# New results for Poisson polynomials

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# Polynomials arising from the Poisson $\phi$ function

The Poisson potential function appears in numerous applied math contexts, ranging from mathematical physics to sharpening iPhone images. A simple 2-D instance is:

$$\phi_2(x, y) = \frac{1}{\pi^2} \sum_{m,n \text{ odd}} \frac{\cos(m\pi x)\cos(n\pi y)}{m^2 + n^2}$$

A 2013 study numerically discovered and then proved that when $x$ and $y$ are rational, then $\phi_2(x, y)$ satisfies

$$\phi_2(x, y) = \frac{1}{\pi} \log \beta(x, y),$$

where $\beta(x, y)$ is an algebraic number.

By computing high-precision numerical values of $\phi_2(x, y)$ for various specific rational $x$ and $y$, and applying an integer relation algorithm, we were able to produce the explicit minimal polynomials for $\beta(x, y)$ in several simple specific cases.


- D. H. Bailey, J. M. Borwein, R. E. Crandall and J. Zucker, "Lattice sums arising from the Poisson equation," *Journal of Physics A: Mathematical and Theoretical*, vol. 46 (2013), 115201.

# A fast algorithm to compute $\phi_2(x, y)$

The original formula for $\phi_2(x, y)$ converges much too slowly for numerical evaluation. But this formula, found by Jonathan Borwein (deceased 2016), is remarkably efficient:

$$\phi_2(x, y) = \frac{1}{2\pi} \log \left| \frac{\theta_2(z, q)\theta_4(z, q)}{\theta_1(z, q)\theta_3(z, q)} \right|,$$

where $q = e^{-\pi}$ and $z = \frac{\pi}{2}(y + ix)$, and where the complex theta functions are computed using these rapidly convergent formulas:

$$\theta_1(z, q) = 2 \sum_{k=1}^{\infty} (-1)^{k-1} q^{(2k-1)^2/4} \sin((2k-1)z),$$

$$\theta_2(z, q) = 2 \sum_{k=1}^{\infty} q^{(2k-1)^2/4} \cos((2k-1)z),$$

$$\theta_3(z, q) = 1 + 2 \sum_{k=1}^{\infty} q^{k^2} \cos(2kz),$$

$$\theta_4(z, q) = 1 + 2 \sum_{k=1}^{\infty} (-1)^k q^{k^2} \cos(2kz).$$

# Using the multipair PSLQ algorithm to find minimal polynomials

Let $X = (x_k)$ be an $(m+1)$-long real or complex vector. An integer relation algorithm such as multipair PSLQ finds a nontrivial integer vector $A = (a_k)$ such that

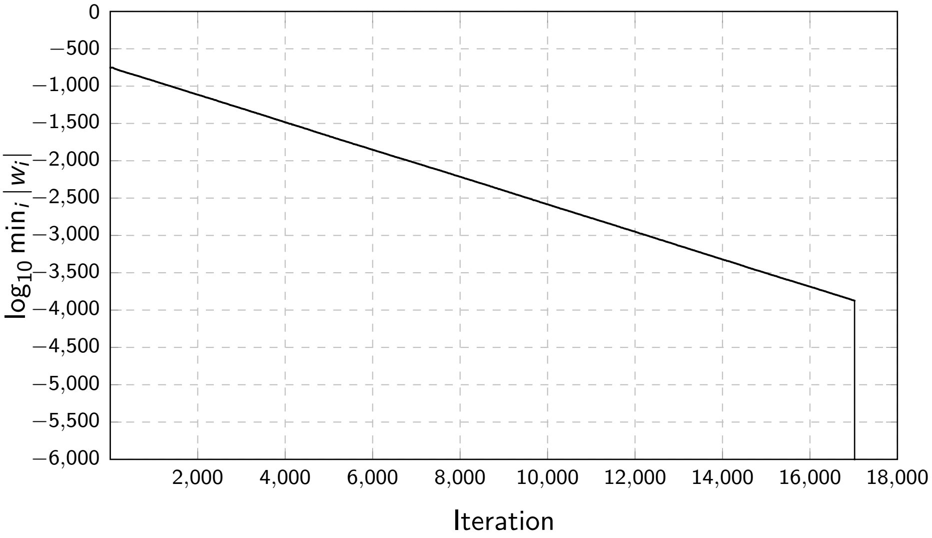
$$a_0 x_0 + a_1 x_1 + \cdots + a_m x_m = 0.$$

These computations typically require floating-point arithmetic with hundreds or thousands of digits, or else the true relation will be lost in a sea of numerical artifacts.

