New computations of Poisson polynomials

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Polynomials arising from the Poisson ϕ function

The Poisson potential function appears in numerous applied math contexts, ranging from mathematical physics to sharpening iPhone images. A simple 2-D instance is:

$$\phi_2(x, y) = \frac{1}{\pi^2} \sum_{m,n \text{ odd}} \frac{\cos(m\pi x)\cos(n\pi y)}{m^2 + n^2}$$

A 2013 study numerically discovered and then proved that when x and y are rational, then $\phi_2(x, y)$ satisfies

$$\phi_2(x, y) = \frac{1}{\pi} \log \beta(x, y)$$

where $\beta(x, y)$ is an algebraic number.

By computing high-precision numerical values of $\phi_2(x, y)$ for various specific rational x and y, and applying variants of the PSLQ program, we were able to produce the explicit minimal polynomials for $\beta(x, y)$ in several simple specific cases.

- D. H. Bailey, J. M. Borwein, R. E. Crandall and J. Zucker, "Lattice sums arising from the Poisson equation," *Journal of Physics A: Mathematical and Theoretical*, vol. 46 (2013), 115201.

Borwein's fast algorithm to compute $\phi_2(x, y)$

The original formula for $\phi_2(x, y)$ converges much too slowly for numerical evaluation. But this formula, found by Jonathan Borwein (deceased 2016), is remarkably efficient:

$$\phi_2(x, y) = \frac{1}{2\pi} \log \left| \frac{\theta_2(z, q)\theta_4(z, q)}{\theta_1(z, q)\theta_3(z, q)} \right|,$$

where $q = e^{-\pi}$ and $z = \frac{\pi}{2}(y + ix)$, and where the complex theta functions are computed using these rapidly convergent formulas:

$$\theta_1(z, q) = 2 \sum_{k=1}^{\infty} (-1)^{k-1} q^{(2k-1)^2/4} \sin((2k-1)z),$$

$$\theta_2(z, q) = 2 \sum_{k=1}^{\infty} q^{(2k-1)^2/4} \cos((2k-1)z),$$

$$\theta_3(z, q) = 1 + 2 \sum_{k=1}^{\infty} q^{k^2} \cos(2kz),$$

$$\theta_4(z, q) = 1 + 2 \sum_{k=1}^{\infty} (-1)^k q^{k^2} \cos(2kz).$$

Using PSLQ to find minimal polynomials

Let $X = (x_k)$ be an $(m+1)$-long real or complex vector. An integer relation algorithm such as PSLQ finds a nontrivial integer vector $A = (a_k)$ such that

$$a_0 x_0 + a_1 x_1 + \cdots + a_m x_m = 0.$$

- PSLQ typically requires floating-point arithmetic with hundreds or thousands of digits, or else the true relation will be lost in a sea of numerical artifacts.
- Fast variations of PSLQ utilize two or even three levels of precision, doing as much computation as possible with only double precision.

To find the minimal polynomial of a computed real number α, just apply PSLQ to the m-long vector $(1, \alpha, \alpha^2, ..., \alpha^{m-1})$.

If a numerically significant result is found, the relation integers are quite possibly the coefficients of a polynomial satisfied by α. By successively repeating with smaller m (or by using *Mathematica* or *Maple*), one can find the minimal polynomial.

- D. H. Bailey and D. J. Broadhurst, "Parallel integer relation detection: Techniques and applications," *Mathematics of Computation*, vol. 70, no. 236 (Oct 2000), 1719–1736.

2013: Some initial Poisson polynomial results

s	Minimal polynomial of $\alpha = \exp(8\pi\phi_2(x, y))$:
5	$1 + 52\alpha - 26\alpha^2 - 12\alpha^3 + \alpha^4$
6	$1 - 28\alpha + 6\alpha^2 - 28\alpha^3 + \alpha^4$
7	$-1 - 196\alpha + 1302\alpha^2 - 14756\alpha^3 + 15673\alpha^4 + 42168\alpha^5 - 111916\alpha^6 + 82264\alpha^7$ $-35231\alpha^8 + 19852\alpha^9 - 2954\alpha^{10} - 308\alpha^{11} + 7\alpha^{12}$
8	$1 - 88\alpha + 92\alpha^2 - 872\alpha^3 + 1990\alpha^4 - 872\alpha^5 + 92\alpha^6 - 88\alpha^7 + \alpha^8$
9	$-1 - 534\alpha + 10923\alpha^2 - 342864\alpha^3 + 2304684\alpha^4 - 7820712\alpha^5 + 13729068\alpha^6$ $-22321584\alpha^7 + 39775986\alpha^8 - 44431044\alpha^9 + 19899882\alpha^{10} + 3546576\alpha^{11}$ $-8458020\alpha^{12} + 4009176\alpha^{13} - 273348\alpha^{14} + 121392\alpha^{15}$ $-11385\alpha^{16} - 342\alpha^{17} + 3\alpha^{18}$
10	$1 - 216\alpha + 860\alpha^2 - 744\alpha^3 + 454\alpha^4 - 744\alpha^5 + 860\alpha^6 - 216\alpha^7 + \alpha^8$

