000000000514113005404055545530314425043343510124134 5
2351125142125134505503545015053522052044340452151505102411552500425130
0511244540010441311500324203032130000000000000000000000000000000000000
0000014212034311121452013525445342113412240220525301054204423552411055
4150155204350414555400310145303033532002534340401301240104453254343502
1420204324150255551010040433000455441145010313314511510144514123443342
3412400551313335045423530553151153501533452435450250055521453054234342
1530350125024205404135451231323245353031534552304115020154242121145201
5422225343403404505301233255344404431033324453321414150142334545424124
3203125340050134150245514404300000000000000000000000000000000000000000
0000000000000000000000000000000000000000000000000000000000000000000000
0000000003133505424444311110555341410520145402134123130014243331331 15


Note the long stretches of zeroes after power-of-three positions, e.g., after 3, 9, 27, 81, 243, 727, etc. This observation can be fashioned into a rigorous proof of nonnormality.

# Two more general nonnormality results on Stoneham constants

1. Given co-prime integers $b \geq 2$ and $c \geq 2$, and integers $p, q, r \geq 1$, with neither $b$ nor $c$ dividing $r$, let $B = b^p c^q r$, and assume $D = c^{q/p} r^{1/p} / b^{c-1} < 1$. Then the constant

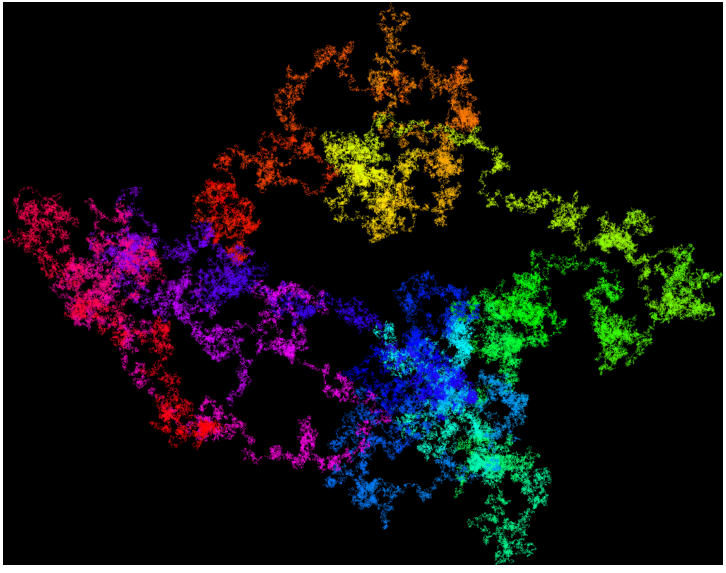
$$\alpha_{b,c} = \sum_{n=1}^{\infty} \frac{1}{c^n b^{c^n}}$$

is $B$-nonnormal. Thus, for example, $\alpha_{b,c}$ is $b$-normal but $bc$-nonnormal. It is not known whether or not this is a complete catalog of the bases for which a Stoneham constant is provably nonnormal.

2. Let $\alpha_{b_1,c_1}$ and $\alpha_{b_2,c_2}$ be two Stoneham constants satisfying the conditions above to be $B$-nonnormal. Assume further there are no integers $s$ and $t$ such that $c_1^s = c_2^t$. Then $\alpha_{b_1,c_1} + \alpha_{b_2,c_2}$ is $B$-nonnormal.

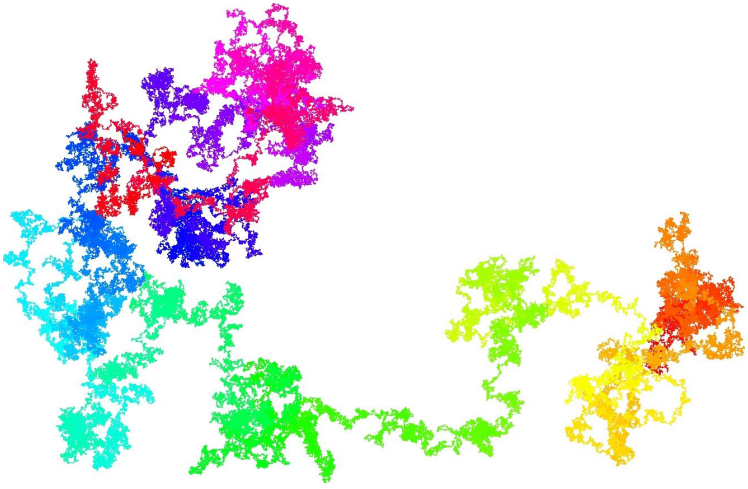

- D. H. Bailey and J. M. Borwein, "Nonnormality of Stoneham constants," *Ramanujan Journal*, vol. 29 (2012), 409–422.

# Walk on the first 100 billion base-4 digits of $\pi$

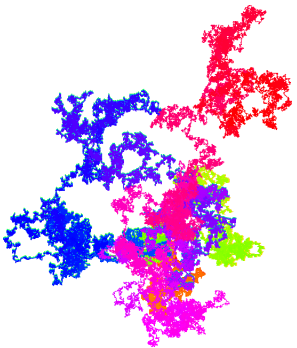


http://gigapan.com/gigapans/106803

# Walk on pseudorandom data



# Walk on $\alpha_{2,3}$: Base 2 (normal) vs base 6 (nonnormal)



# Thanks!

This talk is available at

http://www.davidhbailey.com/dhbtalks/dhb-vienna.pdf

References:


- D. H. Bailey and J. M. Borwein, "Pi day is upon us again, and we still do not know if pi is normal," *American Mathematical Monthly*, March 2014, 191–206.
- D. H. Bailey and M. Misiurewicz, "A strong hot spot theorem," *Proceedings of the American Mathematical Society*, vol. 134 (2006), 2495-2501.
- D. H. Bailey and J. M. Borwein, "Nonnormality of Stoneham constants," *Ramanujan Journal*, vol. 29 (2012), 409–422.
- F. J. Aragon Artacho, D. H. Bailey, J. M. Borwein and P. B. Borwein, "Walking on real numbers," *Mathematical Intelligencer*, vol. 35 (2013), 42–60.