To find the minimal polynomial of degree $m$ of a computed real number $\alpha$, apply an integer relation algorithm to the $m+1$-long vector $(1, \alpha, \alpha^2, ..., \alpha^m)$.


- D. H. Bailey and D. J. Broadhurst, "Parallel integer relation detection: Techniques and applications," *Mathematics of Computation*, vol. 70, no. 236 (Oct 2000), 1719–1736.
- D. H. Bailey, "The two-level multipair PSLQ algorithm," manuscript, updated 17 Feb 2025, https://www.davidhbailey.com/dhbpapers/pslqm2-alg.pdf.

# Multipair PSLQ computer run for $\psi(1/13, 1/13)$



# 2013: Some initial Poisson polynomial results

| $s$ | Minimal polynomial of $\alpha = \exp(8\pi\phi_2(x,y))$: |
|---|---|
| 5 | $1 + 52\alpha - 26\alpha^2 - 12\alpha^3 + \alpha^4$ |
| 6 | $1 - 28\alpha + 6\alpha^2 - 28\alpha^3 + \alpha^4$ |
| 7 | $-1 - 196\alpha + 1302\alpha^2 - 14756\alpha^3 + 15673\alpha^4 + 42168\alpha^5 - 111916\alpha^6 + 82264\alpha^7$ $-35231\alpha^8 + 19852\alpha^9 - 2954\alpha^{10} - 308\alpha^{11} + 7\alpha^{12}$ |
| 8 | $1 - 88\alpha + 92\alpha^2 - 872\alpha^3 + 1990\alpha^4 - 872\alpha^5 + 92\alpha^6 - 88\alpha^7 + \alpha^8$ |
| 9 | $-1 - 534\alpha + 10923\alpha^2 - 342864\alpha^3 + 2304684\alpha^4 - 7820712\alpha^5 + 13729068\alpha^6$ $-22321584\alpha^7 + 39775986\alpha^8 - 44431044\alpha^9 + 19899882\alpha^{10} + 3546576\alpha^{11}$ $-8458020\alpha^{12} + 4009176\alpha^{13} - 273348\alpha^{14} + 121392\alpha^{15}$ $-11385\alpha^{16} - 342\alpha^{17} + 3\alpha^{18}$ |
| 10 | $1 - 216\alpha + 860\alpha^2 - 744\alpha^3 + 454\alpha^4 - 744\alpha^5 + 860\alpha^6 - 216\alpha^7 + \alpha^8$ |

Questions:

1. Given $s$, what is the degree of the corresponding minimal polynomial?
2. Note that when $s$ is even, the polynomial is palindromic, i.e., coefficients $a_k = a_{m-k}$. Does this pattern hold for all even $s$?

# Kimberley's formula for the degree of the polynomial

Based on these preliminary results, Jason Kimberley of the University of Newcastle, Australia observed that the degree $m(s)$ of the minimal polynomial associated with the case $x = y = 1/s$ appears to be given by the following rule:

Set $m(2) = 1/2$. Otherwise for primes $p$ congruent to 1 mod 4, set $m(p) = \mathrm{int}^2(p/2)$, where int denotes greatest integer, and for primes $p$ congruent to 3 mod 4, set $m(p) = \mathrm{int}\,(p/2)(\mathrm{int}\,(p/2) + 1)$. Then for any other positive integer $s$ whose prime factorization is $s = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}$,

$$m(s) = 4^{r-1} \prod_{i=1}^{r} p_i^{2(e_i - 1)}\, m(p_i).$$

This was proven in 2016, as part of an extensive study using large-scale computations.


- D. H. Bailey, J. M. Borwein, J. Kimberley and W. Ladd, "Computer discovery and analysis of large Poisson polynomials," *Experimental Mathematics*, vol. 26 (27 Aug 2016), 349–363, https://www.davidhbailey.com/dhbpapers/poisson-res.pdf.

