New results for Euler sums

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Euler sums

Euler sums are infinite series that involve the classical harmonic function

$$H(k) = 1 + 1/2 + 1/3 + \cdots + 1/k$$

These sums arise in mathematical physics, in the study of the Riemann hypothesis and in numerous other contexts.

Earlier studies have found that many of these sums have closed-form evaluations in terms of π and the Riemann zeta function, for example:

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^3} = \frac{5}{4}\zeta(4) = \frac{1}{72}\pi^4$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{(k+1)^5} = \frac{1}{4}\left(3\zeta(6) - 2\zeta(3)^2\right)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^3}{k^4} = \frac{693}{48}\zeta(7) + 2\zeta(5)\zeta(2) - \frac{51}{4}\zeta(4)\zeta(3)$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{(k+1)^7} = \frac{1}{4}\left(5\zeta(8) - 4\zeta(3)\zeta(5)\right)$$

Mixed Euler sums

In this study, we consider a general class of "mixed Euler sums":

$$M(m, n_0, n_1, n_2, \ldots, n_t) = \sum_{k=1}^{\infty} \frac{H(k)^m}{k^{n_0}(k+1)^{n_1}(k+2)^{n_2}\cdots(k+t)^{n_t}},$$

where $m + n_0 + \cdots + n_t$ is the *order*.

We show that these constants, up to at least order 12, have closed-form evaluations, and we have found analytic and numerical techniques to obtain these formulas. We have also found some very recent results (December 2025) on another general class.

The paper below has details of these results and other related results, too many to mention here, including, for example, results on Stieltjes constants that have connections to mathematical physics.

- R.C. McPhedran and D.H. Bailey, "New results for Euler sums," ArXiv:2311.06294, 19 Jul 2025, https://arxiv.org/abs/2311.06294 or https://www.davidhbailey.com/dhbpapers/NewEulerSums.pdf.

Theorem on mixed Euler sums

Theorem 1: If the order of a mixed Euler sum is 12 or less, then it is expressible as a rational linear sum of terms chosen from the following list of "atomic" constants:

Constants for order 3: 1, $\zeta(2)$, $\zeta(3)$
Additional constant for order 4: $\zeta(4)$
Additional constants for order 5: $\zeta(5)$, $\zeta(2)\zeta(3)$
Additional constants for order 6: $\zeta(6)$, $\zeta(3)^2$
Additional constants for order 7: $\zeta(7)$, $\zeta(2)\zeta(5)$, $\zeta(3)\zeta(4)$
Additional constants for order 8: $\zeta(8)$, $\zeta(2)\zeta(3)^2$, $\zeta(3)\zeta(5)$, $M(2,6)$
Additional constants for order 9: $\zeta(9)$, $\zeta(2)\zeta(7)$, $\zeta(3)\zeta(6)$, $\zeta(4)\zeta(5)$, $\zeta(3)^3$
Additional constants for order 10: $\zeta(10)$, $\zeta(3)\zeta(7)$, $\zeta(3)^2\zeta(4)$, $\zeta(2)\zeta(3)\zeta(5)$, $\zeta(5)^2$, $\zeta(2)$, $M(2,6)$, $M(2,8)$
Additional constants for order 11: $\zeta(11)$, $\zeta(2)\zeta(9)$, $\zeta(3)\zeta(8)$, $\zeta(4)\zeta(7)$, $\zeta(5)\zeta(6)$, $\zeta(2)\zeta(3)^3$, $\zeta(5)\zeta(3)^2$, $\zeta(3)M(2,6)$, $M(3,8)$
Additional constants for order 12: $\zeta(12)$, $\zeta(3)\zeta(9)$, $\zeta(5)\zeta(7)$, $\zeta(2)\zeta(5)^2$, $\zeta(2)\zeta(3)\zeta(7)$, $\zeta(3)\zeta(4)\zeta(5)$, $\zeta(3)^2\zeta(6)$, $\zeta(3)^4$, $\zeta(4)M(2,6)$, $\zeta(2)M(2,8)$, $M(2,10)$, $M(4,8)$

Note that the above list includes $M(2,6)$, $M(2,8)$, $M(3,8)$, $M(2,10)$, $M(4,8)$. We have not been able to reduce these to any other known mathematical constants.