Questions:

- Given s, what is the degree of the corresponding minimal polynomial?
- Note that when s is even, the polynomial is palindromic, i.e., coefficients $a_k = a_{m-k}$. Does this pattern hold for all even s?

These computations were quite expensive, requiring very high numeric precision.
Help! More powerful computational tools are required.

Kimberley's formula for the degree of the polynomial

Based on these preliminary results, Jason Kimberley of the University of Newcastle, Australia observed that the degree $m(s)$ of the minimal polynomial associated with the case $x = y = 1/s$ appears to be given by the following rule:

Set $m(2) = 1/2$. Otherwise for primes p congruent to 1 mod 4, set $m(p) = \mathrm{int}^2(p/2)$, where int denotes greatest integer, and for primes p congruent to 3 mod 4, set $m(p) = \mathrm{int}\,(p/2)(\mathrm{int}\,(p/2) + 1)$. Then for any other positive integer s whose prime factorization is $s = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}$,

$$m(s) = 4^{r-1} \prod_{i=1}^{r} p_i^{2(e_i-1)}\, m(p_i).$$

Questions:

- Does Kimberley's formula hold for larger s?
- Does the palindromic property hold for larger even s?

2016: Improvements to the Poisson polynomial program

1. A new thread-safe arbitrary precision floating-point package; it may optionally utilize the MPFR / GMP packages for even faster performance.
2. A new 3-level multipair PSLQ program:
   a Double precision: approx. 15 digit accuracy.
   b Medium multiprecision: typically 100 digits to 1,000 digits.
   c Full multiprecision: typically 2,500 digits to 25,000 digits.

Using this software, DHB found the minimal polynomials corresponding to $x = y = 1/s$ for all integers s up to 40, and also $s = 42, 44, 45, 46, 48, 50, 64$.

- D. H. Bailey, J. M. Borwein, J. Kimberley and W. Ladd, "Computer discovery and analysis of large Poisson polynomials," *Experimental Mathematics*, 27 Aug 2016, vol. 26, 349-363, `https://www.davidhbailey.com/dhbpapers/poisson-res.pdf`.