# 192-degree minimal polynomial found by multipair PSLQ for $x = y = 1/35$

This distinctive semi-elliptical pattern is strong evidence that the result is correct.

# 2025: Results for the Poisson $\psi$ function

The 2013 study briefly mentioned the closely related function

$$\psi_2(x, y) = \frac{1}{\pi^2} \sum_{m,n \text{ even}} \frac{\cos(\pi m x)\cos(\pi n y)}{m^2 + n^2}.$$

Note this has even indices instead of odd. As with $\phi_2(x, y)$, the authors found that when $x$ and $y$ are rational, then

$$\psi_2(x, y) = \frac{1}{\pi} \cdot \log \beta(x, y),$$

for algebraic $\beta(x, y)$.

A handful of results were given in the 2013 study, but progress has been stymied by an unfortunate error in the fast formulas given in the paper for numerical evaluation.

Sadly, Borwein and Crandall have both passed away, and Zucker is not available.


- D. H. Bailey, J. M. Borwein, R. E. Crandall and J. Zucker, "Lattice sums arising from the Poisson equation," *Journal of Physics A: Mathematical and Theoretical*, vol. 46 (2013), 115201.

# Corrected fast formulas for the Poisson $\psi$ function

DHB has been able to correct these fast formulas, which are as follows:

$$\psi_2(x,y) = -\frac{1}{4\pi}\log\left|2\,\mu(2z)\left(\sqrt{2}\lambda(2z)-1\right)\right|,$$

where $q = e^{-\pi}$, $z = \pi/2 \cdot (y + ix)$, $\mathrm{I}m(z)$ denotes imaginary part, and

$$\mu(z) = \exp\left(-2\,\mathrm{I}m^2(z)/\pi\right)\,\frac{\theta_3^2(z,q)}{\theta_3^2\,(0,q)}, \quad \lambda(z) = \frac{\theta_4^2(z,q)}{\theta_3^2(z,q)}.$$

The complex theta functions are computed using these rapidly convergent formulas:

$$\theta_3(z,q) = 1 + 2\sum_{k=1}^{\infty} q^{k^2}\cos(2kz),$$

$$\theta_4(z,q) = 1 + 2\sum_{k=1}^{\infty}(-1)^k q^{k^2}\cos(2kz)$$

# Degree-36 minimal polynomial found for the case $(x, y) = (1/13, 1/13)$

$+1\ \alpha$

$-102008900\ \alpha^{1}$

$+3386359201083610\ \alpha^{2}$

$-45767430603522450027036\ \alpha^{3}$

$+235847871430876886823255114847\ \alpha^{4}$

$-401808595154612767463343530401906914\ \alpha^{5}$

$+322639319964424434060996969082765345466492\ \alpha^{6}$

$-128935196503678705655858436162015626186093449926\ \alpha^{7}$

$+25436615172069503982520994725239785566535224759940543\ \alpha^{8}$

$-1835635719561759818191190195010167655727243690089160673300\ \alpha^{9}$

$+129524292842384780491187097906105207534455460055928556272450511\ \alpha^{10}$

$-5064396407665154813418840619774597239924756002045577036668076918794\ \alpha^{11}$

$+119351474020211942432679618099379351018638670800160392135616726563072711\ \alpha^{12}$

$-2244708253496400477104375428540337510408970027526179623030216898818137766158\ \alpha^{13}$

$-87078600490613378309622698107527980825554751271865626702148647920682745570459492\ \alpha^{14}$

$-1406389080545060285674447683929282293450220435935950931746315677313494777509752153540\ \alpha^{15}$

$-11385758025544894256777580187724170623548231894685569115823999541381295965034361390310766\ \alpha^{16}$

$-44347113945558467770740466217677813592055855813151818083926242673174104041251410550289108570\ \alpha^{17}$

$-80429185025989342326260671278545693084765048296838517878180425659177548133403135037628619391495\ \alpha^{18}$

$-77609248260769943513410458481207691243839342627537879670899344586106350662197049443332137670614362\ \alpha^{19}$