Sketch of proof of Theorem 1

We first observe that each of the basic Euler sums $M(m,n) = \sum_{1 \le k \le \infty} H(k)^m / k^n$ with order $m + n \le 12$ is reducible to a rational linear sum of the atomic constants. We then argue that any general mixed Euler sum can be reduced to a rational linear combination of $M(m,n)$ of the same order or less by:

1. Changing sums with expressions involving $(k+1)$, $(k+2)$ or $(k+a)$ to sums involving only k, by means of a process akin to completing the square.
2. Applying a partial fraction decomposition: Recall that any rational function can be written uniquely as the sum of terms based on the factorization of the denominator polynomial, as in the example

$$\frac{1}{(k+1)(k+2)^2} = \frac{1}{k+1} - \frac{1}{k+2} - \frac{1}{(k+2)^2}.$$

In practice, these algebraic manipulations are typically rather tedious. But a numerical approach works well for a wide range of cases — see next three pages.

Computing high-precision values of Euler sums

The Euler-Maclaurin formula approximates a sum as an integral with corrections:

$$\sum_{j=a}^{b} f(j) = \int_a^b f(t)\,\mathrm{d}t + \frac{1}{2}(f(a) + f(b)) + \sum_{j=1}^{s} \frac{B_{2j}\left(D^{2j-1}f(b) - D^{2j-1}f(a)\right)}{(2j)!} + R_s(a,b),$$

where B_k is the k-th Bernoulli number, $D^k f(a)$ is the k-th derivative of $f(t)$ evaluated at $t = a$, and $R_s(a,b)$ is a bounded error term.

Applying this to the harmonic function $H(t) = \sum_{j=1}^{t} 1/j$ yields the approximation

$$\hat{H}(t) \approx \gamma + \log(t) + \frac{1}{2t} + \sum_{j=1}^{s} \frac{B_{2j}}{2jt^{2j}},$$

where γ is Euler's constant. In our computations, we set $s = 21$, so this approximation is good to within roughly t^{-44}. See paper for details.

- R.C. McPhedran and D.H. Bailey, "New results for Euler sums," ArXiv:2311.06294, 19 Jul 2025, https://arxiv.org/abs/2311.06294 or https://www.davidhbailey.com/dhbpapers/NewEulerSums.pdf.

Computing high-precision values of Euler sums, continued

Given a mixed Euler sum such as

$$M(m,n,p,q) = \sum_{k=1}^{\infty} \frac{H(k)^m}{k^n (k+1)^p (k+2)^q},$$

denote $\hat{G}(t) = \hat{H}(t)^m / (t^n (t+1)^p (t+2)^q)$. Using the Euler-Maclaurin formula again,

$$M(m,n,p,q) \approx \sum_{j=1}^{k} \frac{H(j)^m}{j^n (j+1)^p (j+2)^q} + \sum_{j=k+1}^{\infty} \hat{G}(j)$$

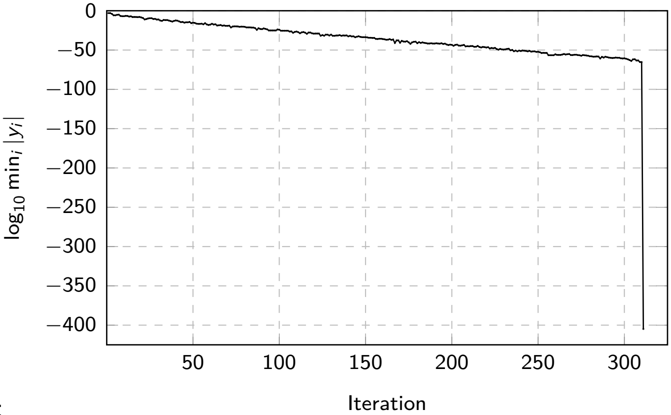
$$\approx \sum_{j=1}^{k} \frac{H(j)^m}{j^n (j+1)^p (j+2)^q} + \int_{k+1}^{\infty} \hat{G}(t)\,\mathrm{d}t + \frac{1}{2}\hat{G}(k+1) - \sum_{j=1}^{s} \frac{B_{2j} D^{2j-1} \hat{G}(k+1)}{(2j)!},$$

where $s = 21$, which is accurate to within roughly k^{-44}. We set $k = 10^9$, so this approximation is correct to within roughly 10^{-396}.