Degree-100 minimal polynomial found for the case $x = y = 1/25$

-1
$-33300\,\alpha^{1}$
$-8029750\,\alpha^{2}$
$+1542379500\,\alpha^{3}$
$-293348778825\,\alpha^{4}$
$-28080295402080\,\alpha^{5}$
$-6629781350330800\,\alpha^{6}$
$+58722280292148000\,\alpha^{7}$
$-7197243181862115100\,\alpha^{8}$
$+182328184490475806640\,\alpha^{9}$
$+1047464861734017883208\,\alpha^{10}$
$+1806185067754657491760\,\alpha^{11}$
$-279458813247177178256270 0\,\alpha^{12}$
$+4313272914482022638686000 0\,\alpha^{13}$
$-25641158089839664383048200 0\,\alpha^{14}$
$+1396700356872030954211266400\,\alpha^{15}$
$+9088870493426588389401095750\,\alpha^{16}$
$-73359906455331763936496182500 0\,\alpha^{17}$
$+137041495749203861918095715885 00\,\alpha^{18}$
$-1578192373620900935457694576413 20\,\alpha^{19}$
$+1321853578066306850140386747623398\,\alpha^{20}$
$-8347129396578224228862993897237280\,\alpha^{21}$
$+38212880291083166937296196441746800\,\alpha^{22}$
$-104022238521780343721711917072676000\,\alpha^{23}$
$-74032065945345957753107951351295900\,\alpha^{24}$
$+2695579175102470750839898367021130608\,\alpha^{25}$
$-17580580889577528003991569683501290200\,\alpha^{26}$
$+74952710236773360685253102982788566000\,\alpha^{27}$
$-239080432366992082679933177546734267500\,\alpha^{28}$
$+594175511524312220492120757330324484000\,\alpha^{29}$
$-1200481201725892359379030176076009068400\,\alpha^{30}$
$+2370791216795342465482695956290874196000\,\alpha^{31}$
$-6958365090143220739303965994873839612375\,\alpha^{32}$
$+29340961737653182271816694472006500014100\,\alpha^{33}$
$-118465202788489603844178657796002545834090\,\alpha^{34}$
$+394388553221080980775811796973709891661972\,\alpha^{35}$
$-1059773119301328191261276484738640092975935\,\alpha^{36}$
$+2274309666161593753722296548558864618267200\,\alpha^{37}$
$-3729748087865447793480560952748424761349600\,\alpha^{38}$
$+3877725636887659946158475367090512632719680\,\alpha^{39}$
$+846723867393661736729146827990176999234824\,\alpha^{40}$
$-15960428173973401824575411855716955373304480\,\alpha^{41}$
$+47118520726980412118527981001595834084077200\,\alpha^{42}$
$-96560391964393397280402148487438432608004000\,\alpha^{43}$
$+159442357501276027171661799158377030358819240\,\alpha^{44}$
$-222707090658310894211715578612740014870494016\,\alpha^{45}$
$+268336339651868593336977017479411555696276960\,\alpha^{46}$
$-280387518545606966350071856817231380200513600\,\alpha^{47}$
$+252610498127566038453865516310244367924136500\,\alpha^{48}$
$-192275255815619622137470846245628244410759600\,\alpha^{49}$
$+117222736648159046124836188054855685036349976\,\alpha^{50}$
$-47209171248166363599083489199392291307559600\,\alpha^{51}$
$-4942922820170238254884354265559936285783500\,\alpha^{52}$
$+36400498321158716446352704351511072445758400\,\alpha^{53}$
$-50714975271028177138621429774290438585502240\,\alpha^{54}$
$+52508039392118222466312798018961036691290304\,\alpha^{55}$
$-45426960236840080028112470330711413632425560\,\alpha^{56}$
$+33189561099972574095810629259221045183420000\,\alpha^{57}$
$-20184727758357355309386195853692979100127600\,\alpha^{58}$
$+10078674543222169818704320601337376637695840\,\alpha^{59}$
$-4216672803412456875435463721793522689401592\,\alpha^{60}$
$+1722566374653535529797841998864588575882560\,\alpha^{61}$
$-940122938034032272236892859372976755103200\,\alpha^{62}$
$+672500761678739947489218957049737685556800\,\alpha^{63}$
$-461584164001204318877039495371276336862015\,\alpha^{64}$
$+265582157437208924810452188771019969018068\,\alpha^{65}$
$-127220613278091048562581392274127224367210\,\alpha^{66}$
$+52420552698515377367407956646216340562900\,\alpha^{67}$
$-19432168360605247012293398573854487548375\,\alpha^{68}$
$+6715976041225158674019447203640258687520\,\alpha^{69}$
$-2160843064282037086551775761827735549808\,\alpha^{70}$
$+617282737944009515075102146542202738080\,\alpha^{71}$
$-145142540389934636970079472194894728300\,\alpha^{72}$
$+25729074826332880513711838275787958000\,\alpha^{73}$
$-3264527861366792560817221584368055000\,\alpha^{74}$
$+473260992168563618404463109451174000\,\alpha^{75}$
$-178858482702935977606858126791577500\,\alpha^{76}$
$+66028976498454634546379541782684000\,\alpha^{77}$
$-15761469487294858226633454090400400\,\alpha^{78}$
$+2480913858831352983037735837103840\,\alpha^{79}$
$-283939423573690705553213818042394\,\alpha^{80}$
$+279240162902683307815577528719 60\,\alpha^{81}$
$-18902649357383087571408191955 00\,\alpha^{82}$
$-1976460509542059894968712490 00\,\alpha^{83}$
$+92058025400867206020845473350\,\alpha^{84}$
$-15898833972285544673912261280\,\alpha^{85}$
$+1776153714893883182147476400\,\alpha^{86}$
$-134561774900933516621588000\,\alpha^{87}$
$+5713502521319481241590900\,\alpha^{88}$
$-280130362788084262599 20\,\alpha^{89}$
$-10740617849680182084536\,\alpha^{90}$
$+416466536564927187120\,\alpha^{91}$
$-5335454032157718300\,\alpha^{92}$
$-61275002107276000\,\alpha^{93}$
$-662049081315760\,\alpha^{94}$
$+123800803998624\,\alpha^{95}$
$-330633806665\,\alpha^{96}$
$-8699478100\,\alpha^{97}$
$+13986250\,\alpha^{98}$
$+6700\,\alpha^{99}$
$1\,\alpha^{100}$