$-98486972960286935058597427475158752725797778389800280769867546432109358430923241788144135117573940304\ \alpha^{20}$

$-19691650910856128072634805952617347998676439947282286453500751513147370138979643245990858532945372205916\ \alpha^{21}$

$+9309918787385892745969832005705814375571916352095853488486481765483651912350877087306889144827987050001\ \alpha^{22}$

$-10666456010828537705779226209323363878808505533903398027164503085831057882487559536838578911597301076755566\ \alpha^{23}$

$+35417729396764306114176298929528437318517945311027155476423311084618888515386368438403158935694752015480023\ \alpha^{24}$

$+116069960465502784558571674771638887485284225126865778906578570502032851030687844459268172813358665411435964\ \alpha^{25}$

$+88132483352713667203878899386387713168899609351477674287372969651847488830739590638623572177021368148034789\ \alpha^{26}$

$+16670738003069103451688809770087989389245830144911912162421647687133703545025155594912685216701709687122038\ \alpha^{27}$

$-704710382216061226800533124363494579736487582187308414415772397548018470271791769799111032904540109122\ \alpha^{28}$

$+19048352579408034314863705586816139277544588932269608909710967729758295491903807063932456206459754\ \alpha^{29}$

$+68580137505147132181935000960252629098842613956757125605920012162434025444432022368938776789\ \alpha^{30}$

$-24285212219813436632015043742897927878464829153650245387685427372653242794182945316036\ \alpha^{31}$

$-150937181998961371940762728692330965504227666747719379967841633525298631117\ \alpha^{32}$

$-967710126182231085867777501101834999529007048488756535637943516\ \alpha^{33}$

$-28762174084177125784616045605304469197998473823997\ \alpha^{34}$

$+34626289697017167900469550986\ \alpha^{35}$

$+1\ \alpha^{36}$

# Degree-32 minimal polynomial found for the case $(x, y) = (1/24, 9/24)$

$-1\ \alpha^{0}$
$+1517633683648\ \alpha^{1}$
$-107032107419961586401460\!8\ \alpha^{2}$
$+16077000969458841149573721377618496\ \alpha^{3}$
$+5325396181883917692148004640782797377183578\!4\ \alpha^{4}$
$+10215415290485951244503558593130092884026825470166547\!2\ \alpha^{5}$
$-355134723946655103535462538871183148710150223902144506789662\!4\ \alpha^{6}$
$-55486391024657149004739748156390810174399401645206681986699499554041\!6\ \alpha^{7}$
$-456068625135300880404526075979695045564225540721629846219568094179542616399 64\ \alpha^{8}$
$+670291922167927614981502689703928928861430088377593305299059116201930373017645536 64\ \alpha^{9}$
$-595905195905694603588462192244385992695205271977913624546982493714892215344375837627560347 2\ \alpha^{10}$
$+1000487402782674880420482568615427003730557764787921720393455297021518218955992297188345142666719 36\ \alpha^{11}$
$-350177363341451941455035658205833596702341009743576501156651113523159226127638586494609376001647233636680\ \alpha^{12}$
$+881472974554592249157201775536177591524823556687278499896398088470381359988152667316050413466631613918078 8672\ \alpha^{13}$
$-1185493123666096196285641499807923131478527019455713554160168788593111092151780212098267643509860456306087188812 00\ \alpha^{14}$
$+1241897692448032242972131719205820982010409163562057582439093003775394934215762410813495185682821857533300214576274240\ \alpha^{15}$
$-1029941865667500303971310595875091680593664425485141780751013565612991722366369026226848897464819005582502252367566607008 6\ \alpha^{16}$
$+557138661401273802639442495321129962275843542497283947791504515464982377779752850427304417976509822775406063132372235007514 24\ \alpha^{17}$
$-41749798666196936951020785485533080031397760552062854934680280903157799895268543941088178814856749264350724193106856105512339888\ \alpha^{18}$
$+96328239089542226712876103551214233785313879278149789598086212861368858783309028210482994157483365531275312895340975849578581070 72\ \alpha^{19}$
$-1155128975791361227645053309061492624429774610937607526975060712156145268497148909729424342569982027851065296509098570077183016788 0\ \alpha^{20}$
$+7581045804115506622055256875598318466133829932267266821931435536201356096214451681981537286482953410270345939351609640051388325440\ \alpha^{21}$
$-1371182737534936834135642993413857462581009354340456641608048876458084499668684556148396166859680121443852981863259063159109019 2\ \alpha^{22}$
$+8943733691315070401772198348673718494825466221863028554303848239310682252790357434697240184663146429806351230614952990614208\ \alpha^{23}$
$-2311684058492760569520846731931893937827152117652399957594431059685705981648312864577721524332531268037685118615839792 92\ \alpha^{24}$
$+72984970347138940337982173846979095932326083435734890396870677687284342480934261962334280838573180256033912270728 32\ \alpha^{25}$
$-3134818145346706676963458117442358136852374304472013186424196754907298248388096600081390851934589757592350 4\ \alpha^{26}$
$-125983281723249988735636152804856965060441480145862833730383442926970841992531803334073340933801024\ \alpha^{27}$
$-108397961801245459860567796763706983264832823455262849002828973202977187842059921304757880\ \alpha^{28}$
$+3199945251234371423259192360483291823016052583628940267793372318292199774395 52\ \alpha^{29}$
$-697418281407673477776633103742501391603178855541498424588651024144\ \alpha^{30}$
$+1555159273351465815964524939895744\ \alpha^{31}$
$-1\ \alpha^{32}$