The first three terms were computed using DHB's multiprecision software; the fourth was computed using *Mathematica*. See paper for details.

Using an integer relation algorithm to find Euler sum formulas

With 400-digit values of the Euler sum and the atomic constants, we use the multipair PSLQ integer relation algorithm to find the formula, as in this example:



Resulting formula:

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^3(k+1)^6} = \frac{1}{4}\left(84\zeta(2) - 108\zeta(3) - 5\zeta(4) - 48\zeta(5) + 24\zeta(2)\zeta(3) - 9\zeta(6) + 6\zeta(3)^2 - 12\zeta(7) + 4\zeta(2)\zeta(5) + 4\zeta(3)\zeta(4)\right)$$

Brief selection of more than 1000 formulas produced by the program

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^3(k+1)^5} = \frac{1}{4}\left(60\zeta(2) - 80\zeta(3) - \zeta(4) - 24\zeta(5) + 12\zeta(2)\zeta(3) - 3\zeta(6) + 2\zeta(3)^2\right)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k^2(k+1)^6} = \frac{1}{8}\left(-144\zeta(3) + 144\zeta(4) - 48\zeta(5) + 32\zeta(2)\zeta(3) + 37\zeta(6) - 24\zeta(3)^2 - 16\zeta(7)\right.$$
$$\left. +16\zeta(2)\zeta(5) - 8\zeta(3)\zeta(4) - 28\zeta(8) + 16\zeta(3)\zeta(5) + 8M(2,6)\right)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^2}{k^7(k+2)^2} = \frac{1}{1536}\left(78 + 42\zeta(2) + 102\zeta(3) - 339\zeta(4) + 420\zeta(5) - 120\zeta(2)\zeta(3) - 776\zeta(6)\right.$$
$$+384\zeta(3)^2 + 1728\zeta(7) - 288\zeta(2)\zeta(5) - 720\zeta(3)\zeta(4) - 384M(2,6) + 3520\zeta(9)$$
$$\left. -1344\zeta(3)\zeta(6) - 960\zeta(4)\zeta(5) - 384\zeta(2)\zeta(7) + 128\zeta(3)^3\right)$$

$$\sum_{k=1}^{\infty} \frac{H(k)^3}{k^2(k+1)^2(k+2)^4} = \frac{1}{64}\left(1312 + 740\zeta(2) + 396\zeta(3) - 1203\zeta(4) - 1018\zeta(5) - 304\zeta(2)\zeta(3)\right.$$
$$\left. +25\zeta(6) - 48\zeta(3)^2 - 119\zeta(7) - 32\zeta(2)\zeta(5) + 132\zeta(3)\zeta(4)\right)$$

See paper for full listing of results. To avoid transcription errors, the LaTeX code for the 1000 formulas was generated automatically from the computer output.

Euler sums with $(2k+1)$ in denominator

By including constants such as $\log(2)$, $\log(2)^2$, $\log(2)\zeta(2)$ in the set of constants for the integer relation search, we found the following formulas, among others:

$$\sum_{k=1}^{\infty} \frac{H(k)}{k(2k+1)} = 2\log(2)^2$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^2(2k+1)} = 2\zeta(3) - 4\log(2)^2$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{(2k+1)^2} = \frac{1}{4}\left(7\zeta(3) - 6\log(2)\zeta(2)\right)$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{k^2(2k+1)^2} = 9\zeta(3) - 6\log(2)\zeta(2) - 8\log(2)^2$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{(2k+1)^3} = \frac{1}{32}\left(45\zeta(4) - 56\log(2)\zeta(3)\right)$$

Euler sums with $H(2k)$ and more complicated denominators

$$\sum_{k=1}^{\infty} \frac{H(2k)}{(2k+1)^4} = -\frac{93}{64}\zeta(5) - \frac{3}{4}\zeta(2)\zeta(3)$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{(2k)^2(2k+1)^2} = \frac{9}{4}\zeta(3) - 2(\log(2))^2 - \frac{3}{2}\log(2)\zeta(2)$$

$$\sum_{k=1}^{\infty} \frac{\mathcal{H}(k)}{(2k)^2(2k+1)^2} = -\frac{9}{8}\zeta(3) + (\log(2))^2 - \frac{1}{2}\zeta(2) + \frac{3}{2}\log(2)\zeta(2)$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{(3k+1)^2} = \frac{1}{18}\left(54 + 2\gamma\psi(1,1/3) + \psi(0,1/3)\psi(1,1/3) - \psi(2,4/3)\right)$$

Here $\mathcal{H}(k) = H(2k) - H(k)$, γ is Euler's constant and $\psi(n,z) = D^{n+1}(\log\Gamma(z))$ denotes the polygamma function.