2016: A proof of Kimberley formula for the case $x = 1/s, y = 1/s$

Analyses of the computed results ultimately led to a proof of Kimberley's formula, in the specific case $x = y = 1/s$, and also of the observed fact that when s is even, the polynomial is palindromic (i.e., coefficient $a_k = a_{m-k}$, where m is the degree).

What about the *much larger* set of cases $(x, y) = (p/s, q/s)$, for $1 \leq p \leq q < s/2$?

Is there a generalization of Kimberley's formula that holds in these other cases? Does the palindromic property hold for these other cases?

A familiar refrain: Help! More powerful computational tools are required.

- D. H. Bailey, J. M. Borwein, J. Kimberley and W. Ladd, "Computer discovery and analysis of large Poisson polynomials," *Experimental Mathematics*, 27 Aug 2016, vol. 26, 349–363, `https://www.davidhbailey.com/dhbpapers/poisson-res.pdf`.

2023: New computations of Poisson polynomials

New multiprecision software: A new high-level multiprecision package that employs FFT-based multiplication; approximately 3X faster than before.

New 3-level multipair PSLQ program:

a Double precision: approx. 15 digit accuracy.
c Medium multiprecision: typically 200 digits to 2,000 digits.
d Full multiprecision: typically 5,000 digits to 50,000 digits.

New computer runs: The new software has been used to compute the minimal polynomials for the entire set of cases $(p/s, q/s)$, where $1 \leq p \leq q < s/2$ and $10 \leq s \leq 36$, and also for $s = 38, 40, 42$ and $s = 50$ (a total of 2,206 cases).

These runs required over six CPU-months of computer time.

2023 results: A modified Kimberley rule for the $(p/s, q/s)$ cases

1. For the cases $x = y = p/s$, Kimberley's formula holds; further, for fixed s, *all these cases share the same minimal polynomial*).
2. For the cases $x = p/s$, $y = q/s$ with s odd, Kimberley's formula holds (except for a few where the correct degree is half Kimberley's rule).
3. For the cases $x = p/s$, $y = q/s$, with s even and both p and q odd, Kimberley's formula holds (except for a few where the correct degree is half Kimberley's rule).
4. For the cases $x = p/s$, $y = q/s$, with s even and one of p or q is even, the correct degree is twice Kimberley's formula (except for a few where the correct degree is equal to Kimberley's rule).

For full details:

- D. H. Bailey, "Large Poisson polynomials: Computation, results and analysis," manuscript, 8 Jul 2023, available at `https://www.davidhbailey.com/dhbpapers/poisson-2023.pdf`.

Sharing of minimal polynomials

One intriguing finding from the latest computations is that many cases for a given s share the same minimal polynomial, even though the numerical values are distinct.

For example, when $s = 17$, the (x, y) cases $(1/17, 1/17), (2/17, 2/17), (3/17, 3/17), (5/17, 5/17), (6/17, 6/17), (7/17, 7/17), (8/17, 8/17)$ all share the same degree-64 minimal polynomial:

$$+ 1 + 6912\alpha^{1} - 1023008\alpha^{2} + 535196800\alpha^{3} + 7742027760\alpha^{4} - 2451239864832\alpha^{5} + 140264665723552\alpha^{6} - 2494265652888704\alpha^{7} + 18453445522215032\alpha^{8}$$
$$+ 216142931589552 64\alpha^{9} - 18404699783816116 80\alpha^{10} + 265601705682887942 40\alpha^{11} - 219265475764921569840\alpha^{12} + 1143759465759937297408\alpha^{13} - 4563932639248948435424\alpha^{14}$$
$$+ 21048406812137688311168\alpha^{15} - 123756069205191278016740\alpha^{16} + 662708878348907477250816\alpha^{17} - 2671051287612630032421280\alpha^{18} + 7693234584556635821267584\alpha^{19}$$
$$- 14862548097474240887146768\alpha^{20} + 11985439092809681992002048\alpha^{21} + 44351668349396581870408736\alpha^{22} - 259625664937972467300807296\alpha^{23}$$
$$+ 803186115899676703948238664\alpha^{24} - 1789602095389051533149533952\alpha^{25} + 3055552833334608777606289376\alpha^{26} - 4156271487999506323835036544\alpha^{27}$$
$$+ 4903963676671959157531751248\alpha^{28} - 6019517253583536219231909888\alpha^{29} + 8780067564346216307831284640\alpha^{30} - 13334548483907481046238812288\alpha^{31}$$
$$+ 17362857489419448630866293318\alpha^{32} - 17345855629600599241800189696\alpha^{33} + 11966489230110362129440701856\alpha^{34} - 3898119322387426442055756416\alpha^{35}$$
$$- 2451983939727870545406928048\alpha^{36} + 4743446591055878746050587136\alpha^{37} - 3881818694457698660972764704\alpha^{38} + 2101492937309911776817793664\alpha^{39}$$
$$- 830074840813669608610951352\alpha^{40} + 269366792757186303037874944\alpha^{41} - 96596567511508184274883040\alpha^{42} + 46311532722057913438161792\alpha^{43}$$
$$- 22155672572673873192657168\alpha^{44} + 8153783303351403692882944\alpha^{45} - 2079969173966458011379616\alpha^{46} + 331427117746835861477504\alpha^{47} - 18856552838875733014756\alpha^{48}$$
$$- 7235322856083561662208\alpha^{49} + 3292609205079608858656\alpha^{50} - 738833647673944491136\alpha^{51} + 76552613117134517712\alpha^{52} - 1424154241008650752\alpha^{53} + 342676113911934816\alpha^{54}$$
$$- 89825284727190400\alpha^{55} + 3891480748650616\alpha^{56} - 154854254425344\alpha^{57} - 3704022727520\alpha^{58} + 404224147840\alpha^{59} - 125943824\alpha^{60} + 62013440\alpha^{61}$$
$$- 670240\alpha^{62} - 1408\alpha^{63} + 1\alpha^{64}$$

December 2024: Results for the Poisson ψ function

The 2013 study briefly mentioned the closely related function

$$\psi_2(x, y) = \frac{1}{\pi^2} \sum_{m,n \text{ even}} \frac{\cos(\pi m x) \cos(\pi n y)}{m^2 + n^2}.$$

As with $\phi_2(x, y)$, the authors found that when x and y are rational, then

$$\psi_2(x, y) = \frac{1}{\pi} \cdot \log \beta(x, y),$$

for algebraic $\beta(x, y)$.

A handful of results were given in the 2013 study, but progress has been stymied by an unfortunate error in the fast formulas for numerical evaluation.

December 2024: Corrected fast formulas for the Poisson ψ function

DHB has found these fast formulas to compute ψ:

$$\psi_2(x, y) = -\frac{1}{4\pi} \log \left| 2\,\mu(2z) \left(\sqrt{2}\lambda(2z) - 1 \right) \right|,$$

where $q = e^{-\pi}$, $z = \pi/2 \cdot (y + ix)$, $\mathrm{Im}(z)$ denotes imaginary part, and

$$\mu(z) = \exp\left(-2\,\mathrm{Im}^2(z)/\pi\right) \frac{\theta_3^2(z, q)}{\theta_3^2(0, q)}, \quad \lambda(z) = \frac{\theta_4^2(z, q)}{\theta_3^2(z, q)}.$$

The complex theta functions are computed using these rapidly convergent formulas:

$$\theta_3(z, q) = 1 + 2\sum_{k=1}^{\infty} q^{k^2} \cos(2kz),$$

$$\theta_4(z, q) = 1 + 2\sum_{k=1}^{\infty} (-1)^k q^{k^2} \cos(2kz).$$

December 2024: Initial results for the Poisson ψ polynomials

DHB has successfully computed the minimal polynomials for $\psi_2(x, y)$ for various rational x and y.

It appears that Kimberley's rule also holds for these polynomials, except that for even s, the polynomials have half the degree (in most cases) of the corresponding $\phi_2(x, y)$.

However, the computations and analysis for $\psi_2(x, y)$ are many times more challenging than with $\phi_2(x, y)$ — some runs have required up to 160,000-digit floating-point arithmetic.

Computations are currently running 24/7 on DHB's 12-processor Mac Studio. A comparable set of cases will require over two CPU-years run time.

A familiar refrain: Help! More powerful computational tools are required.