# Results for the Poisson $\psi$ polynomials

DHB has successfully computed the minimal polynomials for $\psi_2(x, y)$ for the cases $(x, y) = (p/s, q/s)$, where $1 \leq p \leq q < s/2$ and $\gcd(p, q, s) = 1$, for $10 \leq s \leq 25$ and also for $s = 26, 28, 30, 32, 34, 36, 40$, a total of 1017 cases.

These computations were roughly 100X more expensive than with $\phi_2(x, y)$ for comparable cases. Some runs required up to 160,000-digit floating-point arithmetic.

Key findings:

1. The degrees satisfy a generalized Kimberley's rule (see next page).
2. For a given $s$, all the cases $(x, y) = (p/s, p/s)$, where $1 \leq p < s/2$ and $\gcd(p, s) = 1$, share the same minimal polynomial, even though the numerical values are distinct.

   For example, for $s = 36$, the minimal polynomials for the cases $(1, 1), (5, 5), (7, 7), (11, 11), (13, 13), (17, 17)$ are all identical. Note that this represents a complete set of $(p/36, p/36)$ with $1 \leq p < 18$ and $\gcd(p, 36) = 1$.

## A generalized Kimberley rule for the degrees of the $\psi$ polynomials

For the case $(x, y) = (p/s, q/s)$, where $1 \leq p < q < s/2$ and $\gcd(p, q, s) = 1$,

1. If $s$ is even or odd, and $p = q$, then the degree is given by Kimberley's rule.
2. Otherwise if $s$ is odd, then the degree is given by Kimberley's rule, except for a few cases where the degree is half Kimberley's rule.
3. If $s$ is even, and both $p$ and $q$ are odd, then the degree is given by Kimberley's rule, except for a few cases where the degree is half Kimberley's rule.
4. If $s$ is even, with one of $p$ or $q$ even and the other odd, then the degree is given by *twice* Kimberley's rule, except for a few cases where the degree is equal to Kimberley's rule.

# Conclusions

1. Corrected formulas have been found that enable rapid high-precision numerical values for $\psi_2(x, y)$, along the lines of earlier formulas for $\phi_2(x, y)$.
2. Polynomials were computed for a total of 1017 cases.
3. These computations were roughly 100X more expensive, and required much higher numeric precision (up to 160,000 digits), compared with similar $\phi_2(x, y)$ runs.
4. A generalized Kimberley's formula appears to hold for the degrees in all cases, although no proof is yet known.
5. For a given $s$, many cases share the same minimal polynomial even though the numerical values are distinct.

This talk is available here:
`https://www.davidhbailey.com/dhbtalks/dhb-wcnt-2025a.pdf`.

The technical paper is available here:
`https://www.davidhbailey.com/dhbpapers/poisson-psi.pdf`.