The last line above, among others, suggests that the polygamma function may be a fundamental basis for analyzing Euler sums. This led to the result on the next page.

Stop press: A general formula for a large class of Euler sums

Conjecture 1: Let $Q(k,p) = 1/2$ if p is odd and $k = (p-1)/2$, and 1 otherwise. Then for integers $m, p \geq 2$ and for nonzero n with $|n| < m$,

$$\sum_{k=1}^{\infty} \frac{H(k)}{(mk+n)^p} = \frac{p}{2m^p n^p p!} \left(\frac{m^{p+1} p!}{n} + (-1)^p 2n^p (\gamma + \psi(0, n/m)) \psi(p-1, n/m) \right.$$
$$\left. + (-1)^p 2n^p \sum_{k=1}^{\lfloor (p-1)/2 \rfloor} Q(k,p) \binom{p-1}{k} \psi(k, n/m) \psi(p-1-k, n/m) - (-1)^p n^p \psi(p, 1+n/m) \right)$$

We found this via heavy-duty experimentation using *Mathematica*, OEIS and numerical computation. We do not yet have a proof, but we have checked it, using 100-digit arithmetic, for all $\{2 \leq m \leq 10,\ |n| \leq m-1,\ n \neq 0,\ 2 \leq p \leq 10\}$, a set of 810 cases.

Conjecture 2: Any Euler sum whose denominator is a polynomial with rational roots and no zeros at positive integers has a closed-form evaluation in terms of polygammas.

Proof sketch: Apply Conjecture 1 with a partial fraction decomposition (needs work).

Sample of formulas given by Conjecture 1

$$\sum_{k=1}^{\infty} \frac{H(k)}{(3k+2)^3} = \frac{1}{864} \left(243 - 16\psi(1,2/3)^2 - 16\left(\gamma + \psi(0,2/3)\right)\psi(2,2/3) + 8\psi(3,5/3)\right)$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{(3k-2)^3} = \frac{1}{864} \left(243 - 16\psi(1,-2/3)^2 - 16\left(\gamma + \psi(0,-2/3)\right)\psi(2,-2/3) + 8\psi(3,1/3)\right)$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{(4k+1)^5} = \frac{1}{49152} \left(491520 - 6\psi(2,1/4)^2 - 8\psi(1,1/4)\psi(3,1/4) - 2\left(\gamma + \psi(0,1/4)\right)\psi(4,1/4) + \psi(5,5/4)\right)$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{(4k+3)^3} = \frac{1}{6912} \left(512 - 54\psi(1,3/4)^2 - 54\left(\gamma + \psi(0,3/4)\right)\psi(2,3/4) + 27\psi(3,7/4)\right)$$

$$\sum_{k=1}^{\infty} \frac{H(k)}{(5k+2)^4} = \frac{1}{120000} \left(37500 + 96\psi(1,2/5)\psi(2,2/5) + 32\left(\gamma + \psi(0,2/5)\right)\psi(3,2/5) - 16\psi(4,7/5)\right)$$

Curiously, *Mathematica* is able to produce some of these formulas, in simple cases. How is it doing this?

Final conjecture

Conjecture 3: Any Euler sum of the form

$$\sum_{k=1}^{\infty} \frac{H(k)}{P(k)},$$

where $P(k)$ is any polynomial of degree two or greater with integer coefficients and no zeros at positive integers has a closed-form evaluation in terms of polygammas.

We believe this may be true, based on initial experimental evidence, but more investigation is needed.

This talk is available here:
https://www.davidhbailey.com/dhbtalks/dhb-wcnt-2025b.pdf

A technical paper with many of the above results is available here:
https://www.davidhbailey.com/dhbpapers/NewEulerSums.pdf