Degree-36 minimal polynomial found for the case $(x, y) = (1/13, 1/13)$

$$
\begin{array}{l}
+1\ \alpha \\
-102008900\ \alpha^{1} \\
+3386359201083610\ \alpha^{2} \\
-4576743060352245002 7036\ \alpha^{3} \\
+2358478714308768868232551148 47\ \alpha^{4} \\
-401808595154612767463343530401906914\ \alpha^{5} \\
+3226393199644244340609969690827653454 66492\ \alpha^{6} \\
-1289351965036787056558584361620156261860 93449926\ \alpha^{7} \\
+254366151720695039825209947252397855665352247599 40543\ \alpha^{8} \\
-18356357195617598181911901950101676557272436900891 60673300\ \alpha^{9} \\
+1295242928423847804911870979061052075344554600559285562 72450511\ \alpha^{10} \\
-5064396407665154813418840619774597239924756002045577036668 076918794\ \alpha^{11} \\
+1193514740202119424326796180993793510186386708001603921356167265630 72711\ \alpha^{12} \\
-22447082534964004771043754285403375104089700275261796230302168988181377 66158\ \alpha^{13} \\
-870786004906133783096226981075279808255547512718656267021486479206827455 70459492\ \alpha^{14} \\
-1406389080545060285674447683929282293450220435935950931746315677313494777509 752153540\ \alpha^{15} \\
-11385758025544894256777580187724170623548231894685569115823999541381295965034361 390310766\ \alpha^{16} \\
-443471139455584677707404662176778135920558558131518180839262426731741040412514105502 89108570\ \alpha^{17} \\
-8042918502598934232626067127854569308476504829683851787818042565917754813340313503762 8619391495\ \alpha^{18} \\
-776092482607699435134104584812076912438393426275378796708993445861063506621970494433321 37670614362\ \alpha^{19} \\
-98486972960286935058597427475158752725797778389800280769867546432109358430923241788144135 117573940304\ \alpha^{20} \\
-196916509108561280726348059526173479986764399472822864535007515131473701389796432459908585 32945372205916\ \alpha^{21} \\
+930991878738589274596983200570581437557191635209585348848648176548365191235087708730688914 4827987050001\ \alpha^{22} \\
-1066645601082853770577922620932336387880850553390339802716450308583105788248755953683857891 1597301076755566\ \alpha^{23} \\
+354177293967643061141762989295284373185179453110271554764233110846188885153863684384031589356 94752015480023\ \alpha^{24} \\
+1160699604655027845585716747716388874852842251268657789065785705020328510306878444592681728133 58665411435964\ \alpha^{25} \\
+881324833527136672038788993863877131688996093514776742873729696518474888307395906386235721770213 68148034789\ \alpha^{26} \\
+16670738003069103451688809770087989389245830144911912162421647687133703545025155594912685216701 709687122038\ \alpha^{27} \\
-7047103822160612268005331243634945797364875821873084144157723975480184702717917697991110329045401 09122\ \alpha^{28} \\
+190483525794080343148637055868161392775445889322696089097109677297582954919038070639324562 06459754\ \alpha^{29} \\
+685801375051471321819350009602526290988426139567571256059200121624340254444320223689387 76789\ \alpha^{30} \\
-242852122198134366320150437428979278784648291536502453876854273726532427941829453 16036\ \alpha^{31} \\
-150937181998961371940762728692330965504227666747719379967841633525298631117\ \alpha^{32} \\
-967710126182231085867777501101834999529007048488756535637943516\ \alpha^{33} \\
-287621740841771257846160456053044691979984738239 97\ \alpha^{34} \\
+346262896970171679004695509 86\ \alpha^{35} \\
+1\ \alpha^{36}
\end{array}
$$

Conclusions

- New software has been used to compute the minimal polynomials corresponding to $\phi_2(x, y)$, for a total of 2,206 cases. These runs required up to 50,000-digit floating-point and several CPU-months run time.
- A modified Kimberley's formula appears to hold in all cases, although no proof is yet known. For a given s, many cases share the same minimal polynomial.
- Formulas have been found to compute polynomials for $\psi_2(x, y)$.
- Computations are underway to find polynomials associated with $\psi_2(x, y)$, but these require much more computation and much higher numeric precision (up to 160,000 digits), compared with the $\phi_2(x, y)$ runs.
- A familiar refrain: Help! More powerful computational tools are required.

This talk is available at
`https://www.davidhbailey.com/dhbtalks/dhb-wcnt-2024.pdf`.

For full details, see:
`https://www.davidhbailey.com/dhbpapers/poisson-2023.pdf`.